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Classical Optics and Curved Spaces*

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In the eikonal approximation of classical optics, the unit polarization 3-vector of light satisfies an equation that depends only on the index, n, of refraction. It is known that if the original 3-space line element is $d\sigma^2$, then this polarization direction propagates parallely in the fictitious space $n^2 d\sigma^2$. Since the equation depends only on n, it is possible to invent a fictitious curved 4-space in which the light performs a null geodesic, and the polarization 3-vector behaves as the "shadow" of a parallely propagated 4-vector. The inverse, namely, the reduction of Maxwell's equation, on a curved (dielectric free) space, to a classical space with dielectric constant $n=(-g_{00})^{-1/2}$ is well known, but in the latter the dielectric constant ε and permeability μ must also equal $(-g_{00})^{-1/2}$. We calculate the rotation of polarization as light bends around the sun by utilizing the reduction to the classical space. This (non-) rotation may then be interpreted as parallel transport in the 3-space $n^2 d\sigma^2$.

Na aproximação do eikonal da óptica clássica, o vetor unitário de polarização da luz satisfaz uma equação que depende somente do índice de refração, n. Sabe-se que se o elemento de linha, no espaço tridimensio-

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nal original, for $d\sigma^2$, esse vetor de polarização propagar-se-á paralelamente no espaço fictício, $n^2 d\sigma^2$. Como a equação depende somente de n é possível criar um espaço quadridimensional fictício no qual a luz descreve uma geodésica nula, e o vetor unitário de polarização comporta-se como a "sombra" de um quadrivetor paralelamente propagado. O inverso, isto é, a redução das equações de Maxwell, em um espaço curvo, a um espaço clássico com constante dielétrica $n=(-g_{00})^{-1/2}$, é bem conhecido, se bem que neste caso a constante dielétrica ε e a permeabilidade μ são também iguais a $(-g_{00})^{-1/2}$. Utilizando-se da redução ao espaço clássico, estudamos a rotação da polarização da luz quando essa curva-se ao passar nas vizinhanças do sol. Essa (não-) rotação pode ser interpretada como transporte paralelo no espaço tridimensional $n^2 d\sigma^2$.

1. INTRODUCTION

It is well known, and cited in texts on classical optics[1,2] that the behavior of light in a space with dielectric constant ε, permeability μ, and index, n, of refraction in a metric of the form

$$\begin{aligned} ds^2 &= -c^2dt^2 + d\sigma^2 \quad ; \quad n \neq 1 \ , \ \varepsilon \neq 1 \ , \ \mu \neq 1 \ , \\ d\sigma^2 &= \gamma_{\mu\nu}\, dx^\mu\, dx^\nu \quad ; \quad \mu,\nu = 1,\ 2,\ 3 \ , \end{aligned} \qquad (D\text{-space}) \qquad (1.1)$$

which we shall call "dielectric" or D-space, can be redescribed simply, in the eikonal approximation, in a fictitious space ("B"-space, or the barred space) by

$$\begin{aligned} d\bar{s}^2 &= -c^2dt^2 + d\bar{\sigma}^2 \ , \\ d\bar{\sigma}^2 &= \bar{\gamma}_{\mu\nu}\, dx^\mu\, dx^\nu \ , \quad \bar{\gamma}_{\mu\nu} = n^2\, \gamma_{\mu\nu} \ . \end{aligned} \qquad (B\text{-space}) \qquad (1.2)$$

Here, $\gamma_{\mu\nu}$ represents the original 3-space metric in curvilinear coordinates, and it could just as well represent a curved 3-space.

The reduction of Eq. (1.1) to (1.2) is possible only for those properties that depend only on the index, n, of refraction. These are then the trajectory itself and the propagation of the unit 3-vector in the direction of the electric or magnetic field. In terms of the B-space, the trajectory is a geodesic, and the unit polarization vector undergoes parallel transport.

