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Two-Dimensional Massive Quantum Electrodynamics in the Unitary Gauge as a Renormalizable Theory

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We discuss two-dimensional massive quantum electrodynamics both as a superrenormalizable and a renormalizable theory, showing their equivalente up to a renormalization. The Green's functions are explicitly constructed in the zero fermion mass limit.

Discutimos eletrodinâmica quântica massiva em duas dimensões tanto como uma teoria superrenormalizável como uma teoria renormalizável. Mostramos sua equivalência a menos de uma renormalização. As funções de Green são construidas explicitamente no limite em que a massa do fermion tende a zero.

1. INTRODUCTION

The quantum theory of gauge fields has recently received much attention in connection with the unification of electromagnetic and weak interactions. There are also many attempts to incorporate strong interactions in thís scheme, the concept of "asymptotic freedom" having played a central role in their endeavour[1]. It is, therefore, convenient to have a theoretical laboratory at one's disposal in order to study problems connected with gauge invariance. With this idea in mind, we discuss 2-dimensional electrodynamics (QED)[2,3] both as a superrenormalizable and a renormalizable theory. Although this is only an abelian model, we think it is worthwhile to discuss it mainly for pedagogical reasons.

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One of the peculiar features of 2-dimensional QED is that, due to the fact that the phase space d^2k increases only as k^2, for large k, the theory is renormalizable in the so-called unitary gauge and superrenormalizable in the gauge which, in the 4-dimensional world, is called renormalizable. The equivalence of these two formulations can be explicitly studied. Another advantage is, of course, the exact solubility of the theory, in the zero fermion mass limit.

We introduce the usual paraphernalia of Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) perturbation theory[4,5], in the above mentioned two gauges, in Sects. 2 and 3. They include the discussion of Ward identities, equations of motion, and the zero mass limit. In Sect.4, we show the equivalence of the unitary and renormalizable gauges, and in Sect. 5 we make contact with the soluble zero mass limit. The conclusions are contained in Sect.6.

2. THE UNITARY GAUGE

Let us consider the 2-dimensional theory specified by the effective Lagrange density which follows:

$$L_{eff} = \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - M\bar{\psi}\psi - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}m^2 A'^2 + e'\bar{\psi}A\!\!\!/'\psi + \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2$$

$$= L_0 + L_1 , \qquad (2.1)$$

where

$$L_1 = e'\bar{\psi}A\!\!\!/'\psi + \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2 , \quad F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu ,$$

a Lagrangian which, up to the four-fermion interaction, corresponds to massive QED in the so-called unitary gauge. The free meson propagator reads

$$D_{\mu\nu} = -\frac{i}{k^2 - m^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right). \qquad (2.2)$$

Due to the bad asymptotic behaviour of $D_{\mu\nu}$, Eq. (2.1) describes in four dimensions a nonrenormalizable theory. In two dimensions, however, $(\bar{\psi}\gamma_\mu\psi)A'^\mu$ is a super-renormalirable interaction (it has a dimnsion $d = 1 < 2$), and the power counting for a proper graph, γ, constructed from (2.1) and (2.2), gives

$$d(\gamma) = 2 - \frac{F}{2} - B , \tag{2.3}$$

(F = No. of external fermion lines of γ; B = No. of external boson lines of γ), for the degree function, $d(\gamma)$, which measures the superficial divergence of γ. The reason for having included the Thirring interaction[6], $(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi)$, in (2.1), is that one needs it in order to have a renormalizable theory. If not present in zeroth order, this coupling would be induced in order e'^2. Thus the theory turns out to be renormalizable, the divergencies of our graphs being either zero or one.

The renormalization scheme we will adopt is a soft version[7,8] of the BPHZ subtraction procedure. Since it involves changes in the mass parameter, m, it will be convenient to use the following variables[9]:

$$A_\mu = mA'_\mu \ , \ e = m^{-1}e' \ . \tag{2.4}$$

With definition (2.4), we can rewrite (2.1) and (2.2) as

$$L_{eff} = \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - M\bar{\psi}\psi - \frac{1}{4m^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A^2$$

$$+ e\bar{\psi}A\!\!\!/\psi + \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2 , \tag{2.5}$$

$$D_{\mu\nu} = -\frac{i}{k^2-m^2}(m^2 g_{\mu\nu} - k_\mu k_\nu) . \tag{2.6}$$

The Green's functions of the theory are calculated as a finite part of the Gell-Mann and Low formula:

$$G^{(2N,L)}(x_1,\ldots,x_N;y_1,\ldots,y_N;z_1,\ldots,z_n) = \langle T \prod_{i=1}^{N}\psi(x_i)\prod_{j=1}^{N}\bar{\psi}(y_j)\prod_{k=1}^{L}A_{\mu_k}(z_k)\rangle$$

$$= \text{finite part of } {}^{(0)}\langle 0|T \prod_{i=1}^{N} \psi^{(0)}(x_i) \prod_{j=1}^{N} \overline{\psi}^{(0)}(y_j) \prod_{k=1}^{L} A^{(0)}_{\mu_k}(z_k) \exp i\int d^2x L_I^{(0)}(x) |0\rangle^{(0)}, \tag{2.7}$$

where the superscript (0) indicates the free fields as specified by L_0. The finite part prescription consists in the application of Zimmermann's forest formula with two generalized Taylor operators[8], $\tau^{(0)}$ and $\tau^{(1)}$:

$$\tau^{(0)}F(p,m,M) = F(0,\mu,\mu), \quad p = (p_1,p_2,\dots,p_n) \qquad \text{for logarithmically divergence graphs}$$

$$\tau^{(1)}F(p,m,M) = F(0,0,0) + p_i^\mu \left(\frac{\partial F}{\partial p_i^\mu}\right)_{\substack{p=0\\ m=M=\mu}} + M\left(\frac{\partial F}{\partial M}\right)_{\substack{p=0\\ m=M=\mu}} \qquad \text{for linearly divergent graphs.} \tag{2.8}$$

The scheme above is adequate for the derivation of homogeneous parametric differential equations, and has the advantage that the M, and m, dependences of the subtraction terms are trivial, and zero mass limits can be most easily taken. Since we are interested in the soluble $M\to 0$ limit, this subtraction scheme is a very convenient one.

Due to our subtraction scheme (2.8), the vertex functions of this model, $\Gamma^{(2N,L)}(p_i;q_j;m^2,M,\mu)$, where p_i and q_j stand for fermion and meson momenta, respectively, satisfy the following normalization conditions:

$$\Gamma^{(2,0)}(0;\ 0;\ 0,\ 0,\ \mu) = 0, \tag{2.9}$$

$$\frac{\partial}{\partial M}\Gamma^{(2,0)}(0;0;\mu^2,M,\mu)\Big|_{M=\mu} = -i \tag{2.10}$$

$$\frac{\partial}{\partial p^\mu}\Gamma^{(2,0)}(p_\mu;0;\mu^2,\mu,\mu)\Big|_{p=0} = i\gamma_\mu \tag{2.11}$$

$$\Gamma^{(4,0)}(0;0;\mu^2,\mu,\mu)\delta_{\alpha_1\alpha_2\alpha_3\alpha_4} = +i \tag{2.12}$$

$$\Gamma^{(0,2)}(0;0;\mu^2,\mu,\mu) = 0 \ , \tag{2.13}$$

where

$$\delta_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{1}{16}\left(\gamma^{\mu}_{\alpha_1\alpha_4}\gamma_{\mu\alpha_2\alpha_3} - \gamma^{\mu}_{\alpha_1\alpha_3}\gamma_{\mu\alpha_2\alpha_4}\right) \ .$$

Observe that the parameters m, and M, are not the vector-meson, and fermion physical masses. The fermion physical mass, however, goes to zero as M $\to$ 0.

As we see from (2.3), the two-point function of the **meson** field isonly logarithmically divergent. The meson wave function renormalization is, therefore, finite, and accordingly we have not included a counter term of the type $F_{\mu\nu}F^{\mu\nu}$.

