Revista Brasileira de Física, Vol. 6, Nº 3, 1976

Wave-Wave Interactions in a Plasma

AMAURY A. DE ALMEIDA and CALVIN M. BURGOYNE
Instituto Astronômico e Geofísico, Universidade de São Paulo, São Paulo SP

Recebido em 26 de maio de 1976

The problem of the scattering of two longitudinal plasma oscillations into a single transverse wave is considered. The theory is formulated to second order in the Electromagnetic Field using a Vlasov type set of non-linear equations. The emission rate and spectrum are written in terms of the electron distribution function, and the relationship between the observed emission and the dynamics of the plasma is discussed.

Considera-se o problema do espalhamento de duas oscilações de plasma longitudinais resultando em uma única onda eletromagnética transversa. A teoria é formulada incluindo-se os termos de segunda ordem no campo eletromagnético, usando-se um conjunto de equações não lineares do tipo de Vlasov. Escreve-se a taxa de emissão e espectro em termos da função de distribuição de elétrons, e discute-se a relação entre a emissão observada e a dinâmica do plasma.

1. INTRODUCTION

The problem of the emission of radiation with a frequency near twice the plasma frequency has been discussed by several people[1]. We reformulate this theory in such a way that the relationship between the emitted radiation and the particle distribution function is easily analyzed.

The basic idea is to consider the plasma to be described by Maxwell's equations and the collisionless Boltzmann equation. These equations are taken to second order in the fields and the distribution function. We then consider the particle response to the first order fields (in our case, the electric field associated with the longitudinal plasma waves) to be a source current that emits radiation. This method is very similar to that used by Birmingham *et al.*[2], however we do not consider a test particle model since we want all effects to be explicitly written in terms of the particle distribution functions

2. BASIC FORMALISM

We consider a neutral, fully ionized plasma to be described by Maxwell's equations:

$$\vec{\nabla}.\vec{E} = 4\pi\rho \quad , \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\,\frac{\partial\vec{B}}{\partial t} \quad ,$$

$$\vec{\nabla}.\vec{B} = 0 \quad , \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\,\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\;\vec{J}\;, \qquad (1)$$

and by the collisionless Boltzmann equation:

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}.\vec{\nabla}\,f_\alpha + \frac{q_\alpha}{m}\,(\vec{E} + \vec{v}\times\vec{B}).\vec{\nabla}_{\vec{v}}\,f_\alpha = 0 \quad , \qquad (2)$$

where a labels the species of ion or the electrons. These two equations are coupled through the relationship:

$$\rho = \sum_\alpha q_\alpha \int d^3v\; f_\alpha(\vec{v})$$

and

$$\vec{J} = \sum_\alpha q_\alpha \int d^3v\; \vec{v}\; f_\alpha(\vec{v}) \quad . \qquad (3)$$

We begin by writing f, $\vec{E}$, and $\vec{B}$ in the form

$$f = f_0 + \lambda f_1 + \lambda^2 f_2 \quad ,$$

$$\vec{E} = \vec{E}_0 + \lambda\vec{E}_1 + \lambda^2\vec{E}_2 \quad ,$$

$$\vec{B} = \vec{B}_0 + \lambda\vec{B}_1 + \lambda^2\vec{B}_2 \quad , \tag{4}$$

where λ is an expansion parameter that we will later set equal to one. The Boltzmann equation is in terms of Eq. (4) and each order of λ is set equal to zero. That is, by neglecting terms in λ^3 or higher, we get the equations

$$\frac{\partial f_0}{\partial t} + \vec{v} \cdot \frac{\partial f_0}{\partial \vec{X}} + \frac{q}{m}(\vec{E}_0 + \vec{v}\times\vec{B}_0) \cdot \frac{\partial f_0}{\partial \vec{v}} = 0 \ ,$$

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{X}} + \frac{q}{m}(\vec{E}_0 + \vec{v}\times\vec{B}_0) \cdot \frac{\partial f_1}{\partial \vec{v}}$$

$$+ \frac{q}{m}(\vec{E}_1 + \vec{v}\times\vec{B}_1) \cdot \frac{\partial f_0}{\partial \vec{v}} = 0 \ ,$$

and

$$\frac{\partial f_2}{\partial t} + \vec{v} \cdot \frac{\partial f_2}{\partial \vec{X}} + \frac{q}{m}(\vec{E}_0 + \vec{v}\times\vec{B}_0) \frac{\partial f_2}{\partial \vec{v}}$$

$$+ \frac{q}{m}(\vec{E}_1 + \vec{v}\times\vec{B}_1) \frac{\partial f_1}{\partial \vec{v}}$$

$$+ \frac{q}{m}(\vec{E}_2 + \vec{v}\times\vec{B}_2) \frac{\partial f_0}{\partial \vec{v}} = 0 \ . \tag{5}$$