On the other hand, it is well known and cited on texts on General Relativity[3] that Maxwell's equations, in a static curved space, without dielectric constant, permeability, or index of refraction ("G" or "gravitational" space) with metric

$$d\Sigma^2 = g_{00}\, c^2 dt^2 + d\sigma^2 \,,$$

(G-space) (1.3)

$$d\sigma^2 = \gamma_{\mu\nu}\, dx^\mu\, dx^\nu \,,$$

can be redescribed in terms of a space of the type in Eq. (1.1), namely D-space with the identification of the dielectric parameters as

$$\varepsilon = \mu = n = (-g_{00})^{-1/2} \,. \qquad (1.4)$$

It is clear that the latter situation can be turned around to clarify the classical problem: starting with the dielectric space, of Eq. (1.1), without Eq. (1.4) being valid, i.e., without necessarily $\varepsilon = \mu = n$, one can introduce a fictitious curved space of the type in Eq. (1.3), provided, however, that the properties under consideration depend only on n. This then applies to the trajectory and the unit polarization 3-vector. The former will be a null geodesic in the fictitious G-space, and the latter will behave as the "shadow" of a 4-vector that is parallel displaced along the trajectory. Thus, from the point of view of the classical polarization problem (D-space), one can either introduce the fictitious B-space or the fictitious G-space for a simplifying redescription in terms of parallel transport.

It is also clear that if one starts with a real curved space, light propagation, as well as other electromagnetic phenomena, can be understood in terms of a fictitious D-space or B-space. We shall apply this reduction to the special case of the propagation of polarization of light when it bends around the sun.

The basic ideas of this paper are not new. They are implied by the general relativistic eikonal approximation as set forth by Ehlers[4], and by the reduction in Ref. 3 cited above. What we wish to do is clarify their application to some special cases, and in particular to the transition from the classical D-space to the curved G-space.

In Sect.2, the relationship between a 4-space covariant derivative and a 3-space one is discussed briefly. In Sect.3, classical eikonal theory in a dielectric medium is discussed, with a simplified reduction to B-space presented in Sect.4. The redescription of the classical problem, in terms of the fictitious G-space, is contained in Sect. 5. Here, the relation between the geodesic behavior in B-space, and that in G-space, is discussed, and the polarization behavior in G-space derived. In Sect. 6, the use of D-space, to calculate a problern originating in a curved space, is presented.

2. MATHEMATICAL DIGRESSION

In this paper, Latin indices represent values 0,1,2,3, whereas Greek indices represent values 1,2,3. We call, in this Section,

$$g_{00} = -n^{-2} . \tag{2.1}$$

It will be assumed that the metric, in Eqs. (1.2) or (1.3), is static. Then, the Christoffel symbols, appropriate to Eq. (1.3), that are not zero, are

$$\Gamma^{\mu}{}_{\alpha\beta} ,$$

$$\Gamma^{\mu}{}_{00} = - \gamma^{\mu\nu} n^{-2} \frac{\partial}{\partial x^{\nu}} \ln n , \tag{2.2}$$

$$\Gamma^0_{0\mu} = -\frac{\partial}{\partial x^\mu} \ln n .$$

Light will be supposed to be moving along some path described by a spatial parameter q, in the space of Eq.(1.3), or by the distance σ, where

$$d\sigma^2 = \gamma_{\mu\nu}\, dx^\mu\, dx^\nu \tag{2.3}$$

is the space part of Eqs. (1.3) or (1.2). It is assumed that a functional relationship exists, namely,

$$q = q(\sigma) \tag{2.4}$$

(See Section 5 for this function.).

If Γ^i_{jk} represents the Christoffel symbols in the 4-space of Eq.(1.3), then an absolute derivative, along q, is

$$\frac{Da^i}{Dq} = \frac{da^i}{dq} + \Gamma^i_{jk}\, a^j \frac{dx^k}{dq} , \tag{2.5}$$

whereas a 3-space absolute derivative, described by σ, is

$$\frac{{}^3Da^\mu}{D\sigma} = \frac{da^\mu}{d\sigma} + \Gamma^\mu_{\alpha\beta}\, a^\alpha \frac{dx^\beta}{d\sigma} . \tag{2.6}$$

If for "i", in Eq.(2.5), we use a space index, say 1, or μ, the result will not be the same as in Eq. (2.6), since there exist non zero Γ^μ_{00}'s:

$$\frac{Da^\mu}{D\sigma} = \frac{{}^3Da^\mu}{D\sigma} + \Gamma^\mu_{00}\, a^0 \frac{dx^0}{d\sigma} \tag{2.7}$$

$$\frac{Da^0}{Dq} = \frac{da^0}{dq} + \Gamma^0_{0\mu}\left(a^0 \frac{dx^\mu}{dq} + a^\mu \frac{dx^0}{dq}\right) . \tag{2.8}$$

For the gradient, curl and Laplacian, in 3-space, we have in mind the usual weight zero formulation appropriate to general curvilinear coordinates. In what follows, $\gamma_{\mu\nu}$ could represent a space that is already curved.