Normal products up to degree 6=2 are here defined in the usual way[10]. If O, denotes any combination of the basic fields, and its derivatives, of canonical dimension less or equal to two, then the normal product $N_\delta[O]$ is defined by

$$\langle TN_\delta[O]X\rangle = \text{finite part of } {}^{(0)}\langle 0|T{:}O^{(0)}{:}X^{(0)}\exp i\int d^2x L^{(0)}(x)|0\rangle^{(0)} ,$$

$$X = \prod_{i=1}^{N}\psi(x_i)\prod_{j=1}^{N}\psi(y_j)\prod_{k=1}^{L}A_{\mu_k}(z_k) \ , \tag{2.14}$$

with a degree function,

$$\delta(\gamma) = \delta - \frac{F}{2} - B \ , \tag{2.15}$$

for proper subgraphs containing the special vertex $N_\delta[O]$. As we make our subtractions at zero momenta, these normal products satisfy the differentiation formula

$$\partial_\mu \langle TN_\delta[O](x)X\rangle = \langle TN_{\delta+1}[\partial_\mu O](x)\rangle \ . \qquad (2.16)$$

2a. EQUATIONS OF MOTION AND WARD IDENTITIES

Equations of motion for the fermion and meson fields, and Ward identities, can be derived in the standard way[5]. One finds, for example,

$$\partial^\mu \langle A_\mu(x)X\rangle = \sum_{i=1}^{L} \partial_{\nu_i} \delta(x-z_i) \ \langle T X_{\nu_i}\rangle + e \sum_{j=1}^{N} (\delta(x-x_j)-\delta(x-y_j)) \ \langle TX\rangle \qquad (2.17)$$

where

$$X_{\nu_\ell} = \prod_{i=1}^{N} \psi(x_i) \prod_{j=1}^{N} \overline{\psi}(y_j) A_{\nu_1}(z_1)\ldots A_{\nu_{\ell-1}}(z_{\ell-1}) A_{\nu_{\ell+1}}(z_{\ell+1})\ldots A_{\nu_L}(z_L) \, .$$

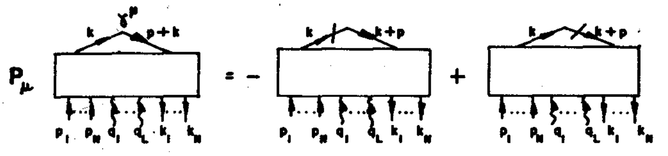
Equation (2.17) can be derived by noting that the line corresponding to the A_μ field can be linked either directly to another meson field (1^{st} term) or to a current vertex (2^{nd} term). In the latter case, one uses current conservation which is expressed by

$$\partial^\mu \langle TN_1(\overline{\psi}\gamma_\mu\psi)(x) \ X\rangle = \sum_{i=1}^{N} [\delta(x-x_i)-\delta(x-y_i)]\langle TX\rangle \ . \qquad (2.18)$$

Equation (2.17) is represented graphically in Fig.1. We sketch derivation of (2.18). First, because of (2.16), we have

$$\partial_\mu \langle TN_1(\overline{\psi}\gamma^\mu\psi)(x) \ X\rangle = \langle TN_2(\partial_\mu(\overline{\psi}\gamma^\mu\psi))(x) \ X\rangle \ . \qquad (2.19)$$

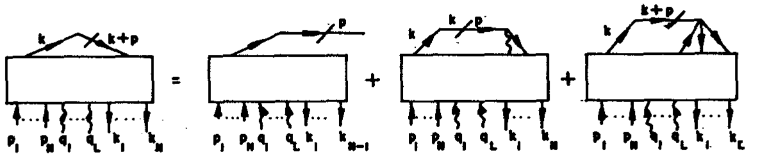
Using now the graphical representation for (2.19), in momentum space, we have


(2.20)

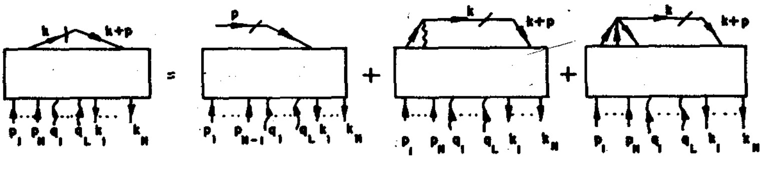
where we used

$$P_\mu\gamma^\mu \equiv \not{P} = \not{k} + \not{p} - M - (\not{k} - M) \; .$$

But


(2.21)

and


(2.22)

and (2.18) follows.

Besides (2.18), we will need the Ward-Takahashi identity for the axial current $N_1(\bar{\psi}\gamma^\mu\gamma^5\psi)$:

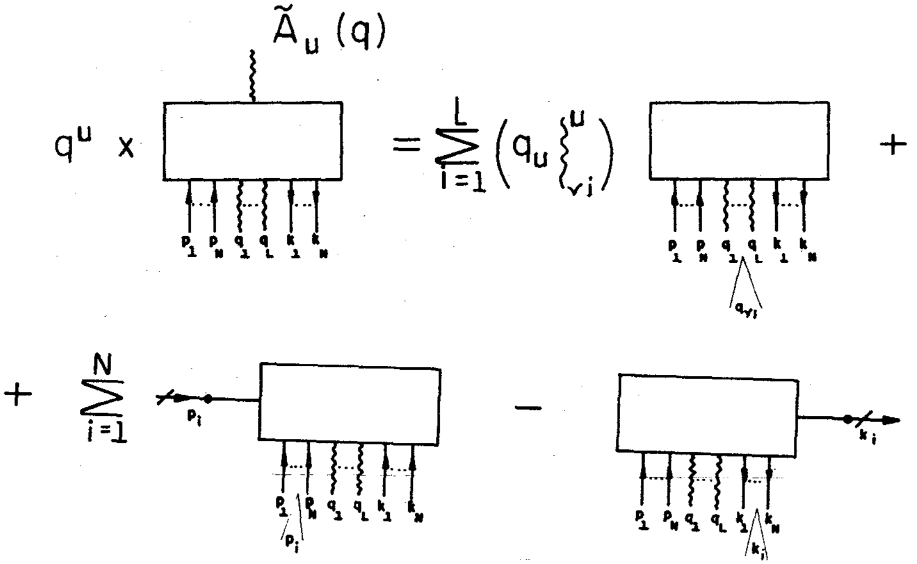


Fig.1 - The $\partial_\mu A^\mu$ line can be attached only to the longitudinal part of the meson line, or to an entering (or leaving) fermion line.

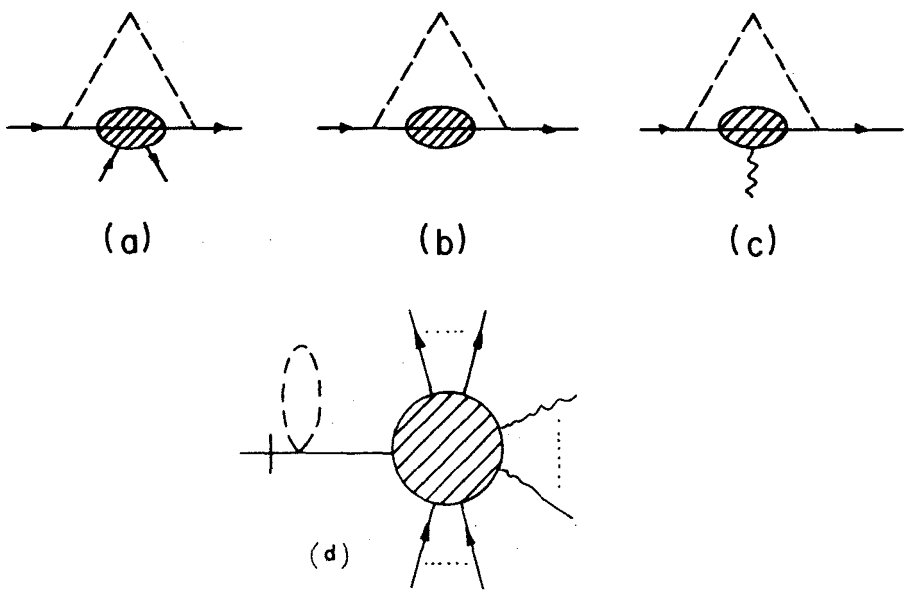


Fig.2 - Graphs a, b, c contributing to anisotropies and graph d, which appears in the iteration of the equation of motion for $(\partial_\mu A^\mu)^2$.

$$\partial^\mu \langle T N_1 [\bar\psi\gamma_\mu\gamma^5\psi](x) X\rangle = 2i\langle T N_2 [M(\bar\psi\gamma^5\psi)](x) X\rangle -$$

$$- \sum_{j=1}^{N} \left[\delta(x-x_j)\gamma^5_{x_j} + \delta(x-y_j)\gamma^{5\,T}_{y_j}\right] \langle 0|TX|0\rangle , \qquad (2.23)$$

which can be shown to be true, by following the same steps that led to Eq. (2.18). (This time however one uses $\not p\gamma^5 = (\not k+\not p-M)\gamma^5+\gamma^5(\not k-M)+2M\gamma^5$).

