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Differential Rotation of Viscous Neutran Matter

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The reaction of a homogeneous sphere of a neutron matter, set in a rotational motion under the influence of an externa1 torque acting on its surface, is investigated. For neutron matter, with a typical neutron star density of 1 0 ~ cm-3 and a temperature varying between 106 and 10' K, originally in uniform rotation, a time dependent differential motion sets in, which lasts a time scale of hours to some decades, resulting finally in co-rotatior During these times, the braking index of a magnetic neutron sphere depends very sensitively on time.

Investiga-se a reação de esfera homogênea de neutrons posta a girar por ação de torque exterior a agir em sua superfície. Para a materia de neutrons, com densidade estelar neutronica de 1015g ~ m - e , temperatura a variar entre 106 e 109 oK, em movimento de rotação uniforme, estabelece-se um movimento diferencial, dependente do tempo, que perdura por um tempo que vai de horas a décadas, resultando em uma co-rotação Durante esse movimento, o índice de freiamento de una esfera de neutrons magnética apresenta dependência mui sensível no tempo.

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1. INTRODUCTION

It is generally accepted that pulsars are rotating neutron stars. As a rule, the interior of such a neutron star contains a liquid phase which consists mainly of neutrons and also contains protons, electrons and negative muons. In more massive neutron stars, with a central density of $\rho_c > 10^{15}$ gcm^{-3} also hyperons become importante constituents (Ruderman[1], Baym and Pethick[2]). However, we are interested in that part of a neutron star which contains normal (i.e. nonsuperfluid) neutron matter. Between densities of about 5×10^{11} gcm^{-3} and 2×10^{14} g cm^{-3} the neutrons are superfluid because of the $^1S_0$ - attraction (Krotscheck[3], Chao et *al.*[4]). The neutrons are supposed to be superfluid again, at densities which exceed the nuclear densities, because of the $^3P_2$ - neutron pairing (Tamagaki[5] , Tamagaki and Takatsuka[6]). It should be pointed out, however, that all these calculations, concerning the $^1S_0$ -pairing as well as the anisotropic $^3P_2$ - superfluidity, should be regarded as first estimates, since the energy gaps depend very sensitively on the nuclear potential used. Detailed calculations performed by Takatsuka[6] confirm that the onset of anisotropic superfluidity is a delicate function of the neutron effective mass. Moreover, it has been investigated by Weyer[7,8] that in the high densities regime, in neutron matter ($\rho \gtrsim 5 \times 10^{14}$ g cm^{-3}), a certain pairing correlation of neutrons in relative singlet states may occur. This pairing is due to the asymmetry of nuclear forces between even and odd states, which suggests the preference of singlet states for neutrons with equal momentum quantum numbers. These correlations build up the so called dineutron clusters , analogously to the a-particle picture. The decision, whether the dineutron correlation or the anisotropic superfluidity prevails in neutron rnatter, still remains unsettled. Therefore we assume[9] - even in the presence of the $^3P_2$ - superfluidity - at least a normal component between - 2×10^{14} and 4×10^{14} g cm^{-3}.

Our following model calculation, which refers to a normal neutronfluid, may be applied to the normal component in the interior of a neutron star.

The question we would like to answer is how the surface angular velocity $\Omega_S(t)$, and the braking index n, defined by

$$n(t) := \Omega_S(t) \,.\, \ddot{\Omega}_S(t) \,.\, \dot{\Omega}_S^{-2}(t) \,, \tag{1}$$

are modified by the existence of viscous normal matter in the interior of a neutron star. This analysis is useful, since the experirnentally found values of n are about 2.5, and special and general relativistic effects have been proven to be too small to give a correction of the desired order (Pfarr[11]).

In the first Section, we give a short epitorne as to the calculation of the first viscosity which we need in the second Section for an approximate solution of the Navier-Stokes equation.

2. VISCOSITY OF NORMAL NEUTRON MATTER

In this Section, we briefly review the essential steps for the evaluation of the first viscosity, in neutron matter, in the framework ofthe Landau theory (Nitsch[12], Heintzmann and Nitsch[9] and the literature cited therein). In doing so, we derive as a first step, a simple representation of Landau's interaction function of the quasi-particles which we need as the most important ingredient, in the Boltzmann-transport equation, to approximate the collision integral. There, this function is related to the forward scattering amplitude of two quasi-particles[13]. The second step only gives a rough sketch of the general assumptions for the evaluation of the first viscosity.