The physical significance of each term in Eq. (5) is the following: f_0 is the distribution function of the particles in an undisturbed state; f_0 is not necessarily an equilibrium state but it does not include short wavelength oscillations. $\vec{E}_0$ and $\vec{B}_0$ are external electric and magnetic field imposed on the plasma. $\vec{E}_1$ and $\vec{B}_1$ are the electric and magnetic fields of a collective osciilation of the plasma, such as an electromagnetic wave or longitudinal plasma oscillation. f_1 gives the motion of the particles due to these collective oscillations. $\vec{E}_2$ and $\vec{B}_2$ are the electric and magnetic fields caused by an interaction between modes of type 1. Finally, f_2 gives the particle response to electromagnetic fields of type 2. Clearly, our formulation is good only in cases where $f_0 \gg f_1 \gg f_2$. Since we are interested in the wave properties of the plasma, it is convenient to Fourier transform all of the basic equations. We note that since f_0, $\vec{E}_0$, and $\vec{B}_0$ are taken to be constant, or at least of wavelength and period much larger than the collective oscillations of the plasma, we only consider the $k = \omega = 0$ Fourier coefficient of f_0, etc. Formally, this will appear as though these "0" quantities remain untransformed while the others are transformed.

The basic equations can then be written in the form

$$\begin{aligned}
\vec{k}.\vec{E}(\vec{k},\omega) &= i4\pi\rho(\vec{k},\omega) \ , \\
\vec{k}.\vec{B}(\vec{k},\omega) &= 0 \ , \\
\vec{k}\times\vec{E}(\vec{k},\omega) &= \frac{\omega}{c}\,\vec{B}(\vec{k},\omega) \ , \\
\vec{k}\times\vec{B}(\vec{k},\omega) &= -\frac{\omega}{c}\,\vec{E}(\vec{k},\omega) + i\frac{4\pi}{c}\,\vec{J}(\vec{k},\omega) \ ,
\end{aligned} \qquad (6)$$

and

$$i\omega f_1(\vec{k},\omega) - i\vec{k}.\vec{v}\, f_1(\vec{k},\omega) + \frac{q}{m}\left[\vec{E}_1(\vec{k},\omega) + \vec{v}\times\vec{B}_1(\vec{k},\omega)\right].\frac{\partial f_0}{\partial\vec{v}} + \frac{q}{m}\left[\vec{E}_0 + \vec{v}\times\vec{B}_0\right].\frac{\partial f_1(\vec{k},\omega)}{\partial\vec{v}} = 0 \ ,$$

$$i\omega f_2(\vec{k},\omega) - i\vec{k}.\vec{v}\, f_2(\vec{k},\omega) + \frac{q}{m}\left[\vec{E}_0 + \frac{\vec{v}}{c}\times\vec{B}_0\right].\frac{\partial f_2(\vec{k},\omega)}{\partial\vec{v}} + \frac{q}{m}\left[\vec{E}_2(\vec{k},\omega) + \frac{\vec{v}}{c}\times\vec{B}_2(\vec{k},\omega)\right].\frac{\partial f_0}{\partial\vec{v}} + \frac{q}{m}\left[(\vec{E}_1 + \frac{\vec{v}}{c}\times\vec{B}_1).\frac{\partial f}{\partial\vec{v}}\right]_{\vec{k},\omega} . \qquad (7)$$

The last term in Eq. (7) is a short hand notation for the expression

$$\frac{q}{m}\int \frac{d^3k'd\omega'}{(2\pi)^4}\left[\vec{E}_1(\vec{k}';\omega') + \frac{\vec{v}}{c}\times\vec{B}_1(\vec{k}';\omega')\right]\cdot\frac{\partial f_1(\vec{k}-\vec{k}',\omega-\omega')}{\partial\vec{v}}\ . \tag{8}$$

We will only consider the special case of no external fields so that from now on we set $\vec{E}_0 = \vec{B}_0 = 0$. Recall that in the linear theory it is convenient to define a conductivity $\overleftrightarrow{\sigma}$ by the equation

$$\vec{J} = \overleftrightarrow{\sigma}\cdot\vec{E}\ , \tag{9}$$

and a dielectric constant $\overleftrightarrow{K}$ by the equation

$$\overleftrightarrow{K} = \underset{\sim}{1} - i\frac{4\pi}{\omega}\overleftrightarrow{\sigma}\ . \tag{10}$$