We now consider Zimmermann's identity:

$$MN_1(\bar\psi\gamma^5\psi) = tN_2(M\bar\psi\gamma^5\psi) + r\tilde\partial^\mu A + sN_2[\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi)] , \qquad (2.24)$$

where $\tilde\partial^\mu = \varepsilon^{\mu\nu}\partial_\nu$, and

$$t = 1 - \frac{\mu}{2}\,\mathrm{Tr}\gamma^5\left(\frac{\partial}{\partial M}\langle 0|TN_1[\bar\psi\gamma^5\psi](0)\tilde\psi(0)\tilde{\bar\psi}(0)\rangle|^{\mathrm{prop}}\right)_{M=m=\mu} ,$$

$$r = i\mu\frac{\partial}{\partial\tilde k_\rho}\langle 0|TN_1[\bar\psi\gamma^5\psi](0)A_\rho(k)|0\rangle\,\Big|^{\mathrm{Prop}}_{\substack{k=0\\ M=m=\mu}} , \qquad (2.25)$$

$$s\,\gamma_\mu\gamma^5 = -i\mu\frac{\partial}{\partial q^\mu}\langle 0|TN_1[\bar\psi\gamma^5\psi](0)\tilde\psi(\tfrac{q}{2})\tilde{\bar\psi}(\tfrac{q}{2})|0\rangle\,\Big|^{\mathrm{Prop}}_{\substack{q=0\\ M=m=\mu}} .$$

This equation can be derived by noting that the difference among vertex functions, containing $MN_1(\bar\psi\gamma^5\psi)$ and $N_2(M\bar\psi\gamma^5\psi)$, comes from subtractions for proper graphs which contain these special vertices. For example, graphs with two external fermion lines will require either the application of $\tau^{(0)}$, or $\tau^{(1)}$, according to whether they contain the degree, one, or the degree two, normal product. This produces an expression, of the type

$$M\left(\frac{\partial}{\partial M}F(0,\mu,M)\Big|_{M=\mu}\right) + p\left(\frac{\partial}{\partial p}F(p,\mu,\mu)\Big|_{p=0}\right) ,$$

times the amplitude for the reduced diagram. Since the reduced diagram will have a special vertex, with two fermion fields, this will give a contribution to the 1st and 3rd terms on the r.h.s. of (2.24). (Charge conjugation properties have already been applied in order to exclude the vertex $\bar{\psi}\gamma_\mu\gamma^5\overleftrightarrow{\partial}^\mu\psi$ from (2.24)). The second term, on the r.h.s. of (2.24), can be explained by a similar reasoning.

Observe the absence of a four fermion vertex on the r.h.s. of (2.24); as is well known, this results from Fermi statistics, and from specific properties of the two dimensional Dirac matrices, as well. With the information contained in (2.24), and (2.23), we rewrite the axial vector Ward identity as

$$(1-h)\ \partial_\mu <TN_1[\bar{\psi}\gamma^\mu\gamma^5\psi](x)X> = \frac{2Mi}{t} <TN_1[\bar{\psi}\gamma^5\psi]\ (x)X>$$
$$+ R<T\partial^\mu\tilde{A}_\mu(x)X> - \sum_{j=1}^{N}\ [\delta(x-x_j)\gamma^5_{x_j} + \delta(x-y_j)\gamma^{5\,T}_{y_j}]<TX> \quad , \qquad (2.26)$$

with

$$h = \frac{2s}{it}\ ,\ R = \frac{2r}{it}\ . \qquad (2.27)$$

Note that both h and R are mass independent, due to (2.24) and (2.25).

2b. HOMOGENEOUS PARAMETRIC EQUATIONS

The derivation of homogeneous parametric equations is greatly simplified by the introduction of the following differential vertex operations (D.V.O.)[11]:

$$\Delta_1 = \frac{i}{2}\int d^2x N_2\,[A_\mu A^\mu](x)\ ,\qquad \Delta_2 = \frac{-i}{4m^2}\int d^2x N_2\,[F_{\mu\nu}F^{\mu\nu}](x)\,,$$

$$\Delta_3 = i\int d^2x N_2\,[M\bar{\psi}\psi](x)\ ,\qquad \Delta_4 = \frac{1}{2}\int d^2x N_2\,[\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi](x)\ ,$$

$$\Delta_5 = i\int d^2x N_2[\bar{\psi}A\!\!\!/\psi](x)\ ,\qquad \Delta_6 = \frac{i}{2}\int d^2x N_2\,[(\bar{\psi}\gamma_\mu\psi)^2](x)\ . \qquad (2.28)$$

With this notation, the Lagrangian (2.5) can be rewritten as

$$iL_{\mathrm{eff}} = \Delta_4 - \Delta_3 + \Delta_2 + \Delta_1 + e\Delta_5 + g\Delta_6 \quad . \tag{2.29}$$

Notice that $F_{\mu\nu}F^{\mu\nu}$ is a soft[12] operator, since it cancels out the longitudinal part of the vector meson propagator (2.6) . We have, therefore, two soft insertions:

$$\Delta_0 = -i \int d^2x N_1 [\overline{\psi}\psi]\ (x)\ , \quad \Delta'_0 = \frac{-i}{4m^2} \int d^2x N_0 [F_{\mu\nu}F^{\mu\nu}]\ (x)\, . \tag{2.30}$$

Due to our subtraction scheme, (2.9), it is easy to derive the following relations for the vertex functions $\Gamma^{(2N,L)}$.

$$m^2 \frac{\partial}{\partial m^2} \Gamma^{(2N,L)} = - \Delta'_0\ \Gamma^{(2N,L)}\ , \tag{2.31}$$

$$\frac{\partial}{\partial M}\ \Gamma^{(2N,L)} = - \Delta_0\ \Gamma^{(2N,L)}\ , \tag{2.32}$$

$$M\Delta_0\,\Gamma^{(2N,L)} = \sum_{i=1}^{6} s_i \Delta_i \Gamma^{(2N,L)}\ , \quad s_2 = 0\ , \tag{2.33}$$

$$\Delta'_0\Gamma^{(2N,L)} = \sum_{i=1}^{6} t_i \Delta_i \Gamma^{(2N,L)}\ , \quad t_2 = 1\ . \tag{2.34}$$

The peculiar form of (2.31) is a direct consequence of our change of variables, (2.4). The μ-dependence of $\Gamma^{(2N,L)}$ is given by

$$\mu \frac{\partial}{\partial \mu} \Gamma^{(2N,L)} = \sum_{i=1}^{7} \alpha_i \Delta_i \Gamma^{(2N,L)}\ , \tag{2.35}$$

where the coefficients α_i are mass independent. They can be determined directly by observing that μ enters only via the subtraction terms. For example,

$$\alpha_4 = \frac{\mu}{4i} \,\mathrm{Tr}\, \frac{\partial}{\partial\mu} \{ \gamma^\alpha \frac{\partial}{\partial p^\alpha} \Gamma^{(2,0)}(p,-p)\Big|_{\substack{p=0\\ m=M=\mu}} \},$$

$$\alpha_5 = \frac{\mu}{4i} \,\mathrm{Tr}\, \frac{\partial}{\partial\mu} \{ \gamma^\alpha \Gamma^{(2,1)}_\alpha(0,0,0) \Big|_{m=M=\mu} \}. \tag{2.36}$$

The counting identities

$$N\Gamma^{(2N,L)} = (-2\Delta_3 + 2\Delta_4 + 2e\Delta_5 + 4g\Delta_6)\Gamma^{(2N,L)} ,$$
$$L\Gamma^{(2N,L)} = (2\Delta_1 + 2\Delta_2 + e\Delta_5)\Gamma^{(2N,L)} \tag{2.37}$$

can be derived by integrating the equations of motion

$$\langle TN_2[\overline{\psi}(i\not{\partial}-M)\psi](x)X\rangle = - \langle TN_2[e\overline{\psi}\not{A}\psi + g(\overline{\psi}\gamma_\mu\psi)^2](x)X\rangle + \sum_{k=1}^{N} \delta(x-y_k)\langle TX\rangle \quad , \tag{2.38}$$

$$\langle TN_2[A_\nu \frac{\partial^\mu\partial^\nu}{m^2} A_\mu - \frac{1}{m^2} A_\nu\partial^2 A^\nu - A^2](x)X\rangle = -i \sum_{k=1}^{L} \delta(x-z_k)\langle TX\rangle.$$

Making use of Eqs.(2.31-37), and (2.39) which follows,

$$\frac{\partial}{\partial g} \Gamma^{(2N,L)} = \Delta_6 \Gamma^{(2N,L)} ,$$
$$\frac{\partial}{\partial e} \Gamma^{(2N,L)} = \Delta_5 \Gamma^{(2N,L)} , \tag{2.39}$$

one can establish a homogeneous parametric differential equation of the Weinberg[13] type

$$\{ \mu \frac{\partial}{\partial\mu} + \rho_1 m^2 \frac{\partial}{\partial m^2} + \rho_2 M \frac{\partial}{\partial M} + \beta_1 \frac{\partial}{\partial g}$$

$$+ \beta_2 \frac{\partial}{\partial e} - 2N\gamma_1 - L\gamma_2 \} \Gamma^{(2N,L)} = 0 . \tag{2.40}$$