3. CLASSICAL EIKONAL THEORY

We follow here the discussion of Born and Wolf[5]. In an isotropic dielectric medium, with $D = \varepsilon E$, $B = \mu H$, and

$$n^2 = \varepsilon\mu \ , \tag{3.1}$$

and in a space of the type in Eq, (1.1), the electric field satisfies

$$\nabla^2\vec{E} - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} + (\nabla \ln\mu) \times (\nabla \times \vec{E}) + \nabla\left[\vec{E}\cdot\nabla\ln\varepsilon\right] = 0 \ . \tag{3.2}$$

Solving, the lowest order eikonal approximation, consists in the expansion in k^{-1} of the assurned form

$$\vec{E} = \vec{e}(\vec{r})\ e^{ik_0 S(\vec{r})}\ e^{-i\omega t} \ , \tag{3.3}$$

where S is the eikonal scalar, and $k_0 = \omega/c$. Eq.(3.3) in (3.2) gives

$$\vec{K} + \vec{L}/(ik_0) + \vec{M}/(ik_0)^2 = 0 \tag{3.4}$$

where $\vec{K}$, $\vec{L}$ and $\vec{M}$ are independent of k_0.
The eikonal approximation, in lowest order, sets $\vec{K} = 0$, then $\vec{L} = 0$:

$$\vec{K} = \left[n^2 - (\nabla S)^2\right]\vec{e} = 0, \tag{3.5}$$

$$\vec{L} = \left[\nabla S\cdot\nabla\ln\mu - \nabla^2 S\right]\vec{e} - 2(\vec{e}\cdot\nabla\ln n)\nabla S - 2(\nabla S\cdot\nabla)\vec{e} = 0. \tag{3.6}$$

S is regarded as a wave front, with its normals representating rays . Thus, from Eq. (3.5), the identification

$$n\,\frac{dx^\mu}{d\sigma} = \gamma^{\mu\nu}\,\frac{\partial S}{\partial x^\nu} \tag{3.7}$$

is made. Squaring both sides gives $n^2=(\nabla S)^2$, which is Eq.(3.5).

Eq. (3.7) is the ray equation. Or, if desired, S can be eliminated, by taking a derivative[6], to obtain

$$\frac{^3D}{D\sigma}\left(n\,\frac{dx^\mu}{d\sigma}\right) = \gamma^{\mu\nu}\,\frac{\partial n}{\partial x^\nu}\ . \tag{3.8}$$

The polarization is more complicated. However, if the unit vector u^α,

$$u^\mu = \frac{e^\mu}{(e^\nu\ .\ e^\nu)}\ ,\quad u_\mu u^\mu = 1\ , \tag{3.9}$$

is introduced, then u^α satisfies a relatively simple equation, depending only on n, derived from Eq. (3.6) (Ref.7):

$$\frac{^3Du^\mu}{D\sigma} + \left[u^\alpha\,\frac{\partial}{\partial x^\alpha}\,\ln n\right]\frac{dx^\mu}{d\sigma} = 0\ . \tag{3.10}$$

Eqs. (3.8) and (3.10) are the basic equations of the eikonal approximation. One can also get information about intensity, from squaring (3.3) and finding $\vec{e}$, but that is irrelevant to this paper.

4. INTRODUCTION OF THE BARRED SPACE

As mentioned earlier, Eq.(3.8) can be interpreted as a geodesic in the barred space of Eq.(1.2), and Eq.(3.10) can be interpreted as parallel propagation in the same space. In this Section, we present a simplified derivation of these results. The reduction is possible since Eqs. (3.8) and (3.10) depend only on n, and not, say, on ε specifically.