The total energy, E, of an interacting system is a functional of the distribution function $n_\sigma(p)$ of the quasi-particles. If the function $n_\sigma(p)$ is sufficiently close to the ground-state distribution function, $n_\sigma^0(p)$, we carry out an expansion of $E[n]$ (Pines and Nozières[13]), viz.,

$$E[n] - E_0 = \sum_{\sigma,p} \varepsilon_\sigma(p)\,\delta n_\sigma(p) + (2!)^{-1} \sum_{\sigma;\sigma';p,p'} f_{\sigma\sigma'}(p,p')\,\delta n_\sigma(p)\,\delta n_{\sigma'}(p') \tag{2}$$

where the quasi-particle energy, $\varepsilon_\sigma(p)$, is the first, and the interaction function between the quasi-particles, $f_{\sigma\sigma'}(p,p')$, the second, variational derivative of the total energy $E[n]$, i.e.,

$$\varepsilon_\sigma(p) := \delta E[n] / \delta n_\sigma(p) \quad , \tag{3}$$

and

$$f_{\sigma\sigma'}(p,p') := \delta^2 E[n] / \delta n_\sigma(p) \delta n_{\sigma'}(p') . \tag{4}$$

The deviation, $\delta n_o(p)$, from $n^0_\sigma(p)$ is defined by

$$\delta n_\sigma(p) := n_\sigma(p) - n^0_\sigma(p) \quad . \tag{5}$$

A simple approximation of the quantities $\varepsilon_o(p)$ and $f_{\sigma\sigma'}(p,p')$, by means of the Hartree-Fock theory, gives

$$E_{HF}[n] = \sum_{\sigma;p} p^2 (2m)^{-1} n_\sigma(p) + (2!)^{-1} \sum_{\sigma,\sigma';p,p'} \langle p\sigma p'\sigma' | V | p\sigma p'\sigma' \rangle_a n_\sigma(p) n_{\sigma'}(p'), \tag{6}$$

with

$$\varepsilon_\sigma(p) = p^2 (2m)^{-1} + \sum_{\sigma';p'} \langle p\sigma p'\sigma' | V | p\sigma p'\sigma' \rangle_a n_{\sigma'}(p'), \tag{7}$$

and

$$f_{\sigma\sigma'}(p,p') = \langle p\sigma p'\sigma' | V | p\sigma p'\sigma' \rangle_a \quad . \tag{8}$$

We describe the interacting forces between the neutrons using the unitarily transformed (Mittelstaedt *et al.*[14]) Gammel-Christian-Thaler potential (Gammel *et al.*[15]).

in the considerations above, we dealt with stable, homogeneous distributions, for which the function $n_\sigma(p)$ neither depended on time nor on the relative position of the quasi-particles. In a more general case, however, we consider a weak time dependent inhornogeneous perturbation of the ground state of our system. As a consequence, the distribution function of the quasi-particles becomes, in the classical limit, explicit dependent on time and position: $n_\sigma = n_\sigma(p,r,t)$.

We determine $n_\sigma(p,r,t)$ by solving the Boltzmann equation

$$\partial n/\partial t + \{n,\tilde{\varepsilon}\} = I(n), \tag{9}$$

where $I(n)$ is the collision integral of the quasi-particles, which we approxirnate, in the case of binary collisions, specified by

$$p_1 + p_2 \rightarrow p_1' + p_2' \quad , \tag{10}$$

in terms of Born collision cross sections using the interaction function of Eq.(8). The local excitation energy of a quasi-particle is equal to

$$Eo(p,r) = \varepsilon_\sigma(p) + \sum_{\sigma';p'} f_{\sigma\sigma'}(p,p')\,\delta n_{\sigma'}(p',r). \tag{11}$$

Once the collision integral is known, we can study the transport properties of the system such as viscosity, thermal conductivity or spin diffusion. If we impose on the system an inhomogeneous static perturbation, containing a velocity gradient, this gradient induces a flow of momentum which is only limited by the collisions between quasi-particles, and which is - in the co-mving coordinate system - proportional to the imposed velocity gradient. Thus, the first viscosity, η, is defined as the proportionality coefficient between the momentum flux density tensor Π_{ik} and the expression

$$\{\partial v_i/\partial x_k + \partial v_k/\partial x_i - (2/3)(\partial v_i/\partial x_i)\delta_{ik}\} . \tag{12}$$

