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A Generalized Eckart Wave Function*

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A general form of the Eckart wave function is proposed for the l^{th} partial wave. Analytic expressions for the integrals usually encountered are also presented.

Propõe-se uma forma geral para a função de Eckart para a onda parcial l. Apresentam-se também expressões analíticas para integrais frequentemente encontradas.

The Eckart wave function[1] has been in use for a long time[2-8], as an approximate analytic form for the relative wave function of a single particle outside a given core. Due to this wide variety of problems where Eckart wave functions are used and, even more recently[10], in the study of the important low energy reaction n+p → d + 2γ, a generalization of the Eckart wave function is here presented. Analytic expressions for the integrals which mostly occur are given. The wave function was suggested phenomenologically from the behavior near the origin as well as the behavior in the asymptotic region to be of the form[e]

$$\psi(r) = N\left(1 - e^{-r/R}\right)^{n+1} \frac{e^{-kr}}{r}, \qquad (1)$$

where N is a normalization constant and R, n, k are parameters with the following physical significance: R is the cut off radius, below which

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the first factor dominates and gives a r^n dependence near the origin . On the other hand, for $r \gg R$, the asymptotic form is e^{-kr}/r, indicating that k is the wave number corresponding to the separation energy, E_{sep}:

$$k^2 = 2\mu E_{sep}/\hbar^2 \quad ; \quad \mu = (\text{ reduced mass }) \tag{2}$$

Hence one has essentially two free parameters n and R. In principle , if we can guess the form of the wave function near the origin, we may choose a specific value of n and thus be left with only one adjustable parameter R.

The form e^{-kr}/r is the correct asymptotic form for the s partial wave $(l=0)$. However, one can easily show that the correct asymptotic formfor the l^{th} partial wave should be[9]

$$\psi_l(r) \xrightarrow[r \to \infty]{} \frac{e^{-kr}}{kr} \left[1 + \sum_{m=1}^{l} \frac{1}{m!\,(2kr)^m} \prod_{n=1}^{m} (l+n)\ (l+1-n) \right] . \tag{3}$$

The form (3) is the exact bound solution for the l^{th} partial wave for $V(r) \to 0$ (with no Coulomb interaction). Thus Eq. (3) is a more refined asymptotic form than Just e^{-kr}/r. Hence we propose a refined Eckart wave function for the l^{th} partial wave as

$$\psi_l(r) = N \sum_{m=0}^{l+1} a_m \left(1 - e^{-r/R}\right)^{l+m+1} \frac{e^{-kr}}{r^{m+1}} , \tag{4}$$

where

$$a_m = \frac{1}{m!\,2^m k^{m+1}} \prod_{n=1}^{m} (l+n)(l+1-n) , (m \neq 0) \tag{5}$$

and

$$a, = l/k .$$

The factor N is a normalization constant. The wavefunction (4) has only one adjustable parameter, *viz.*, R. This gives to ψ_l the correct dependence near the origin, $\psi_l(r) \xrightarrow[r \to 0]{} r^L$, as well as in the asymptotic region.

The form (4) of the Eckart wavefunction is also very convenient for evaluating many often encountered integrals in analytic form. A similar form has already been used by us to propose a suitable analytic wave function for the deuteron $D(l=2)$ state[10]. Write Eq. (4) in the form

$$\psi_l(r) = N \sum_{m=0}^{l} a_m \phi_{lm}(r) \ . \tag{6}$$

We need to evaluate integrals of the general form

$$\int_0^\infty \phi_{lm'}(r) e^{-\sigma r} r^j \phi_{lm}(r)\, dr \ , \tag{7}$$

where j is an integer, positive or negative. For the functional form of $\phi_{lm}(r)$, we need to evaluate only integrals of the form

$$I(\alpha,\beta,p,q) = \int_0^\infty (1-e^{-\beta r})^p \frac{e^{-\alpha r}}{r^q}\, dr \ , \tag{8}$$

where p, q are integers and q can be positive as well as negative. With appropriate definition of α, β, p, q, the integral of the form (7) can be expressed in terms of $I(\alpha,\beta,p,q)$. Differentiating Eq. (8) with respect to β, we get a recurrence relation

$$\frac{\partial I(\alpha,\beta,p,q)}{\partial \beta} = pI(\alpha+\beta,\beta,p-1,q-1) \ . \tag{9}$$

Integrating over β

$$I(\alpha,\beta,p,q) = p \int_0^\beta I(\alpha+\beta',\beta',p-1,q-1)\, d\beta' \ , \tag{10}$$

since $I(\alpha,0,p,q) = 0$. For the special case $p=q$, we have

$$I(\alpha,\beta,0,0) = \frac{1}{\alpha} \ , \quad I(\alpha,\beta,1,1) = \log\left(\frac{\alpha+\beta}{\alpha}\right) \ . \tag{11}$$

We can now show by induction that for $p \geq 1$,

$$I(\alpha,\beta,p,p) = \frac{1}{(p-1)!} \sum_{i=0}^{p} \binom{p}{i} (-)^{p-i} (\alpha+i\beta)^{p-1} \log(\alpha+i\beta). \tag{12}$$

To obtain the general form $p \geq q$, we write

$$I(\alpha,\beta,p,q) = \int_0^\infty (1-e^{-\beta r})^{p-q} (1-e^{-\beta r})^q \frac{e^{-\alpha r}}{r^q} dr$$

$$= \sum_{j=0}^{p-q} \binom{p-q}{j} (-1)^j \int_0^\infty (1-e^{-\beta r})^q \frac{e^{-(\alpha+j\beta)r}}{r^q} dr$$

$$= \sum_{j=0}^{p-q} \binom{p-q}{j} \frac{(-1)^j}{(q-1)!} \sum_{i=0}^{q} \binom{q}{i} (-1)^{p-i} (\alpha+j\beta+i\beta)^{q-1} \log(\alpha+j\beta+i\beta), \tag{13}$$

substituting from Eq. (12) in the last step. Introducing a new dummy index $s = i + j$ and utilizing the elementary relation

$$\sum_{j=0}^{p-q} \binom{p-q}{j} \binom{q}{s-j} = \binom{p}{s} , \tag{14}$$

we obtain the final form

$$I(\alpha,\beta,p,q) = \frac{1}{(q-1)!} \sum_{s=0}^{p} \binom{p}{s} (-1)^{q-s} (\alpha+s\beta)^{q-1} \log(\alpha+s\beta) . \tag{15}$$

(for $p \geq q \geq 1$) .

For q negative or zero, we can simply expand $(1-e^{-\beta r})^p$ and use elementary integral formulas. The final expression for $I(\alpha,\beta,p,q)$ is very simple and convenient for investigating the properties of the extended Eckart wave function.

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