First consider Eq. (3.8). A geodesic in the barred space is defined by

$$\frac{^3D}{D\bar\sigma}\,\frac{dx^\mu}{d\bar\sigma} = \frac{d}{d\bar\sigma}\,\frac{dx^\mu}{d\bar\sigma} + \bar\Gamma^\mu{}_{\alpha\beta}\,\frac{dx^\alpha}{d\bar\sigma}\,\frac{dx^\beta}{d\bar\sigma} = 0\ , \tag{4.1}$$

where $\bar{\Gamma}$ is computed from the $\bar{\gamma}_{\mu\nu}$, where $\bar{\gamma}_{\mu\nu} = n^2\gamma_{\mu\nu}$. From the definition, we get

$$\bar{\Gamma}^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta} + \delta^{\mu}_{\alpha} \frac{\partial}{\partial x^{\beta}} \ln n + \delta^{\mu}_{\beta} \frac{\partial}{\partial x^{\alpha}} \ln n - \gamma_{\alpha\beta}\gamma^{\mu\lambda} \frac{\partial}{\partial x^{\lambda}} \ln n \,. \tag{4.2}$$

Eq. (4.1) becomes, with this,

$$\frac{d^2 x^{\mu}}{d\bar{\sigma}^2} + 2\left(\frac{\partial}{\partial x^{\nu}} \ln n \right) \frac{dx^{\nu}}{d\bar{\sigma}} \frac{dx^{\mu}}{d\bar{\sigma}} = \frac{1}{n^2} \gamma^{\mu\nu} \frac{\partial}{\partial x^{\nu}} \ln n \,. \tag{4.3}$$

But this is easily seen to be Eq. (3.8), if in the latter we write $d\sigma = d\bar{\sigma}/n$. Thus Eq. (3.8) can be interpreted as a geodesic in the[4] barred 3-space.

For the polarization equation, (3.10), we must take into account that if the vector is described as u^{α}, in the 3-space of Eq. (1.1), then the same vector has components

$$\bar{u}^{\alpha} = n^{-1}\, u^{\alpha} \tag{4.4}$$

in the barred 3-space. The simplest way to see this is to notice that, in both spaces, it is the same vector and must be normalized:

$$1 = u_{\mu}u^{\mu} = \bar{u}_{\mu}\bar{u}^{\mu} = \gamma_{\mu\nu}u^{\mu}u^{\nu} = \bar{\gamma}_{\mu\nu}\bar{u}^{\mu}\bar{u}^{\nu}. \tag{4.5}$$

If $\bar{u}^{\alpha}$ is parallel propagated, in the barred 3-space, then

$$\frac{{}^{3}D\bar{u}^{\mu}}{D\bar{\sigma}} = \frac{d}{d\bar{\sigma}}\, \bar{u}^{\mu} + \bar{\Gamma}^{\mu}_{\alpha\beta}\, \bar{u}^{\alpha} \frac{dx^{\beta}}{d\bar{\sigma}} = 0 \,. \tag{4.6}$$

Using Eq. (4.2), this equation reduces to Eq. (3.10) if, from previous work on the wave equation, the fact that u^{α} and $dx^{\alpha}/d\sigma$ are perpendicular is also used.

Thus the unit vector $\bar{u}^{\alpha}$ is parallel transported in the barred space.

5. REDESCRIPTION IN TERMS OF GSPACE

Just as in the previous section, Eqs. (3.8) and (3.10) were redescribed in a simple manner, in the fictitious barred space, we wish here to show that a similar result occurs in the G-space of Eq. (1.3). Again, the reduction is possible only because Eqs. (3.8) and (3.10) depend only on n

We first wish to show that Eq. (3.8) corresponds to a null geodesic in G-space:

$$\frac{D}{Dq}\frac{dx^i}{dq} = 0 \ , \ d\Sigma = 0 \ . \tag{5.1}$$

To do this, we first must find the relation between the parameter q and the path length σ , in the 3-space. This can be done by letting $i = 0$ in Eq (5.1), using Eqs. (2.8) and (2.2):

$$\frac{d}{dq}\left(\frac{1}{n^2}\frac{dx^0}{dq}\right) = 0 \ . \tag{5.2}$$

However, if $d\Sigma = 0$, from Eq. (1.3) we have

$$\frac{1}{n^2}\left(\frac{dx^0}{d\sigma}\right)^2 = +1. \tag{5.3}$$

The derivative of the square root of Eq. (5.3) is zero, a result incompatible with Eq. (5.2), unless

$$\frac{d\sigma}{dq} = n \ , \ \frac{dx^0}{d\sigma} = n \ . \tag{5.4}$$

This is the desired relation.