Since the calculation of η is somewhat lengthy and cumbersome, we here only quote the final result[16]:

$$\eta = \omega(\rho) \,.\, T^{-2} . \tag{13}$$

The density dependent function $\omega(\rho)$ contains mainly all those quantities which arise from the interaction of the neutrons such as forward scattering amplitudes, and the effective mass m^* of the quasi-particles. The explicit expression for $\omega(\rho)$, as well as a Table for some η-values, are given in the work by Heintzmann and Nitsch[3]. However, the result (13) can be easily understood by some qualitative arguments: since the neutron-neutron scattering is restricted to lie within a layer of width $(k_B T)$, around the Fermi surface (i.e. all elementary excitations of interest are to be found in this layer), the transition probability for the process (10), in the "thermal limit"[13], is of order $(k_B T)^2$. This leads to a quasi-particle lifetime, τ_p, proportional to T^{-2}. Moreover, τ_p represents a qualitative measure for the collision time of quasi-particles which is proportional to the first viscosity, η, according to the elementary kinetic theory of gases. We shall see, in the next Section, that the temperature dependence of η very sensitively affects the duration of the differential rotation in neutron matter.

3. DIFFERENTIALLY ROTATING NEUTRON FLUID

We want, first, to describe our model. We consider a uniformly rotating sphere, of neutron fluid, with a solid outer layer. At a certain time, t_0, we apply a torque from outside to the surface of the sphere, by switching on a homogeneous magnetic field in its interior. This torque causes a braking at the surface and thereby produces - because of the viscosity η - a velocity gradient within the star matter (Fig.1). We determine this velocity field by solving the "special" Navier-Stokes equation,

$$dv/dt = \partial v/\partial t + (v\nabla)v = -\rho^{-1}\nabla p - \nabla\Phi + \rho^{-1}\eta\nabla^2 v, \qquad (14)$$

where we have already assumed that the density, ρ, as well as the viscosity, η, does not depend on position and time. For $v = 0$, Eq.14 reduces to the well-known condition for the hydrostatic equilibrium:

$$\nabla p = -\rho\nabla\Phi, \qquad (15)$$

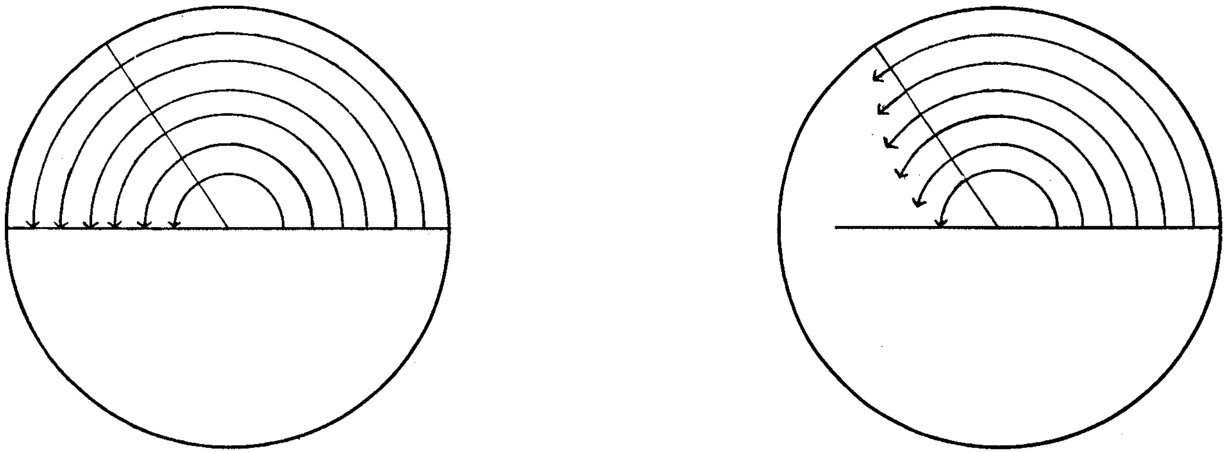

Fig. 1a

Fig. 1b

Fig. 1 - Velocity fields; before (a) and after (b), switching on the magnetic dipole field.