The **proof** of (2.40) **is** standard[5]. One substitutes the above equations into **(2.35),** and equates to zero the **coefficient** of each D.V.O., Δ_i, $i = 1 2,\ldots,7$. This gives the **following** system of equations for the ρ's, γ's, and β's:

$$\alpha_1 - \rho_1 t_1 - \rho_2 s_1 - 2\gamma_2 = 0 , \tag{2.41}$$

$$\rho_1 + 2\gamma_2 = 0 , \tag{2.42}$$

$$\alpha_3 - \rho_1 t_3 - \rho_2 s_3 - 2\gamma_1 = 0 , \tag{2.43}$$

$$\alpha_4 + \rho_1 t_4 - \rho_2 s_4 - 2\gamma_1 = 0 , \tag{2.44}$$

$$\alpha_5 - \rho_1 t_5 - \rho_2 s_5 + \beta_2 - 2\gamma_1 e - \gamma_2 e = 0 , \tag{2.45}$$

$$\alpha_6 - \rho_1 t_6 - \rho_2 s_6 + \beta_1 - 4g\gamma_1 = 0 . \tag{2.46}$$

This system always has a solution in perturbation theory, since its determinant is non vanishing in zeroth order.

From the equations above, we have

$$\beta_2 = e\gamma_2 . \tag{2.47}$$

To see that, one uses the Ward identity

$$\Gamma_\mu^{(2,L)}(p,-p;0) = e \frac{\partial}{\partial p^\mu} \Gamma^{(2,0)}(p,-p) , \tag{2.48}$$

which follows directly from (2.18). Equation (2.48) implies that $a_5 = ea_4$, $s_5 = es_4$, $t_5 = et_4$, and thus, using (2.44) and (2.45), we obtain (2.47).

We can now show that several parameters, occurring in (2.40), do vanish, namely,

$$\beta_1 = \rho_1 = \beta_2 = \gamma_2 = 0. \qquad (2.49)$$

In order to show that $\gamma_2 = 0$, we use that

$$\left[D - 2N\gamma_1 - L\gamma_2 + 2\gamma_2 \right] \Delta_0' \Gamma^{(2N,L)} = 0, \qquad (2.50)$$

where

$$D = \mu \frac{\partial}{\partial\mu} + \rho_1 m^2 \frac{\partial}{\partial m^2} + \rho_2 M \frac{\partial}{\partial M} + \beta_1 \frac{\partial}{\partial g} + \beta_2 \frac{\partial}{\partial e}, \qquad (2.51)$$

which is easily derived, since Δ_0' is an integrated zeroth order normal product. Now, the derivative of (2.40), with respect to m^2, gives

$$\left[D - 2N\gamma_1 - L\gamma_2 \right] m^2 \frac{\partial}{\partial m^2} \Gamma^{(2N,L)} = 0 . \qquad (2.52)$$

Thus, comparing (2.51) with (2.52), it follows that $\gamma_2=0$. From (2.42) and (2.47), we have, then, $\rho_1=\beta_2=0$.

To show that $\beta_1=0$, we follow the recipe of Ref.14. Here, the following notation, for proper functions, containing only one normal product vertex, will be used:

Normal product	Notation
$N_1[\bar\psi\gamma_\mu\psi](x)$	Γ_μ
$N_1[\bar\psi\gamma^5\psi](x)$	Γ_5
$N_1[\bar\psi\gamma_\mu\gamma^5\psi](x)$	$\Gamma_{\mu 5}$

(2.53)

Then following the same steps as above we can derive

$$[D - 2N\gamma_1 - L\gamma_2 + 2\gamma_1]\Gamma_\mu^{(2N,L)} = 0, \qquad (2.54)$$

$$[D - 2N\gamma_1 - L\gamma_2 + 2\gamma_1 + u]\Gamma_5^{(2N,L)} = 0 \qquad (2.55)$$

Note the additional term in Eq.(2.55). If the β's are zero, it is related to the so called binding dimension, which is a contribution to the anomalous dimension of the $N_1[\overline{\psi}\gamma^5\psi]$ field, produced in the process of joining $\overline{\psi}(x)\gamma^5$, and $\psi(y)$, to form the composite object. Because of current conservation, the corresponding term is absent from (2.54). We now apply the operator D to the Eqs. (2.18) and (2.23), and use (2.54) and (2.55), together with the relation $\gamma^\mu\gamma^5 = \varepsilon^{\mu\nu}\gamma_\nu$, to obtain

$$Dh = 0, \qquad (2.56)$$

$$DR = 0, \qquad (2.57)$$

$$D\left(\frac{M}{t}\right) = u\,\frac{M}{t}\,. \qquad (2.58)$$

As we have seen, h does not depend on the masses. Thus, from (2.57), we have

$$\beta_1\,\frac{\partial h}{\partial g} = 0\,. \qquad (2.59)$$

But $\frac{\partial h}{\partial g} \neq 0$, as a simple calculation shows. Hence,

$$\beta_1 = 0\,. \qquad (2.60)$$

We can understand these results, perhaps more easily, by using the infinite counter term approach[15]. In that language, the ρ's, β's and γ's are associated with infinite mass, coupling and wave function renormalization, respectively. y , for example, is zero, because the meson two-point function is only logarithmically divergent, implying the absence of infinite wave funtion renormalization for A^μ.

In computing this logarithmic divergence, of the vector-meson propagator, one can set $M=0$, since terms proportional to M are already finite. But because of the property of the two-dimensional Dirac algebra,

$$\gamma_\alpha \gamma^\mu \gamma^\alpha = 0 \ , \tag{2.61}$$

and symmetric integration, the vector-meson mass renormalization is finite, implying the vanishing of ρ_1 . Since in gauge theories, $\beta_2 = e\gamma_2$, it follows that $\beta_2=0$. That $\beta_1=0$ is, finally, a consequence of the fact that the interaction, as $M\to 0$, is of the form $:j^\mu j_\mu:$, with j_μ divergenceless, and a combination of free fields, as it will be shown later (Sect.5).

3. THE SUPERRENORMALIZABLE GAUGE

In four dimensions, the nonrenormalizability of the model, of the previous Section, is solved by a gauge principle: instead of (2.5), one considers a new Lagrangian

$$L' = (1+d)i\ (\bar\psi \frac{\overleftrightarrow{\not\partial}}{2} - ie\not A)\psi - (1-c)M\bar\psi\psi - \frac{1}{4m^2} F_{\mu\nu}F^{\mu\nu}$$

$$+ \frac{1}{2} A^2 + \frac{1}{2} (g+f)(\bar\psi\gamma^\mu\psi)^2 - \frac{1}{2m_0^2} (\partial_\mu A^\mu)^2 \tag{3.1}$$

where the finite counterterms will be fixed below and the addition of the term $(\partial_\nu A^\mu)^2$ has the effect of improving the ultraviolet behaviour of the vector-meson propagator. We have

$$D'_{\mu\nu} = \frac{-i}{k^2-m^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) m^2 + \frac{-i}{k^2-m_0^2}\ \frac{k_\mu k_\nu}{k^2} m_0^2 \ . \tag{3.2}$$

With m_0 finite, (3.2) is a meson propagator in an indefinite metric Hilbert space. As, in four dimensions, only gauge invariant (i.e., m_0 independent) objects can have physical relevance.

The power counting adequate for (3.1) gives

$$\delta(\gamma) = 2 - \frac{1}{2} F - N(\mathrm{ve}), \tag{3.3}$$

where $N(ve)$ is the number of vertices, of the type $\bar{\psi} \not{A} \psi$, in γ. Observe, from (3.3), that the vertices, $A_\mu A^\mu$ and $(\partial_\mu A^\mu)^2$, are trivial from the renormalization point of view: either they belong to a IPR (one particle reducible) graph, or to a finite graph. Thus, in the Lagrangian (3.1), these vertices are well defined as ordinary products. If one uses the renormalization scheme, (2.8). then the vertex functions of this model will satisfy normalization conditions of the type (2.9) - (2.13), with the additional requirement that $e=0$ in these formulas.