The remainder of the proof consists in writing down the space parts of Eq. (5.1), using Eq. (2.7):

$$\frac{D}{D\sigma}\frac{dx^\mu}{dq} = \frac{{}^3D}{D\sigma}\left(\frac{dx^\mu}{n}\right) + \Gamma^\mu_{00}\, n\, \frac{dx^0}{d\sigma}\frac{dx^0}{d\sigma} = 0 \ , \tag{5.5}$$

where dx^i/dq has been replaced by $n dx^i/d\sigma$, according to Eq. (5.4). Using Eqs. (2.2) and (5.4), in the last term, we easily see that Eq. (5.5) reduces to Eq. (3.8).

The connection between the null geodesic behavior in the G-space, and the space geodesic in the barred space, can be seen directly from the relation $d\Sigma = 0$ which can be written

$$\frac{cdt}{dq} = \left[n^2 \gamma_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{dq} \right]^{1/2} = \frac{d\bar{\sigma}}{dq} . \tag{5.6}$$

If the light goes between points A and B, the time elapsed will be

$$t_{AB} = \int_A^B \frac{dt}{dq}\, dq = \frac{1}{c} \int_A^B d\bar{\sigma} . \tag{5.7}$$

Thus, minimizing the time, in G-space, corresponds to minimizing the 3-space distance, in B-space, This is why both spaces give geodesics for the motion.

We now turn to Eq. (3.10). How is this to be described in G-space? We shall argue that the equation satisfied by u^α is the same equation that is satisfied by the projection w^a of a 4-space vector $a^{\bar{}}$ perpendicular to the path $v^\alpha = dx^\alpha/d\sigma$, that is, by

$$w^\mu = a^\mu - v^\mu v_\nu a^\nu , \quad v^\mu = dx^\mu/d\sigma , \quad v^\mu v_\mu = 1 , \tag{5.8}$$

where a^i is to have the following properties: it is propagated parallely along and orthogonal to the ray in G-space, and its space parts mustnot beparallel to v^a. The latter condition is needed, for if $a^a \sim v^\alpha$, then since $v_a v^a = 1$, Eq. (5.8) gives $w^\alpha = 0$. In other words, the polarization unit vector in 3-space, u^a, propagates as a "shodow" or "image" of a parallely propagated vector in G-space.

To prove that Eq. (5.8) satisfies Eq. (3.10), consider first the properties of a^i. Since it is orthogonal to the ray in G-space,

$$a_i \frac{dx^i}{dq} = n \left[a_\mu \frac{dx^\mu}{d\sigma} + n\, a_0 \right] = 0 \quad . \tag{5.9}$$

Since it is to be parallely propagated, its magnitude must be constant (we can choose this do be 1)

$$a_i a^i = \text{const } (\to 1) \ , \tag{5.10}$$

and its equations of motion, from Eqs. (2.7) and (2.8), are

$$\frac{{}^3Da^\mu}{D\sigma} = \frac{a^0}{n} \gamma^{\mu\nu} \frac{\partial}{\partial x^\nu} \ln n \quad , \tag{5.11}$$

$$\frac{da^0}{d\sigma} - a^0 \frac{d}{d\sigma} \ln n - a^\mu n \frac{\partial \ln n}{\partial x^\mu} = 0 \ . \tag{5.12}$$