With these definitions and the continuity equation, $\rho = \vec{k}.\vec{J}/\omega$, we find that Maxwell's equations become

$$\begin{aligned} &\vec{k}.\overleftrightarrow{K}.\vec{E} = 0 \quad , \quad \vec{k}\times\vec{E} = \frac{\omega\vec{B}}{c}\ ,\\ &\vec{k}.\vec{B} = 0 \quad , \quad \vec{k}\times\vec{B} = -\frac{\omega\overleftrightarrow{K}}{c}\cdot\vec{E}\ , \end{aligned} \tag{11}$$

and the dielectric constant is given by the expression

$$\overleftrightarrow{K} = \underset{\sim}{1} + \sum_\alpha \frac{4\pi q_\alpha^2}{m_\alpha\omega}\int d^3v\ \frac{\vec{v}(\partial f_{\alpha 0}/\partial\vec{v})}{\omega - \vec{k}.\vec{v}}\ . \tag{12}$$

It is then simple to derive the dispersion relationship

$$\vec{k}\times(\vec{k}\times\vec{E}) = -\frac{\omega^2}{c^2}\overleftrightarrow{K}.\vec{E}\ . \tag{13}$$

This linear theory is well known so we will not consider this part further but proceed to the non-linear case. From Eq. (7), we see that $f_2(\vec{k},\omega)$ can be written in the form

$$f_2(\vec{k},\omega) = i\frac{q}{m}\ \frac{\left[(\vec{E}_1 + \frac{\vec{v}}{c}\times\vec{B}_1)\cdot\frac{\partial f_1}{\partial\vec{v}}\right]_{\vec{k},\omega} + \vec{E}_2(\vec{k},\omega)\cdot\frac{\partial f_0}{\partial\vec{v}}}{\omega - \vec{k}.\vec{v} + i\varepsilon}\ , \tag{14}$$

where we have neglected $\vec{B}$ with respect to $\vec{E}$, and ε is a term that has been introduced for mathematical convenience which will later be set equal to zero. From Eq. (6), we can write the equation

$$\vec{k}.\vec{E}_2 = i\, 4\pi\, \rho_2$$

$$= i\, 4\pi \sum_\alpha q_\alpha \int d^3v\, f_{\alpha 2}$$

$$= -\frac{4\pi e^2}{m}\int d^3v\, \frac{\left[(\vec{E}_1 + \frac{\vec{v}}{c}\times\vec{B}_1).\frac{\partial f_1}{\partial\vec{v}}\right]_{\vec{k},\omega} + \vec{E}_2(\vec{k},\omega).\frac{\partial f_0}{\partial\vec{v}}}{\omega - \vec{k}.\vec{v} + i\varepsilon} \quad . (15)$$

In Eq. (15), we have taken only electrons since the electron mass, m, is much less than any ion mass. We note that $\vec{E}_2$ can be written in the form

$$\vec{E}_2 = \frac{1}{k^2}(\vec{E}_2.\vec{k})\vec{k} - \frac{1}{k^2}\vec{k}\times(\vec{k}\times\vec{E}_2)$$

$$= \frac{1}{k^2}\left[(\vec{E}_2.\vec{k})\vec{k} - \vec{k}\times\left(\frac{\omega}{c}\vec{B}_2\right)\right]$$

$$= \frac{1}{k^2}\left[(\vec{E}_2.\vec{k})\vec{k} + \frac{\omega^2}{c^2}\vec{E}_2 - i\frac{4\pi\omega}{c^2}\vec{j}_2\right] \quad .$$

From Eq. (16), we then see that $\vec{E}_2$ has components given by the expressions

$$\vec{k}.\vec{E}_2 = \frac{i4\pi}{\omega}\vec{k}.\vec{j}_2 \quad ,$$

$$\vec{k}\times\vec{E}_2 = \frac{\omega^2}{k^2c^2}\vec{k}\times\vec{E}_2 - \frac{i4\pi\omega}{k^2c^2}\vec{k}\times\vec{j}_2 \quad . \qquad (17)$$

We first note that if $\vec{E}$ is parallel to $\vec{k}$, then for these longitudinal waves we have the relationship

$$(\vec{k}.\vec{E}_2)\left(\vec{k}.\frac{\partial f}{\partial \vec{v}}\right) = k^2\ \vec{E}_2 . \frac{\partial f}{\partial \vec{v}} \quad , \tag{18}$$

so that we can rewite Eq.(15) in the form

$$\vec{k}.\vec{E}_2 = -\frac{i4\pi\rho_s}{D_L} \quad , \tag{19}$$

where ρ_s is given by the equation

$$\rho_s = -\frac{i}{4\pi}\ \omega_p^2 \int d^3v\ \frac{\left[(\vec{E}_1 + \frac{\vec{v}}{c}\times\vec{B}_1)\frac{\partial f_0}{\partial\vec{v}}\right]_{\vec{k},\omega}}{\omega - \vec{k}.\vec{v} + i\varepsilon} \quad , \tag{20}$$

and D_L is given by the equation

$$D_L = \left(1 + \frac{\omega_p^2}{k^2}\int d^3v\ \frac{\vec{k}.\frac{\partial f_0}{\partial\vec{v}}}{\omega - \vec{k}.\vec{v} + i\varepsilon}\right) . \tag{21}$$

In Eqs.(20) and (21), we have used the usual definition of the plasma frequency, ω_p, given by the expression

$$\omega_p = \left(\frac{4\pi e^2 n}{m}\right)^{1/2} \quad , \tag{22}$$

and f is now normalized to one instead of n.