The derivation of Ward identities, and homogeneous parametric equations, can be pursued similarly to what was done in Section 2. The gauge criteria for this model, however, deserve some comment. We have

$$\partial_\mu \langle TA^\mu(x)X\rangle = -\sum_{\ell=1}^{L} m_0^2 \partial_{\nu_\ell} \Delta_F(x-z_\ell) m_0^2 \langle TX_{\nu_\ell}\rangle +$$

$$+ iem_0^2 \sum_{i=1}^{N} \left[\Delta_F(x-x_i,m_0^2) - \Delta_F(x-y_i,m_0^2)\right]\langle TX\rangle \ , \qquad (3.4)$$

which shows that $\partial_\mu A^\mu$ is a free field of mass m_0. Furthermore, because of the superrenormalizability of the interaction $\bar{\psi}\not{A}\psi$, the discussion of the m_0 independence of physical quantities is greatly simplified. We have (for the Green's functions $G^{(2N,L)}$):

$$\frac{\partial}{\partial m_0^2} G^{(2N,L)} = i \int d^2x \ N\left[\frac{\partial}{\partial m_0^2}(1+d)\ i(\bar{\psi}\frac{\overleftrightarrow{\not{\partial}}}{2} - ie\not{A})\psi - \frac{\partial}{\partial m_0^2}((1-c)M)\bar{\psi}\psi + \right.$$

$$\left. + \frac{\partial}{\partial m_0^2}\frac{(g+f)}{2}(\bar{\psi}\gamma^\mu\psi)^2 + \frac{1}{2m_0^4}(\partial_\mu A^\mu)^2\right](x)\ G^{(2N,L)} \qquad (3.5)$$

where the normal product is subtracted according (3.3). By using the equations of motion it is a simple matter to verify that

$$\frac{i}{2m_0^4}\int d^2x\ N(\partial_\mu A^\mu)^2(x)\ G^{(2N,L)} = \Delta_0\ G^{(2N,L)} + \frac{\rho N}{m_0^2} G^{(2N,L)} \qquad (3.6)$$

where Λ is the D.V.O. given by (4.8) and the term containing ρ comes from the graphs of Fig. (2.d).

Using the analog of (2.37) one sees that if the counterterrns are fixed as

$$\begin{aligned}(1 - c) &= (1 - c_0)\ (m_0)^{2\rho}\\ (1 + d) &= (1 - d_0)\ (m_0)^{2\rho}\\ (g + f) &= (g + f_0)\ (m_0)^{4\rho}\end{aligned} \tag{3.7}$$

where the m_0 independent constants c_0, d_0, f_0 are used to fix the fermion mass-shell and the normalization condition on the four point function, one obtains m_0-independent Green's functions for transversal meson and on-shell fermion fields. There will be no anisotropic normal products[5] coming up, since graphs with one internal meson line are already convergent. By an extension, composite objects, having degree less or equal to two, will be gauge invariant, if they satisfy both the equations (3.4) and (3.5).

4. AN EQUIVALENCE THEOREM

In the previous Sections, we have seen two formulations of the theory of a massive vector boson interacting with a massive spinor field in two dimensions. The possibility of a formulation, directly without ghost fields, is a peculiarity of the two-dimensional world, and in this Section we want to investigate the equivalence of theories that differ by the presence, or absence, of the ghost field. We will show that, for gauge invariant quantities the theories, of Sections 2 and 3 are equivalent up to a renormalization. To this end, we consider the class of theories specified by a parameter λ, $0 \le \lambda \le 1$:

$$L_\lambda = \lambda i N\left[\bar\psi\left(\frac{\overleftrightarrow{\partial\!\!\!/}}{2} - ie A\!\!\!/\right)\psi\right] + (1-\lambda) i \bar N\left[\bar\psi\left(\frac{\overleftrightarrow{\partial\!\!\!/}}{2} - ie A\!\!\!/\right)\psi\right] -$$

$$- \frac{1}{4m^2}(1-b) N_2\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{2}(g+f) N_2\left[(\bar\psi\gamma_\mu\psi)^2\right] +$$

$$+ \frac{1}{2} N_2\left[A^2\right] - \frac{1}{2}\frac{1}{m_0^2} N_2\left[(\partial_\mu A^\mu)^2\right] + (1-e) N_2\left[M\overline{\psi}\psi\right] +$$

$$+ idN\left[\frac{1}{2}\overline{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - ie\overline{\psi}A\!\!\!/\psi\right] . \tag{4.1}$$

The degree function, which determines the number of subtractions to be made, for proper subgraphs, is given by

$$\delta(\gamma) = 2 - \frac{F}{2} - \sum_a (2 - \delta_a), \tag{4.2}$$

where, excepting the vertex $\overline{\psi}A\!\!\!/\psi$, the degree δ_a, for the normal products of the Lagrangian (4.1), is 2. In the case of the vertex $\overline{\psi}A\!\!\!/\psi$, we define $\delta_a=1$ for the corresponding normal N-product, whereas, for $\overline{N}[\overline{\psi}A\!\!\!/\psi]$,

$$\delta_a = \begin{cases} 1, \text{ if } v_a \text{ is an external vertex, i.e., it has an external } A_\mu \text{ attached to } \overline{N}[\overline{\psi}A\!\!\!/\psi], \\ 2, \text{ otherwise.} \end{cases}$$

Thus,

$$\delta(\gamma) = 2 - \frac{1}{2}F_\gamma - \overline{B} - v , \tag{4.3}$$

where

v : n? of vertices $N[\overline{\psi}A\!\!\!/\psi]$

$\overline{B}$: n? of external A_μ fields attached to $\overline{N}[\overline{\psi}A\!\!\!/\psi]$.

Up to renormalization (two theories are equal up to renormalizations, if they differ only by the values of their counter terms), we see that the case $\lambda=1$ corresponds to the superrenormalizable theory of Section 3, while the case $\lambda=0$ corresponds, in the limit $m_0\to\infty$, to the theory described in Section 2.

In order to obtain a gauge invariant S-matrix, the Green's functions will have to satisfy[16]

$$\frac{\partial}{\partial m_0^2} G^{(2N,L)} = \Delta_0 G^{(2N,L)}, \tag{4.4}$$

with Δ_0 some D.V.O. normalized on-mass shell. This can be established by conveniently adjusting the counter terrns, in (4.1), as we will show now. First, we have

$$\frac{\partial}{\partial m_0^2} = \frac{1}{m_0^4}\Delta_6 - \frac{\partial c}{\partial m_0^2}\Delta_3 + \frac{\partial d}{\partial m_0^2}\Delta_4 + \frac{\partial f}{\partial m_0^2}\Delta_5 - \frac{\partial b}{\partial m_0^2}\Delta_2 \ , \tag{4.5}$$

where we are employing the notation

$$\Delta_1 = \frac{i}{2}\int d^2x \mathrm{N}_2\left[A^2\right](x) \ , \qquad \Delta_2 = \frac{-i}{4m^2}\int d^2x \mathrm{N}_2\left[F_{\mu\nu}F^{\mu\nu}\right](x) \ ,$$

$$\Delta_3 = i\int d^2x \mathrm{N}_2\left[M\overline{\psi}\psi\right](x) \ , \qquad \Delta_4 = -\int d^2x \mathrm{N}\left[\overline{\psi}\left(\frac{1}{2}\overleftrightarrow{\partial\!\!\!/} - ie A\!\!\!/\right)\psi\right](x) ,$$

$$\overline{\Delta}_4 = -\int d^2x \overline{\mathrm{N}}\left[\overline{\psi}\left(\frac{1}{2}\overleftrightarrow{\partial\!\!\!/} - ie A\!\!\!/\right)\psi\right](x) , \quad \Delta_5 = \frac{i}{2}\int d^2x \mathrm{N}_2\left[(\overline{\psi}\gamma^\mu\psi)^2\right](x) ,$$

$$\Delta_6 = \frac{i}{2}\int d^2x \mathrm{N}_2\left[(\partial_\mu A^\mu)^2\right](x) \ . \tag{4.6}$$

Now, we want to prove the identity

$$\frac{1}{m_0^4}\Delta_6 = \Delta_0 + \sum_{i=2}^{5}\sigma_i\Delta_i + \overline{\sigma}_4\overline{\Delta}_4 \ , \tag{4.7}$$

where

$$\Delta_0 G^{(2N,L)} = i\int d^2x \{ \sum_{i,j=1}^{L} \partial_{\nu_i}\Delta_F(x-z_i,m_0^2)\, \partial_{\nu_j}\Delta_F(x-z_j,m_0^2) \langle TX_{\nu_i\nu_j}\rangle -$$

$$- ie \sum_{i,j=1}^{L} \partial_{\nu_i}\Delta_F(x-z_i,m_0^2)\left(\Delta_F(x-x_j,m_0^2) - \Delta_F(x-y_j,m_0^2)\right)\langle TX_{\nu_i}\rangle -$$

$$-ie^2 \sum_{i\neq j}^{N} \left(\Delta_F(x-x_i,m_0^2)\Delta_F(x-x_j,m_0^2) + \Delta_F(x-y_i,m_0^2)\Delta_F(x-y_j,m_0^2)\right)\langle TX\rangle$$

$$+ e^2 \sum_{i,j}^{N} \Delta_F(x-x_i,m_0^2)\Delta_F(x-y_j,m_0^2)\langle TX\rangle \} \ . \tag{4.8}$$

The term $\sigma_1\Delta_1$ is absent from the r.h.s. of Eq. (4.7), because σ_1 is given by $\Delta\Gamma^{(0,2)}(0,0)$. Current conservation makes $\Delta_0\Gamma^{(0,2)}(k_1,k_2)$ transverse in its external meson lines, and it thus vanishes at $k_1=k_2=0$.