where p is the pressure of the neutron matter, and Φ is the Newtonian gravitational potential. From the form of Eq.14, we get a rough estimate of the dissipation time of the viscous forces, which is

$$\tau_{vis} = \rho L^2 \eta^{-1} , \tag{16}$$

where L is the characteristic length scale of the velocity field. This time, τ_{vis}, is independent of the initial condition for the differential equation (14). For $L \sim R$, where R is the radius of the neutron star, and for a density $\rho = 10^{15}$ gcm^{-3} and a temperature $T = 10^8$K, we get $\tau_{vis} \approx 10^8$ sec. $\approx$ 3 years. Afterwards, the matter is rigidly co-rotating again.

We simplify (14) using the following *Ansatz* (cf. Heintzmann et *al.*[17]) for the velocity field $v(r,t)$,

$$v(r,t) = \Omega(r,t) r \sin\theta \, e_\phi . \tag{17}$$

In spherical coordinates, we obtain from (14) the following partial differential equation for $\Omega(r,t)$:

$$\partial\Omega(r,t)/\partial t = \rho^{-1} \eta\{\partial^2\Omega(r,t)/\partial r^2 + 4r^{-1}\, \partial\Omega(r,t)/\partial r\} . \tag{18}$$

Here, we have already assumed $\nabla p \approx - \rho\nabla\Phi$. According to the results of Heintzmann *et al.*[17,18], this assumption is justified.

For the complete solution of Eq.18, we need initial and boundary conditions, which we define as follows:

Initial condition:

$$\Omega(r,0+) =: \Omega_0 ;$$

boundary condition:

(i) The solution has to be regular at the origin, $r = 0$.
(ii) $\Omega(R,t) = \Omega_S(t)$ is a function of time, with $\Omega_S(0+) = \Omega_0$, which will be specified by the torque equilibrium condition exhibited below Eq.(24).

The variable R denotes the radial coordinate at the surface of the star. The solution of the Laplace-transform of the differential equation (18) reads

$$\Omega_L(r,s) = s^{-1}\,\Omega_0 + (\omega_0(s) - s^{-1}\Omega_0)\left(\frac{R}{r}\right)^3 \frac{\sinh(r(s/\nu)^{1/2}) - r(s/\nu)^{1/2}\cosh(r(s/\nu)^{1/2})}{\sinh(R(s/\nu)^{1/2}) - R(s/\nu)^{1/2}\cosh(R(s/\nu)^{1/2})}, \tag{19}$$

with

$$\omega_0(s) := L\{\Omega_S(t)\}\ ,$$

and the quantity $\nu = \rho^{-1}\eta$ is the kinematic viscosity. The torque, at the surface of the sphere, induced by the velocity field ofthe viscousfluid given by Eq. 19 can easily be given by (cf. Landau and Lifschitz[19])

$$D_L(s) = \int \eta R(\partial\Omega_L(r,s)\,|\,\partial r)\Big|_{r=R} R\sin\theta . 2\pi R^2 \sin^2\theta d\theta = \frac{8}{3}\pi R^4 \eta(\partial\Omega_l\,|\,\partial r)\Big|_{r=R} \quad . \tag{20}$$

Using Eq. (19), we explicitly get, for $D_L(s)$, the expression

$$D_L(s) = \frac{8}{3}\,\pi R^3\,\eta(\omega_0(s) - s^{-1}\Omega_0)\ \left\{\frac{-R^2(s/\nu)}{1 - R(s/\nu)^{1/2}\coth(R(s/\nu)^{1/2})} - 3\right\}\quad . \tag{21}$$

As already mentioned in the Introduction, we are interested in the influente of the viscous forces on the braking index, $n(t)$. Using the balance condition for the torques we evaluate $\Omega_S(t)$ in the following way. The function $D_L(s)$, in Eq. (21), cannot be transformed analytically into the original space of the Laplace-transform. We, therefore, give the representation of D (t) in the two limiting cases:

a) $\delta_t \ll R$ and b) $\delta_t \gg R$, (22) here δ_t means the depth of penetration of the velocity field, at time t (cf. Landau and Lifschitz [19]).