In the case of transve'rse waves, we can use Eqs. (17), (14) and (3) to give the relationship

$$\vec{k}\times\vec{E}_2 = -\frac{i\ 4\pi\ \omega\ \vec{k}\times\vec{j}_s}{k^2c^2D_T(\vec{k},\omega)} \quad , \tag{23}$$

where D_T and $\vec{j}_s$ are defined by the equations

$$D_T(\vec{k},\omega) = 1 - \frac{\omega_p^2}{k^2c^2} + \frac{\omega_p^2\,\omega}{k^2c^2}\int d^3v \frac{f_0}{\omega - \vec{k}.\vec{v} + i\varepsilon} , \tag{24}$$

and

$$\vec{j}_s = -i\,\frac{\omega_p^2}{4\pi}\int d^3v \frac{\vec{v}\left[(\vec{E}_1 + \frac{\vec{v}}{c}\times\vec{B}_1)\cdot\frac{\partial f_1}{\partial\vec{v}}\right]_{\vec{k},\omega}}{\omega - \vec{k}.\vec{v} + i\varepsilon} \tag{25}$$

It is important to note that we have taken $\vec{j}_2$ to be given by two parts, $\vec{j}_s$ and a part proportional to $\vec{E}_2$. The part proportional to $\vec{E}_2$ is a particle response to the wave $\vec{E}_2$, and $\vec{j}_s$ is a current due to $\vec{E}_1$, and f_1 serves as a source current to emit waves $\vec{E}_2$

3. EMISSION OF WAVES

We now compute the energy, W, emitted by the source $\vec{j}_s$. That is, we compute

$$W = \lim_{T\to\infty}\int_{-T}^{T} dt \int d^3r\, \vec{E}_2(\vec{r},t).\vec{j}_s(\vec{r},t). \tag{26}$$

By Parseval's theorem Eq. (26) can be written as the equation

$$W = \frac{\mathrm{Re}}{2(2\pi)^4}\int d\omega \int d^3k\, \vec{E}_2(\vec{k},\omega).\vec{j}_s^{\,*}(\vec{k},\omega), \tag{27}$$

and the energy per unit frequency, W_ω, is given by the equation

$$W_\omega = \frac{\mathrm{Re}}{(2\pi)^4}\int d^3k\, \vec{E}_2(\vec{k},\omega).\vec{j}_s^{\,*}(\vec{k},\omega) \tag{28}$$

We have restricted ourselves to the case $\omega > 0$; however, the Fourier transforms are defined for all w. We make this change by multiplying Eq. (27) by 2 and interpreting ω as $|\omega|$.

By combining Eqs. (16) and (28), we can write the equation

$$W_\omega = \text{Re}\, \frac{1}{(2\pi)^4} \int d^3k \left[\frac{(\vec{E}_2 . \vec{k})\vec{k} . \vec{j}_s^{\,*}}{k^2} - \frac{\vec{k}\times(\vec{k}\times\vec{E}_2) . \vec{j}_s^{\,*}}{k^2} \right] \tag{29}$$

From Eqs. (19) and (23), we then can write Eq. (29) in the form

$$W_\omega = \frac{1}{(2\pi)^4} \text{Re} \int d^3k \left[\frac{(-i4\pi\rho_s)\vec{k} . \vec{j}_s^{\,*}}{k^2 D_L(\vec{k},\omega)} + \frac{(-i4\pi\omega)\left[(\vec{k}\times\vec{j}_s)\times\vec{k}\right] . \vec{j}_s^{\,*}}{k^2 e^2 D_T(\vec{k},\omega) k^2} \right]$$

$$= \frac{1}{4\pi^3} \text{Im} \int d^3k \left[\frac{|\vec{k} . \vec{j}_s|^2}{\omega k^2 D_L(\vec{k},\omega)} - \frac{\omega(\vec{k}\times\vec{j}_s)^2}{k^4 e^2 D_T(\vec{k},\omega)} \right] . \tag{30}$$