The identity (4.7) is proved by iterating the Ward identity,

$$<T\partial_\mu A^\mu(x)X> = -\sum m_0^2\partial_{\nu_i}\Delta_F(x-z_i,m_0^2)<TX_{\vartheta_i}> +$$

$$+\, iem_0^2\sum_{i=1}^{N}\left[\Delta_F(x-x_i,m_0^2)-\Delta_F(x-y_i,m_0^2)\right]<TX> \qquad , \qquad (4.9)$$

and taking into account the additional terms, coming from anisotropies in subtractions, for the graphs in Fig. 2.a, b, c and the contribution from the graph of Fig. 2.d. Observe that these graphs must contain at least one vertex $\bar{N}$.

The $\bar{\Delta}$ insertion can be eliminated from (4.7), if one uses

$$\bar{\Delta}_4 G^{(2N,L)} - \Delta_4 G^{(2N,L)} = \left(\xi_3\Delta_3+\xi_4\Delta_4+\xi_5\Delta_5\right)G^{(2N,L)}, \qquad (4.10)$$

where the coefficients $\xi_i(g,e,\lambda,\mu)$, $i=1,2,3,4$, are associated with subtractions present in graphs containing $\bar{\Delta}$, but absent in those containing A. Note that the vertex $N_2[A^2]$ is absent from the r.h.s. of (4.10), by the same reason as in Eq.(4.7). Using (4.10), Eq.(4.7) can be rewritten as

$$\Delta_6 = \Delta_0 + \sum_{i=2}^{5}\eta_i\Delta_i\ . \qquad (4.11)$$

From (4.5) and (4.11), we see that in order to satisfy Eq.(4.4) the counter terms must be chosen as

$$\frac{\partial b}{\partial m_0^2} = \eta_2 \quad , \ b = b_0 - \int_{\mu^2}^{m_0^2}\eta_2\, d\bar{m}_0^2 \qquad ,$$

$$\frac{\partial c}{\partial m_0^2} = \eta_3 \quad , \quad c = c_0 + \int_{\mu^2}^{m_0^2} \eta_3 \; d\bar{m}_0^2 \quad ,$$

$$\frac{\partial d}{\partial m_0^2} = -\eta_4 \quad , \quad d = d_0 - \int_{\mu^2}^{m_0^2} \eta_4 \, d\bar{m}_0^2 \quad , \tag{4.12}$$

$$\frac{\partial f}{\partial m_0^2} = -\eta_5 \quad , \quad f = f_0 - \int_{\mu^2}^{m_0^2} \eta_5 \, d\bar{m}_0^2 \quad .$$

Thus we still have, at our disposal, the constants (independent of m_0) b_0, c_0, d_0, and f_0, which will be fixed by imposing the λ-independence of the S-matrix. We now have

$$\frac{\partial}{\partial\lambda} G^{(2N,L)} = \left[\Delta_4 - \bar{\Delta}_4 - \frac{\partial b}{\partial\lambda} \Delta_2 - \frac{\partial c}{\partial\lambda} \Delta_3 + \frac{\partial d}{\partial\lambda} \Delta_4 + \frac{\partial f}{\partial\lambda} \Delta_5 \right] G^{(2N,L)} . \tag{4.13}$$

Using (4.10), (4.13) becomes

$$\frac{\partial}{\partial\lambda} G^{(2N,L)} = \left[- \left(\frac{\partial c}{\partial\lambda} + \xi_3\right) \Delta_3 - \frac{\partial b}{\partial\lambda} \Delta_2 \right.$$

$$\left. + \left(\frac{\partial d}{\partial\lambda} - \xi_4\right) \Delta_4 + \left(\frac{\partial f}{\partial\lambda} - \xi_5\right) \Delta_5 \right] G^{(2N,L)} . \tag{4.14}$$

The remaining step is to rewrite (4.14) in terms of gauge invariant normal products $\tilde{N}_2[O]$. These are linear combinations of the $N_2[O]$ normal products

$$\tilde{\Delta}_i = \sum_j v_{ij} \Delta_j \quad , \quad i,j = 2,3,4,5 \; , \tag{4.15}$$

satisfying

$$\frac{\partial}{\partial m_0^2} \tilde{\Delta}_i G^{(2N,L)} = \Delta_0 \tilde{\Delta}_i G^{(2N,L)} \tag{4.16}$$

Observe that, only formally, gauge invariant products O_i can appear in (4.15). The matrix $[v]_{ij}$ certainly has an inverse $[w]_{ij}$ in perturbation theory, and, therefore, (4.14) can be expressed in terms of the $\tilde{\Delta}_i$ as

$$G^{(2N,L)} = \sum_{j=2}^{5} \Big[-\left(\frac{\partial c}{\partial\lambda} + \xi_3\right) w_{3j}\tilde{\Delta}_j - \frac{\partial b}{\partial\lambda} w_{2j}\tilde{\Delta}_j + $$

$$+ \left(\frac{\partial d}{\partial\lambda} - \xi_4\right) w_{4j}\tilde{\Delta}_j + \left(\frac{\partial f}{\partial\lambda} - \xi_5\right) w_{5j}\tilde{\Delta}_j \Big] G^{(2N,L)} \qquad (4.17)$$

The coefficients, in (4.17), must be m_0-independent, since, on the fermion mass-shell both $G^{(2N,L)}$ and $\tilde{\Delta}_i$ are such; they can be evaluate by chmsing $m_0 = \mu$. Thus, imposing A-independence of $G^{(2N,L)}$, it will result in the following system of equations

$$-\left(\frac{\partial c_0}{\partial\lambda} + \xi_3\right) w_{32} - \frac{\partial b_0}{\partial\lambda} w_{22} + \left(\frac{\partial d_0}{\partial\lambda} - \xi_4\right) w_{42} + \left(\frac{\partial f_0}{\partial\lambda} - \xi_5\right) w_{52} = 0 ,$$

$$-\left(\frac{\partial c_0}{\partial\lambda} + \xi_3\right) w_{33} - \frac{\partial b_0}{\partial\lambda} w_{23} + \left(\frac{\partial d_0}{\partial\lambda} - \xi_4\right) w_{43} + \left(\frac{\partial f_0}{\partial\lambda} - \xi_5\right) w_{53} = 0,$$

$$-\left(\frac{\partial c_0}{\partial\lambda} + \xi_3\right) w_{34} - \frac{\partial b_0}{\partial\lambda} w_{24} + \left(\frac{\partial d_0}{\partial\lambda} - \xi_4\right) w_{44} + \left(\frac{\partial f_0}{\partial\lambda} - \xi_5\right) w_{54} = 0,$$

$$-\left(\frac{\partial c_0}{\partial\lambda} + \xi_3\right) w_{35} - \frac{\partial b_0}{\partial\lambda} w_{25} + \left(\frac{\partial d_0}{\partial\lambda} - \xi_4\right) w_{45} + \left(\frac{\partial f_0}{\partial\lambda} - \xi_5\right) w_{55} = 0,$$

(4.18)

which can be solved perturbatively for $\frac{\partial b_0}{\partial\lambda}$, $\frac{\partial c_0}{\partial\lambda}$, $\frac{\partial d_0}{\partial\lambda}$ and $\frac{\partial f_0}{\partial\lambda}$.

This concludes the proof of

$$\frac{\partial}{\partial\lambda} G^{(2N,L)} = 0. \qquad (4.19)$$

Let us now discuss the relation, of the theories constructed in this Section, to the ones of Sections 2 and 3. Due to (4.19), we get the same Green's functions, for any value of A. For example, for $\lambda=1$, which corresponds, up to renormalizations, to the superrenormalizable case, the Lagrangian (4.1) contains no D.V.O. of the type $\overline{\Delta}_4$. Thus, the anisotropies are absent, and the counter terms b,c,d and f, are m_0-independent . Since, for $\lambda=1$, the number of subtractions is the same as that of the superrenormalizable case, the limit $m_0 \to \infty$ will not exist, except for gauge-invariant quantities on the mass shell, which are already m_0-independent . When we talk about equivalence, up to renormalizations, we always exclude these gauge invariant objects.