Also

$$a^0 = - n^2 a_0 \ . \tag{5.13}$$

With these properties of a^i, the behavior of w^α, in Eq. (5.8), can be determined. First, we see that it is orthogonal to v^a:

$$w^\mu v_\mu = 0 \ . \tag{5.14}$$

This follows immediately from the definition in Eq. (5.8). Second, its magnitude is aaintained at a constant value

$$w^\mu w_\mu = a^\mu a_\mu - (a^\nu v_\mu)^2 = a^i a_i = 1 . \tag{5.15}$$

The squared term is evaluated using Eqs. (5.9) and (5.13). Third, its equation of motion is obtained by taking ${}^3D/Dq$ of Eq. (5.8):

$$\frac{{}^3Dw^\mu}{D\sigma} = \frac{{}^3Da^\mu}{D\sigma} - \frac{{}^3Dv^\mu}{D\sigma} (-na_0) - v^\mu \frac{d}{d\sigma} (-na_0) , \tag{5.16}$$

where, for $v^\alpha a_\alpha$, we used Eq. (5.9).

For the first term, on the right in Eq. (5.16), use Eq. (5.11). For the second, use Eq. (5.5). For the third, use Eq. (5.12). Almost everything cancels and we are left with

$$\frac{{}^3Dw^\mu}{D\sigma} = - v^\mu w^\nu \frac{\partial}{\partial x^\nu} \ln n \; . \tag{5.17}$$

But this is just Eq. (3.10), which was what was to be proved.

Thus, the light propagation, in terms of the fictitious G–space, is that its path is a null-geodesic, and the 3-space unit polarization vector behaves as a "shadow" of a vector parallely transported.

6. REDUCTION OF A GRAVITATIONAL CALCULATION

In this Section, we start from light propagation in a curved space without dielectric constant or index of refraction, i.e. G-space. In this space, a real field tensor, F_{ij}, exists related to real electric and magnetic fields, E_μ and B_μ, by

$$\begin{aligned} E_\mu &= \gamma_{\mu\nu} E^\nu = F_{\mu 0} \; , \quad B_{\mu\nu} = F_{\mu\nu} \; , \\ B^\mu &= (2\gamma^{1/2})^{-1} \varepsilon^{\mu\alpha\beta} F_{\alpha\beta} = \gamma^{\mu\nu} B_\nu . \end{aligned} \tag{6.1}$$

\

As indicated in these equations, the 3-vectors are considered to be true vectors, in the 3-space. Here, $\varepsilon^{\mu\alpha\beta}$ is the alternating symbol. If, in addition, the following fictitious quantities are <u>defined</u>:

$$D^\mu = \gamma^{\mu\nu} D_\nu = (-g_{00})^{1/2} F^{0\mu} = (-g_{00})^{-1/2} E^\mu , \tag{6.2}$$

$$H^\mu = \gamma_{\mu\nu} H^\nu = (-g_{00})^{1/2} \frac{1}{2} \gamma^{1/2} \varepsilon_{\mu\alpha\beta} F^{\alpha\beta} = (-g_{00})^{1/2} B_\mu \; ,$$

then it is well known[3] that Maxwell's equations, in the curved but dielectrically empty G-space, have the <u>form</u> of Maxwell's equations in a dielectric medium with Eq. (1.4) defining the dielectric parameters. Of course, the 3-space curls, divergences, etc., must be defined in generalized curvilinear coordinates, corresponding to the possibly curved $\gamma_{\mu\nu}$ tensor.

What this means as far as light is concerned is that light propagation in G-space can be redescribed by light propagation in the D-space of Eq. (1.1 using 1 . In addition, since B-space is equivalent to *D*-space in this respect, light can be regarded as propagated as if in the fictitious B-space.

For calculation purposes, it is frequently easier to compute equations in the classical terminology. We wish to complete this Section by calculating the rotation of polarization, as light bends around the sun, using Eq. (3.10). The metric in G-space is taken to be the Schwarzchild metric, whose space part is[8]

$$d\sigma^2 = n^2 dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \,, \tag{6.3}$$

using n^{-2} to represent g_{00}:

$$n^2 = -\frac{1}{g_{00}} = \frac{1}{1-2m/r} \,, \tag{6.4}$$

with $m = GM/c^2$, M the mass of the sun.