The D_L^{-1} term is an emitted longitudinal wave, and the D_T^{-1} term is an emitted transverse wave. Since only a transverse wave can be emitted from the plasma, we restrict ourselves to the second term in Eq. (30). We recall that, for transverse waves, w is of the order kc, so that $\vec{k}.\vec{v} << \omega$ in Eq. (24) and then, by setting $\epsilon = 0$, we get the relationship

$$D_T(\vec{k},\omega) \cong 1 - \frac{\omega^2}{c^2k^2} + \frac{\omega_p^2}{c^2k^2} . \tag{31}$$

With these approximations we can then write Eq. (30) in the form

$$W_\omega = -\frac{\omega}{4\pi^3} \text{Im} \int d^3k \frac{|\vec{k}\times\vec{j}_s|^2}{k^2\left[k^2c^2 - \omega^2 + \omega_p^2\right]}$$

$$= -\frac{\omega}{4\pi^3 c^2} \text{Im} \int d\Omega_k \int_0^\infty dk \frac{k^2 |\vec{j}_s(\vec{k},\omega)|^2 \sin^2\theta}{\left[k^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2}\right]} \tag{32}$$

It is convenient to define a wave vector, $\vec{k}_T$, defined by the equation

$$k_T^2 \equiv \left(\frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}\right) , \qquad (33)$$

and note that the denominator of Eq. (32) can then be written in the form

$$\frac{1}{k^2-k_T^2} = \frac{1}{2k_T}\left[\frac{1}{k-k_T} - \frac{1}{k+k_T}\right] . \qquad (34)$$

By combining these equations we are finally able to write

$$W_\omega = -\frac{\omega}{4\pi^2 c^2}\int d\Omega_k \sin^2\theta \, . \int_0^\infty dk \, \frac{k^2|\vec{j}_s(\vec{k},\omega)|^2}{2k_T}\,\delta(k-k_T)$$

$$= -\frac{\omega}{8\pi^2 c^2}\int d\Omega_k \sin^2\theta \, . \, k_T|\vec{j}_s(\vec{k}_T,\omega)|^2$$

$$= -\frac{\omega(\omega^2-\omega_p^2)^{1/2}}{8\pi^2 c^3}\int d\Omega_k \sin^2\theta \, . \, |\vec{j}_s(\vec{k}_T,\omega)|^2 \, . \qquad (35)$$

The factor $\vec{j}_s(\vec{k}_T,\omega)$ must now be calculated. In Eq. (25) the factor $\vec{v}\times\vec{B}_1$ is neglected as being a relativistic correction and we take the long wavelength limit so that to first order in $\vec{k}.\vec{v}/\omega$ we get the equation

$$\vec{j}_s = -i\frac{\omega_p^2}{4\pi\omega}\int d^3v \, \vec{v}\left[\vec{E}_1 \, . \, \frac{\partial f_1}{\partial\vec{v}}\right]_{\vec{k},\omega}\left(1 + \frac{\vec{k}.\vec{v}}{\omega}\right)$$

$$= -i\frac{\omega_p^2}{4\pi\omega}\int d^3v \, \vec{v}\int\frac{d^3k'}{(2\pi)^3}\int\frac{d\omega'}{2\pi}\,\vec{E}_1(\vec{k}';\omega) \, . \, \frac{\partial f_1(\vec{k}-\vec{k}';\omega-\omega')}{\partial\vec{v}} \, . \left(1 + \frac{\vec{k}.\vec{v}}{\omega}\right) . \qquad (36)$$

It is possible to write f_1 in terms of f_0 from Eq. (7). We see that f_1 and $\partial f_0/\partial \vec{v}$ are related by the equation

$$f_1(k-k';\omega-\omega') = i\,\frac{q}{m}\,\frac{\vec{E}_1(\vec{k}-\vec{k}';\omega-\omega')\,.\,\frac{\partial f_0}{\partial \vec{v}}(v)}{(\omega-\omega') - (\vec{k}-\vec{k}').\vec{v}} \quad . \tag{37}$$

If f_1 from Eq.(37) is put into Eq. (36) we then get the following expression for $\vec{j}_S$ to first order in $\vec{k}.\vec{v}/\omega$:

$$\vec{j}_s(\vec{k},\omega) = \frac{\omega_p^2}{4\pi m\omega^2}\int\frac{d^3k'}{(2\pi)^3}\int\frac{d\omega'}{(2\pi)}\int d^3v\ v\ .\ \frac{1}{\omega-\omega'}\left[1+\frac{(\vec{k}-\vec{k}').\vec{v}}{\omega-\omega'}\right]\ .$$

$$.\ \frac{\partial}{\partial\vec{v}}\left[\frac{\partial f_0(v)}{\partial\vec{v}}\ \vec{E}_1(\vec{k}-\vec{k}';\omega-\omega')\right].\ \vec{E}_1(\vec{k};\omega)\ \left(1+\frac{\vec{k}.\vec{v}}{\omega}\right) \quad . \tag{38}$$