Since our Green's functions are A-independent, the limit $m_0 \to \infty$, cannot either exist for $\lambda=0$, for gauge-dependent objects. In this case, however, we did make the same number of subtractions as in the renormalizable unitary gauge. Thus, now, the m_0-dependent counter terms diverge in the $m_0 \to \infty$ limit. We conclude that in this limit, in which the equivalence, up to renormalizations, obviously continues to hold, one needs an infinite renormalization to go from the theories of this Section to the unitary gauge.

5. THE SOLUBLE ZERO MASS LIMIT

Two dimensional QED is known to be soluble, if the mass of the fermion is zero, even when the vector field has a bare mass different from zero. Actually, this model is an example of a dynamical generation of mass, in which the vector field gets a mass through the interaction. We want to consider, here, the limit $M\to 0$ of the model of Section 3. Due to the presence of vertices, of the superrenormalizable type, in (3.1), some remarks are needed.

i. Due to the renormalization condition (2.9), with $e=0$, reduced graphs with vertices with two fermion lines will have a momentum factor, which improves the infrared convergence of the integral, in the loop momenta of these lines (see Fig.3). This is necessary if one wants to avoid infrared divergencies arising from the fact that, we have two legs with zero mass in the unsubtracted integrand[17] .

ii. Increasing the number of vertices, of the type $\bar{\psi}\gamma_\mu\psi A^\mu$, in a graph, does not introduce infrared problems if the mass of the vector boson is maintained different from zero. This will not be true, in general, if $m'=0$. Even in the Landau gauge $(m_0=0)$, there will occur divergencies associated with graphs of the type of Fig.4, and the perturbation series in e' will not exist. However, because of the mass generation, an exact solution will exist. To obtain this solution, one should first take the limit $M\to 0$, maintaining m_0, and m' different from zero; next, sum the perturbative series to get the exact solution, and then discuss the other zero mass limits for gauge invariant quantities.

Let us begin discussing the $M\to 0$ limit. From (3.4), the vector meson propagator satisfies

$$\partial_\mu \langle TA^\mu(x)A_\nu(y)\rangle = -\frac{m_0^2}{m'^2}\,\partial_\nu \Delta_F(x-y;m_0^2) \quad , \qquad (5.1)$$

whereas for the curl of A_μ, we have

$$\begin{aligned}
&\tilde{\partial}_\mu \langle TA^\mu(x)A_\nu(y)\rangle \\
&= -\tilde{\partial}_\nu \Delta_F(x-y;m_0^2) + e'\int d^2x'\Delta_F(x-x';m'^2)\,\tilde{\partial}_\lambda \langle Tj^\lambda(x')A_\nu(y)\rangle \\
&= -\tilde{\partial}_\nu \Delta_F(x-y;m_0^2) - \alpha\int d^2x'\Delta_F(x-x';m'^2)\,\tilde{\partial}_\lambda \langle TA_\lambda(x')A_\nu(y)\rangle \quad , \qquad (5.2)
\end{aligned}$$

with

$$\Delta_F(x-x';m'^2) = \int e^{-ik(x-x')}\,\frac{1}{k^2-m'^2}\,\frac{d^2k}{(2\pi)^2} \; ,$$

$$j_\mu(x) = N_1[\bar{\psi}\gamma_\mu\psi](x) \; .$$

In obtaining (5.2), we used the axial-vector current conservation:

$$\langle T\tilde{\partial}^\mu j_\mu(x)X\rangle = -\frac{\alpha}{e}\langle T\partial_\lambda A^\lambda(x)X\rangle + \beta\sum_{j=1}^{N}\left\{\delta(x-x_j)\gamma^5_{x_j} + \delta(x-y_j)\gamma^{5\,T}_{y_j}\right\}\langle TX\rangle \; ,$$

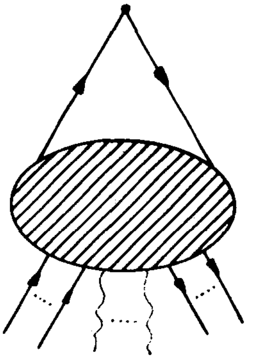


Fig.3 - The reduced vertex has a momentum factor which improves the infrared behaviour of this graph.

(a)

(b)

Fig.4 - Graphs which diverge if $m = M = 0$.

where a and β are known functions of the masses, and coupling constants.

Equation (5.2) can be easily integrated, yielding

$$\tilde{\partial}_\mu \langle TA^\mu(x) A_\nu(y)\rangle = - \tilde{\partial}_\nu \Delta_F(x-y; m'^2 + \alpha), \qquad (5.3)$$

which shows, explicitly, that $\tilde{\partial}_\mu A$ is a free field, of mass $m^2 + \alpha$. The generation of mass is, as we see, a direct consequence of the anomaly in the axial vector Ward identity. Using the identity

$$a^\mu = - \partial^\mu \int d^2y D(x-y) \partial^\nu a_\nu(y) + \tilde{\partial}^\mu \int d^2y D(x-y) \tilde{\partial}^\nu a_\nu(y) \;, \qquad (5.4)$$

with

$$\Box D(x) = - \delta(x) \;,$$

in (5.4) the vector a being expressed in terms of its divergence and curl[18], we obtain

$$\langle TA_\mu(x) A_\nu(y)\rangle = - \frac{\partial_\mu \partial_\nu}{m'^2} \left(D(x-y) - \Delta_F(x-y; m_0^2) \right)$$

$$- \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu}{m'^2 + \alpha} \left(D(x-y) - \Delta_F(x-y; m'^2 + \alpha) \right) . \qquad (5.5)$$

Other Green's functions, with at least one vector meson, can be obtained in a similar way. If

$$Y = \prod_{i=1}^{N} \psi(x_i) \prod_{j=1}^{N} \overline{\psi}(y_j) ,$$

then we have, for example,

$$\langle TA_\mu(x) Y\rangle$$

$$= \frac{e'}{m'^2} \sum_{i=1}^{N} \partial^\mu \left(D(x-x_i) - D(x-y_i) + \Delta_F(x-y_i; m_0^2) - \Delta_F(x-x_i; m_0^2) \right) \langle TY\rangle$$

$$- \frac{e}{m'^2+\alpha} \sum_{i=1}^{N} \tilde{\partial}^{\mu} \Big((D(x-x_i) - \Delta_F(x-x_i;m'^2_0+\alpha)\gamma^5_{x_i}$$

$$+ (D(x-y_i) - \Delta_F(x-y_i;m'^2+\alpha))\gamma^{5\,T}_{y_i} \Big) \langle TY \rangle \ . \tag{5.6}$$

The above formulas **indicate** that A_μ can be written as

$$A_\mu = \partial_\mu \phi_1 + \tilde{\partial}_\mu \phi_2 \ , \tag{5.7}$$

whit $\phi_1 = \phi_{10} + \phi_{11}$, **and** $\phi_2 = \phi_{20} + \phi_{21}$; ϕ_{10} and ϕ_{20} are zero mass scalar fields, **while** ϕ_{11} and ϕ_{21} are scalar fields of (mass)^2 m_0^2, and m'^2+ a , respectively.

We can **now** integrate the Ward identities, for the vector and **axial-vec-** tor currents, to obtain

$$\langle T j^{\mu}(x) Y \rangle$$

$$= - \partial^{\mu} \sum_{i=1}^{N} (D(x-x_i) - D(x-y_i)) \langle TY \rangle$$

$$+ \frac{\alpha}{m'^2+\alpha} \tilde{\partial}^{\mu} \sum_{i=1}^{N} \Big((D(x-x_i) - \Delta_F(x-x_i;m'^2+\alpha) \Big) \gamma^5_{x_i} +$$

$$+ \Big((D(x-y_i) - \Delta_F(x-y_i;m'^2+\alpha))\gamma^{5\,T}_{y_i} \Big) \langle TY \rangle$$

$$+ \beta \sum_{j=1}^{N} \tilde{\partial}^{\mu} \Big(D(x-x_j)\gamma^5_{x_j} + D(x-y_j)\gamma^{5T}_{y_j} \Big) \langle TY \rangle \ . \tag{5.8}$$

Green's functions, containing only fermion fields, need a **little** bit more of discussion. We start from the Dirac equation

$$i\partial\!\!\!/ \langle T\psi(x) Y \rangle$$

$$= i \sum_{k=1}^{N} \delta(x-y_k) \langle TY_{\hat{y}_k} \rangle (-1)^{N+k} - e \langle T(A\!\!\!/\psi)(x) Y \rangle$$

$$- g\langle TN_{3/2}\left[(\bar{\psi}\gamma_\mu\psi)\gamma^\mu\psi\right](x)Y\rangle\ , \tag{5.9}$$

and use the Wilson[19] identity

$$\langle T:N(\bar{\psi}\gamma_\mu\psi)(x+\varepsilon)\gamma^\mu\psi(x):X\rangle = a_1\langle TN_{3/2}\left[(\bar{\psi}\gamma_\mu\psi)\gamma^\mu\psi\right](x)X\rangle$$

$$+ a_2\not\partial\langle T\psi(x)X\rangle + a_3\langle T\psi(x)X\rangle + a_4\langle T\not A(x)\psi(x)X\rangle\ . \tag{5.10}$$