For $\delta_t << R$, we get from Eq.21, the result

$$D(t) = \frac{8}{3}\pi R^4 (\pi^{-1}\eta\rho)^{1/2} \int_0^t d\tau (d\Omega_S(\tau)|d\tau).(t-\tau)^{-1/2}, \quad (23)$$

which, likewise, represents the solution of the analogous "plane" problem[19]. In Table 1, we exhibit some values of times when the relation (a) is valid. If we assume now that, at time t_0, the braking at the

$T[K]$	$\eta[10^{17}$ poise$]$	t_c	$n(t_c)$
10^6	1.6×10^6	1/2 hour	3×10^6
10^7	1.6×10^4	2 days	3×10^4
10^8	1.6×10^2	1/2 year	3×10^2
10^9	1.6	50 years	3

Table 1. All values are, here, given for constant density, $\rho=10^{15}gcm^{-3}$. The time parameter, t_c, is calculated by equating the radius R and the depth of penetration $\delta_t := 2(\eta\rho^{-1}t)^{1/2}$: $t = 4^{-1}\eta^{-1}\rho R$. The two cases, (a) and (b), in the text, refer to $t<<t_c$ and $t>>t_c$, respectively.

surface of our fluid sphere is caused by the torque of a magnetic dipole field, i.e.,

$$D_{elm}(\Omega_S) = -(2/3c^3)\Omega_S^3 \mu^2 \sin^2\chi = D_{vis}(\Omega_S), \quad (24)$$

we get the following integro-differential equation for $\Omega_S(t)$:

$$-(2/3c^3)\Omega_S^3\ \mu^2\sin^2\chi = \frac{8\pi}{3} R^4(\pi^{-1}\eta\rho)^{1/2} \int_0^t d\tau\,(d\Omega_S|d\tau).(t-\tau)^{-1/2} . \tag{25}$$

Here, χ means the angle between the axis of rotation and the magnetic dipole axis; $\mu = B_0R^3$ is the magnetic dipole moment, and c the velocity of light. Two-fold iterative integration of (25), with the starting *Ansatz* $\Omega_S^{(1)}(t) = \Omega_0 + \text{const.}\,(t)^{1/2}$, leads to an approximate solution, for small values of **t**:

$$\Omega_S^{(2)}(t) = \Omega_0 - (\Omega_0^3|c_\eta\pi)\ t^{1/2}(1 - \tfrac{3}{2}\,\Omega_0^2\, c_\eta^{-1} t^{1/2}), \tag{26}$$

with

$$c_\eta : = \frac{8}{3}\pi R^4(\pi^{-1}\eta\rho)^{1/2}\ (3c^3|2\mu^2).$$

Hence, it follows for the braking index, n, in the limit $t\to 0$, the expression:

$$n(t) = c_\eta\pi\,\Omega_0^{-2}\ t^{-1/2} - 1 . \tag{27}$$

Obviously, the braking index, n, diverges to $+\infty$ as **t** goes to nought. And putting $\Omega_0 \approx \Omega_{crit} = (R^{-3}GM)^{1/2} \approx 0.8 \times 10^4\ \sec^{-1}$ (Heintzmann et *al.*[18]) we see that n decreases from $+\infty$ to values of about 100 to 10, within the allowed time scales (cf. Table 1).

In case (b), the final state of a rotating viscous sphere of neutron star matter has already been studied by Heintzmann et *al.*[17]. We, however, are interested in a dynamical process which corresponds to this final state.

For $\delta_t >> R$, the expansion of (21), and the following transforrnation into the original space, leads to the result:

$$D(t) = (8/15)\pi R^5\rho\,\{\dot\Omega_S(t) + (35)^{-1}R^2\eta^{-1}\rho\ddot\Omega_S(t)\} . \tag{28}$$