In principle, we can now find the emitted energy from Eqs. (35) and (38), if we know f_0 and the associated $\vec{E}_1$. However, we are then faced with two problems. First, there are an infinite number of f_0's that solve the 0'th order Vlasov equation so that a choice of f depends on the method of preparation of the plasma or its initial conditions. Second, we must have some model to find $\vec{E}_1$. The linear theory will give us a dispersion relationship from which we can find the relationship between $\vec{k}$ and ω , for the field $\vec{E}_1$, but we also need a model to find the amplitude of this electric field. It is then impossible to give a complete solution of Eq. (35). However, as we shall see in the next Section, some very general results can be derived without detailed knowledge of f_0 and the exact spectrum of $\vec{E}_1$.

4. EMISSION FROM TWO PLASMA WAVES

A simple calculation of the energy and momentum of each type of wave shows that the only way to produce a transverse wave is by the scattering of two longitudinal plasma oscillations. We, therefore, consider only the case of two plasma waves scattering to give one transverse wave. The $\vec{E}_I$'s in Eq. (38), therefore, will be taken to be due to longitudinal plasma waves. From Eqs. (12) and (13), we see that for $\vec{k}\times\vec{E} = 0$ and by taking $\vec{k}$ to be in the z-direction, we can write the z-z-component of the dielectric constant as

$$\mathrm{Re}\ k_{zz} = 1 + \frac{\omega_p^2}{\omega^2}\int d^3v\ v_z \frac{\partial f_0}{\partial v_z}\left[1 + \frac{kv_z}{\omega} + \left(\frac{kv_z}{\omega}\right)^2 + \ldots\right]$$

$$= 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2}{\omega^2}<v_z^2>\right) , \tag{39}$$

and

$$\mathrm{Im}\ k_{zz} = \frac{\pi\omega_p^2}{\omega k}\int dv_z\ \frac{\partial f_0}{\partial v_z}\ v_z\ \delta\left(\frac{\omega}{k} - v_z\right) = \frac{\pi\omega_p^2}{k^2}\ \frac{\partial f_0(v_z)}{\partial v_z}\bigg|_{\omega/k} . \tag{40}$$

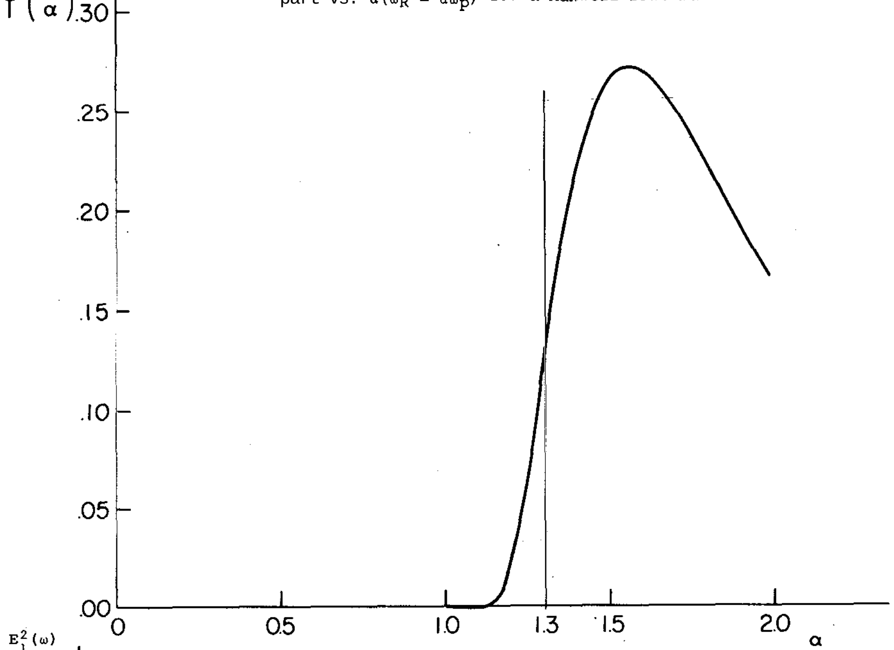
In Eq. (39), we have taken f_0 isotropic in $\vec{v}$ so that the first order term in v_z goes to zero. If we take $\vec{k}$ to be real and use the dispersion relationship, we get for complex $\omega = \omega_R + i\omega_I$ the expression

$$0 = (\omega_R^2 - \omega_I^2 + 2i\ \omega_R\omega_I) - \omega_p^2\left[1 + \frac{3k^2<v_z^2>}{\omega_R^2-\omega_I^2 + 2i\ \omega_R\omega_I}\right]$$

$$+ i\frac{\pi\omega_p^2}{k^2}(\omega_R^2 - \omega_I^2 + 2i\ \omega_R\omega_I)\frac{\partial f_0(v_z)}{\partial v_z}\bigg|_{\omega/k} . \tag{41}$$

If $\omega_I << \omega_R$, which is required if the idea of a plasmawave is to be meaningful, then Eq. (41) can bewritten for real and imaginary parts in the form

Figure 1. The ratio of the imaginary part of the frequency to the real part vs. $\alpha(\omega_R = \alpha\omega_p)$ for a Maxwell-Boltzmann distribution.