Note that a_1, a_2 and a_4 are independent of e', M, and m', while a_3 is linear in e'. Moreover, $a_3=0$ because, in the zero mass limit, it is given by

$$\langle T:N_1(\bar{\psi}\gamma^\mu\psi)(0)\gamma^\mu\psi(0):\tilde{\bar{\psi}}(0)\rangle^{prop}\Big|_{M=0,\ e'=0}\ , \tag{5.11}$$

since it results from the first subtraction term, for linearly divergent graphs. But using the normalization condition, (2.9), and

$$\langle TN_{3/2}[\mathcal{O}(0)]\tilde{\bar{\psi}}(p)\rangle^{prop}\Big|_{\substack{p=0\\ M=0\\ e'=0}} = \text{contribution of the trivial graph}\ ,$$

in Eq. (5.10), we obtain the result that (5.11) is equal to zero. Substituting (5.10) into (5.9), it follows that

$$Z_1(\varepsilon)\not\partial\langle T\psi(x)Y\rangle =$$

$$i\sum_k(-1)^{N+k}\,\delta(x-y_k)\langle TY\hat{y}_k\rangle - e'Z_2(\varepsilon)\langle T(\not A\psi)(x)Y\rangle$$

$$- gZ_3(\varepsilon)\langle:N_1(\bar{\psi}\gamma_\mu\psi)(x)\gamma^\mu\psi(x):Y\rangle\ . \tag{5.12}$$

Applying $\mu(\partial/\partial\mu)$ to (5.11), and using (5.48), we obtain

$$\mu\frac{\partial}{\partial\mu}\left(\frac{Z_2}{Z_1}\right) = 0\ ,\quad \mu\frac{\partial}{\partial\mu}\left(\frac{Z_3}{Z_1}\right) = 0\ ,\quad \mu\frac{\partial}{\partial\mu}\left(\frac{1}{Z_1}\right) = 2\gamma_2\frac{1}{Z_1}\ ,$$

which shows that as $\varepsilon\to 0$, Z_2/Z_1, and Z_3/Z_1, are finite constants, but $Z_1 = c_1(\mu^2\varepsilon^2)^{\gamma_2}$, with c_1 a finite constant.

Using these results, we can rewrite **(5.12)** as

$$i\not\partial\langle T\psi(x)Y\rangle =$$

$$i\sum_{k=1}^{N}(-1)^{N+k}\,\delta(x-y_k)\langle TY_{\hat{y}_k}\rangle - \bar{e}\langle T(\not A\psi)(x)Y\rangle - \bar{g}\langle T:(j^{\mu}\gamma_{\mu}\psi)(x):Y\rangle, \quad \textbf{(5.13)}$$

where the Z_1 factor has been absorbed into ψ; $\bar{e} = (Z_2/Z_1)e'$, $\bar{g}=(Z_3/Z_1)g$.

From (5.6) and **(5.8)**, we have[20]

$$\langle T(A_{\mu}\psi)(x)\bar{\psi}(y)\rangle$$

$$= \frac{e'}{m'^2}\left[\partial_{\mu}(\Delta_F(x-y;m_0^2) - D(x-y))\right]\langle T\psi(x)\bar{\psi}(y)\rangle$$

$$- \frac{e'}{m'^2+\alpha}\,\tilde{\partial}_{\mu}\left(D(x-y) - \Delta_F(x-y;m'^2+\alpha)\right)\gamma_y^{5T}\langle T\psi(x)\bar{\psi}(y)\rangle,$$

and

$$\langle T:(j_{\mu}\psi)(x):\bar{\psi}(y)\rangle =$$

$$= \partial_{\mu}D(x-y) + \frac{\alpha}{m'^2+\alpha}\,\tilde{\partial}_{\mu}\left[D(x-y) - \Delta_F(x-y;m'^2+\alpha)\right]\gamma_y^{5T}\langle T\psi(x)\bar{\psi}(y)\rangle$$

$$+ \beta\,\tilde{\partial}_{\mu}D(x-y)\gamma^{5T}\langle T\psi(x)\bar{\psi}(y)\rangle\;. \quad (5.14)$$

Thus, the fermion two-point function is given by

$$\langle T\psi(x)\bar{\psi}(y)\rangle = e^{-iF(x,y)}\;{}^{(0)}\langle T\psi^{(0)}(x)\bar{\psi}^{(0)}(y)\rangle^{(0)}, \quad (5.15)$$

where

$$F(x,y) = \frac{e'\bar{e} - \alpha\bar{g}}{m'^2+\alpha}\,(D(x-y) - \Delta_F(x-y;m'^2+\alpha))$$

$$+ \frac{e'\bar{e}}{m'^2}\,(D(x-y) - \Delta_F(x-y;m_0^2)) - \bar{g}(1+\beta)\,D(x-y)\;. \quad (5.16)$$

Green's functions, with more than two fermion fields, can be similarly constructed.

From (5.5), and (5.16), we can verify Eq. (3.5). Furthermore, we can explicitly see that the m_0 -dependence can be gauged away.

6. CONCLUSION

We have shown how to construct Green's functions, in gauges which differ in the high energy behavior of the photon propagator. Yet, they all lead to the same S-matrix, due to the presence of suitable counter terms which, in the $m_0 \to \infty$ limit, become infinite, in order to absorb the difference between a superrenormalizable, and a renormalizable theory. Observables are, of course, m_0 -independent.

The considerations of this paper can be extended to four dimensions[21], where one has an infinite number of counter terms, whose presence is to ensure that the renormalizable, and the nonrenormalizable theory, give rise both to the same S-matrix.

REFERENCES and NOTES

1. See, e.g., S. Coleman, *Secret Symmetry: An Introduction to Spontaneous Symmetry Breakdown and Gauge Fields;* Lectures given at the 1973 International School of Physics Ettore Majorana.
2. J. Schwinger, Phys. Rev. 128, 2425(1962); *Theoretical Physics,* Trieste Lectures, 1962, pg.89; I.A.E.A., Vienna, 1963.
3. J.H. Lowenstein and J.A. Swieca, Ann. Phys.,68, 172 (1971).
4. W. Zimmermann, Commun. Math. Phys. 15, 208 (1969); ibid. 11, 1 (1968).
5. J.H. Lowenstein, *Normal Product Methods in Renormalized Perturbation Theory*, Univ. of Maryland Technical Report N? 73-068 (1972); *BPHZ Renormalization,* Lectures given at the International School of Math.Phys., Erice, Sicily, 1975.
6. W. Thirring, Ann. Phys. 3, 91 (1958).

7. M. Gomes, J.H. Lowenstein and W. Zimrnermann, Commun. Math. Phys. 39, 81 (1974).

8. M. Gomes and B. Schroer, Phys. Rev. D10, 3525 (1974).

9. The introduction of the variables, e and m, is only a matter of convenience and everything can be done without them.

10. We will not use normal products with degree higher than two; if needed, they can be defined as usual.

11. J.H. Lowenstein, Commun. Math. Phys. 24, 1 (1971).

12. By soft, we mean an operator with very good short distance behaviour; in the case at hand it means that $:F^{\mu\nu} F_{\mu\nu}:$ is already well defined.

13. S. Weinberg, Phys. Rev. D8, 3497 (1973).

14. M. Gomes and J. H. Lowenstein, Nucl. Phys. B45, 1635 (1971).

15. See, e.g., N.N. Bogoliubov and D.W. Shirkow, *Introduction to the Theory of Quantized Fields*, New York, Interscience Publ. 1959.

16. J.H. Lowenstein and B. Schroer, Phys. Rev. D6, 1553 (1972).

17. J.H. Lowenstein, *Convergence Theorems for Renormalized Feynman Integrals with Zero-Mass Propagators*, to be published in Commun. Math. Phys.. Note that graphs, which are already finite due to the vertex $\bar{\psi}A.\gamma\psi$, do not create infrared divergencies, since the conservation of the axial-vector current excludes the generation of a fermion mass.

18. K. Johnson, N.Cimento 20, 773 (1961).

19. K. Wilson, Phys. Rev. 179, 1499 (1969).

20. Here we have used

$$\lim_{x \to x_i} \Delta_F(x-x_i ; m_0^2) = D(x-x_i) \ , \quad m_0^2 \text{ finite.}$$

21. M. Gomes and R. Köberle, to be published.