In the limit $\eta\to\infty$, we get the torque of a rigidly rotating sphere with a moment of inertia

$$I : = (8/15)\pi R^5\rho . \tag{29}$$

Using the torque equilibrium condition, $D_{vis}(\Omega_S) = D_{elm}(\Omega_S)$, we now obtain an ordinary differential equation of the second order, for $\Omega_S(t)$, which reads

$$I\{\dot{\Omega}_S(t) + a_\eta \ddot{\Omega}_S(t)\} = -\alpha\, \Omega_S^3(t) \ , \tag{30}$$

where

$$a_\eta : = (35)^{-1} R^2 \eta^{-1} \rho \ , \quad \text{and} \quad \alpha : = (3c^3)^{-1} . \ 2\mu^2 \sin^2\chi \ . \tag{31}$$

Those times for which (30) is a valid approximation can be taken from Table 1. It is useful to write (30) in terms of dimensionless parameters and variables. For that purpose, we define the following quantities

$$b : = (a\ I)^{-1} \alpha (t_A \Omega_A)^2 . \tag{32}$$

$$x : = t/t_A \ , \quad \Omega_A : = \Omega_S(t_A), \quad y : = \Omega_S/\Omega_A \ , \quad a : = a_\eta^{-1} t_A \ .$$

The constant time, t_A, defines the onset of the validity for the limiting case (b). Equation (30) reads, now

$$d^2y|dx^2 + a\, dy|dx + by^3 = 0 \ . \tag{33}$$

The coefficients, a and b, are of the same order of magnitude, but both are very large compared to 1 (see Table 2). For that reason, we inter-

$T[K]$	t_A [sec]	Ω_A [sec^{-1}]	a	b/a
10^7	10^8	10^4	6×10^3	1.5
	3×10^{10}	2×10^2	2×10^6	1.8×10^{-1}
10^8	10^8	10^4	6×10	1.5
	3×10^{10}	2×10^2	2×10^4	1.8×10^{-1}

Table 2. The coefficients, a and b, of the differential equation (33) are given for some special values of ternperatures, and initial data t_A and Ω_A.

pret the contribution due to the second derivative, in Eq. (33), as a perturbation of the differential equation

$$ady|dx + by^3 = 0 \, . \tag{33'}$$

Its solution corresponds to the slow down law of the rigidly rotating magnetic dipole and leads to the braking index $n = 3$. The approximate solution of Eq. (33) is $y_0' = -a^{-1}b\,y_0^3$, and

$$y = y_0 + a^{-1} y_0^3(1 + a^{-1}b \ln y_0^3), \tag{34}$$

whence we derive the braking index

$$n(t) = 3(1 + (35)^{-1} 2\alpha I^{-1} R^2 \eta^{-1} \rho\Omega_S^2(t)) \, . \tag{35}$$

Since we cannot give the complete solution $\Omega_S(t)$, for all values of t, we are not able to give the exact initial value Ω_A. If we assume the period of the crab pulsar to have the value $\Omega_S(t)$, the additive term in (35) is of the order of 10^{-2} to 10^{-4}. It only gives a positive contribution of the order of percents to the braking index of the magnetic dipole.

4. RESULTS AND DISCUSSION

As we have seen in the first Section, the first viscosity of neutron matter depends on the nuclear forces and on the density of the matter. It is proportional to T^{-2}. Especially, the temperature influences the duration of viscous forces in the case of differential rotation of the neutron matter. Within a more detailed investigation, in the second Section, we come to know that viscous forces cannot be regarded as acting during astronomical time scales ($\tau_{vis}^{max} \approx 100$ years). From Table 2, we see that the coupling of the differential rotation to the rigid one takes time scales of years to decades. However, in the above treatment we did not consider turbulences as a possible mechanism to destroy the velocity field[17]. In Section 2, we have also answered the question of how far the presence of differentially rotating viscous matter influences the slowdown law of a pulsar. The resulting fact that the braking index, $n(t)$, is always larger than 3 is a consequence of our model calculation: while the surface of the star is already braked by the dipole radiation, the interior maintains its original angular velocity and therefore hurries on in advance of the surface (Fig. Ib). As a consequence, angular momentum is transported from the inside to the surface due to collisions of the particles within the viscous fluid. This transported angular momentum leads to a braking index larger than 3.

The interaction of the rnagnetic field, in the interior of the star,with the neutrons, may be neglected because of the large Fermi energy of the neutrons in comparison to (μB) (cf. Pfarr[20]). A more detailed discussion would have to include the existence of electrons, protons and negative muons, as constituents of a neutron star, together with their interactions with the magnetic field.

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