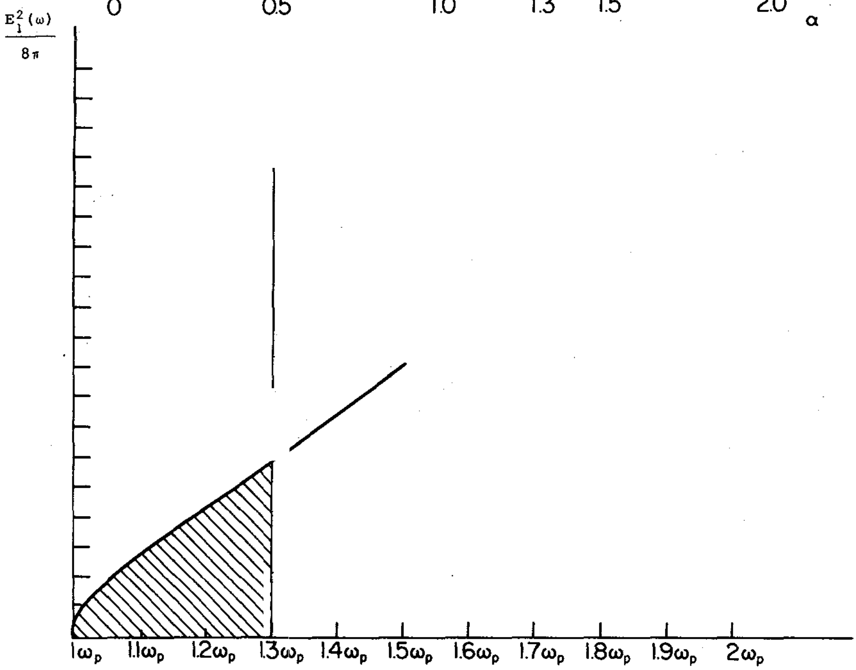


Figure 2. The energy density of longitudinal waves as a function of ω. Since $E_1(\omega)$ is a function of T we have plotted only the functional form of the energy density and thus the units of $E_1^2(\omega)/8\pi$ are arbitrary.

$$\omega_R^2 = \omega_p^2 \left(1 + \frac{3k^2 \langle v_z^2 \rangle}{\omega_R^2}\right)$$

and

$$\omega_I \cong - \frac{\pi \omega_p^2 \omega_R}{k^2} \left. \frac{\partial f_0(v_z)}{\partial v_z} \right|_{v_z = \omega/k} \quad . \tag{42}$$

If we define a by the equation

$$\omega_R \equiv \alpha \omega_P \quad , \tag{43}$$

then the condition that $\omega_I \ll \omega_R$ becomes

$$-v_0^2 \left. \frac{\partial f_0(v_z)}{\partial v_z} \right|_{v_0} \ll \frac{1}{\pi} \quad , \tag{44}$$

where v_0 is defined as

$$v_0 = \left[\frac{3\langle v_z^2 \rangle}{\alpha^2(\alpha^2 - 1)} \right]^{1/2} \tag{45}$$

We note that a $\geq$ 1 and that as a goes to 1 the left-hand side of Eq.(44) goes to zero for any normalizable distribution function. Also the left--hand ride of Eq. (44) will be a maximum near $v_0 = \langle v_z^2 \rangle^{1/2}$ or for $a \approx 1.5$ and for larger α, the long wavelength approximations we have used start to break down. In Fig. 1, we show the special cases of f_0 being a Maxwell-Boltzmann distribution.

The period of a wave is given by $T = 2\pi/\omega_R$ and the decay time $t_d = 1/\omega_I$. For a wave to exist, the period must be less than the decay time or the wave will be damped before we get a complete oscillation. The requirement $T < t_d$ is then the same as requiring that $(\omega_I/\omega_R) < 0.16$. From Fig.