The deflection of light can, without loss of generality, be considered to take place in the plane $\theta = \pi/2$. Eqs. (3.10) become, using the Christoffel symbols, derived from Eq. (6.3):

$$\frac{du^r}{d\sigma} = -2 \frac{dr}{d\sigma}\frac{n'}{n} u^r + r\, n^{-2} \frac{d\phi}{d\sigma} u^\phi \,, \tag{6.5}$$

$$\frac{du^\phi}{d\sigma} = \left(-\frac{d\phi}{d\sigma}\frac{n'}{n} - \frac{1}{r}\frac{d\phi}{d\sigma}\right)u^r - \frac{1}{r}\frac{dr}{d\sigma} u^\phi \,, \tag{6.6}$$

$$\frac{du^\theta}{d\sigma} = -\frac{1}{r}\,\frac{dr}{d\sigma}\,u^\theta \;, \quad n' \equiv \frac{dn}{dr} \tag{6.7}$$

The first two of these are coupled equations for u^r and u^ϕ. The third is an equation for u^ℓ, whose solution is irnmediate

$$ru^\ell = \text{const} . \tag{6.8}$$

In tensor notation, a vector is written in terms of non-unit basic vectors $\hat{e}_\mu$:

$$\vec{u} = u^\mu\,\hat{e}_\mu \quad , \quad d\vec{r} = dx^\mu\,\hat{e}_\mu \quad . \tag{6.9}$$

In terms of the unit vectors, $\hat{b}_\mu$, the same vector is written

$$\vec{u} = U^\mu\,\hat{b}_\mu \;, \quad \hat{b}_\mu \cdot \hat{b}^\mu = 1 \;, \quad \hat{b}_\mu = \frac{\hat{e}_\mu}{(\hat{e}_\mu \cdot \hat{e}_\mu)} \quad . \tag{6.10}$$

For the metric, in Eq. (6.3), this means that

$$\hat{e}_\theta = r\,\hat{b}_\theta \;, \quad U^\theta = ru^\theta \quad , \tag{6.11}$$

whence, from Eq. (6.8),

$$U^\theta = \text{const} . \tag{6.12}$$

This result enables us to obtain an answer to the rotation of polarization without solving Eqs. (6.8) and (6.9). For we know that u^α is perpendicular to $dx^\alpha/d\sigma$, and can therefore be resolved into components in the $\hat{b}_\theta$ and $\hat{b}_k$ directions, where $\hat{b}_k$ is perpendicular to both $\hat{b}_\theta$ and $dx^\mu/d\sigma$:

$$u_\mu u^\mu = u_\theta u^\theta + u_k u^k \quad . \quad \text{(no sums on right)} \tag{6.13}$$

But the lefthand side is constant along the trajectory (see Eq. (3.9)) , and the first term on the right is constant (see Eq. (6.15) or (6.8)) . Hence, the other term must be constant." This means that the ratio, U^θ/U^k , evaluated at an infinite distance, before the light passes around the sun, must be the same as when evaluated, at an infinite distance, after passage around the sun. In other words, there is no rotation of polarization around the propagation direction. This is a rigorous conclusion, from the lowest order eikonal theory.

7. CONCLUSIONS

In this note, we have shown that, just as the classical propagation of light in a dielectric medium can be interpreted simply in the fictitious 3-space of Eq. (1.2), so too it can be interpreted simply in the fictitious 4-space of Eq. (1.3), where the polarization unit 3-vector propagates as a kind of "shadow" of a parallely propagated vector there.

Also, it was shown that calculations, involving light in a curved space, can be treated in a routine manner by conversion to the equations of classical optics by means of the well-known identification of $(-g_{00})^{-1/2}$ with n, the dielectric constant. As an application, it was shown that the polarization unit 3-vector will not rotate around the propagation direction as light is bent around the sun. At least, not in the lowest order eikonal approximation. According to the equivalence of the spaces in Eqs. (1.1), (1.2) and (1.3), in this example, the non-rotatíon can be understood as parallel propagation of the unit 3-vector in the B-space of Eq. (1.2).

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4, J. Ehlers, A. f. Naturf. 22a, 1328, (1967).
5, Ref. 1, pg 10 and Sec. 3.1.1.
6, Ref. 1, pg 122.
7, Ref. 1, pg 119.
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