1, we see that $\alpha \lesssim 1.3$ in order to satisfy the above requirernents. Although Fig.1 was made for the special case of a Maxwell-Boltzmann distribution, the results are essentially the same for any isotropic, continuou~, motonomically decreasing function of v_z. That is, for most reasonable distribution functions the frequency of the longitudinal waves is restricted to the interval $w_p < \omega < 1.3\ w_p$. In a similar way, we can argue that the wave vector, k, is restricted to the range

$$k^2 \langle v_z \rangle \lesssim 0.2\ \omega_p^2 \ .$$

Although we know the range of frequencies of $\vec{E}_1(\vec{k};\omega)$, we still need an estimate of its amplitude in order to calculate the total emission rate. Again, it is impossible to answer the question of the amplitude of $\vec{E}_1$, in general, for all possible situations; however, for all cases in which the concept of electron temperature is valid, the amplitude of $\vec{E}_1$ is given by its thermal value.

The energy density per unit frequency of the longitudinal waves is given by the formula[3]

$$\frac{u_E(\omega)}{V} = \frac{E_1^2(\omega)}{8\pi} = \left(\frac{\omega^2 \kappa T}{2\pi^2 c^3}\right) \left(\frac{1}{3\sqrt{3}}\right) \left(\frac{c}{u_0}\right)^3 \cdot \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} , \qquad (46)$$

where u_0 is defined by the equation

$$u_0^2 = \frac{\kappa T}{m} \ . \qquad (47)$$

By combining Eqs. (46) and (47), we get the expression for $E_1^2(\omega)$

$$E_1^2(\omega) = \left\{ \frac{4\ \omega^2}{\pi\sqrt{3}\ c^3} \right\} \left\{ \frac{c^3 m^{3/2}}{(\kappa T)^{1/2}} \right\} \left\{ 1 - \frac{\omega_p^2}{\omega^2} \right\}^{1/2} . \qquad (48)$$

In order to find the total energy of the longitudinal waves u_T we integrate $E_1^2/8\pi$ over all frequency in the allowed interval $w_p \leq w \leq \alpha\omega_p$, to get the expression

$$u_T = \frac{V}{8\pi}\int_{\omega_p}^{\alpha\omega_p} E_1^2(\omega)\,d\omega = \frac{V}{8\pi}\left[\frac{4(mc^2)^{3/2}\ \omega_p^3(\alpha^2-1)^{3/2}}{9\sqrt{3}\ \pi\ c^3(\kappa T)^{1/2}}\right] \tag{49}$$

We note that we have used the classical limit, in Eq. (46), so that Eq. (49) is only valid if $\hbar\omega_p \ll \kappa T$. From Eq. (48), we can find the amplitude of $E_1(\omega)$ so that we can now, in principle, solve for $j_s(\omega)$.

The problem now is simply one of doing integrals for various f_0's toget $j_s(\omega)$ and then the emission rate, W_ω; however these integrals will, in general, require numerical solutions. Since we are not interested in a particular f_0 but in the general behavior of emitting plasmas we choose to simplify the problem in such a way that all the essential physics is retained but the integrations are made simple. To do this, we note that $E_1(\omega)$ is restricted to the small interval $\omega_p < \omega < \alpha\omega_p$, where $\alpha \cong 1.3$ and that E_1 is an increasing function of ω inside this interval (see Fig.2).

We, therefore, can approximate $\vec{E}_1(\omega)$ by the function

$$\vec{E}_1(\vec{k}';\omega') = 2\pi E_1(\vec{k}')\,\delta(\omega'-\alpha\omega_p)\hat{k}' \ , \tag{50}$$

where $\vec{E}_1(\vec{k}')$ is defined in such a way that the total energy density is still given by Eq. (49). We then can write the expression

$$u_T T = \int \frac{d^3k'}{(2\pi)^3}\ \frac{E_1^2(\vec{k}')}{8\pi}\ (2\pi)^2 \int_0^\infty \frac{d\omega'}{2\pi}\ \delta^2(\omega'-\alpha\omega_p) \ ,$$

$$u_T T = \int \frac{d^3k'}{(2\pi)^3}\ \frac{E_1^2(\vec{k}')}{8\pi}\ 2\pi \int_0^\infty \frac{d\omega'}{2\pi}\ (\omega'-\alpha\omega_p)T \ ,$$

$$u_T = \int \frac{d^3k'}{(2\pi)^3}\ \frac{E_1^2(\vec{k}')}{8\pi} \ , \tag{51}$$

where T is interpreted as some long time in which all our emission processes take place, and we will require that

$$T >> \frac{2\pi}{\omega'} \quad , \tag{52}$$

so that we are looking only at time average energy densities. From Eqs. (49) and (51), we then have the expression

$$\int \frac{d^3k'}{(2\pi)^3} E_1^2(\vec{k}') = V\left[\frac{4(mc^2)^{3/2} \omega_p^3(\alpha^2-1)^{3/2}}{9\sqrt{3}\,\pi\, c^3 \quad (\kappa T)^{1/2}}\right] . \tag{53}$$

Tha source current, $\vec{j}_s(\vec{k},\omega)$, is then given by Eqs. (38) and (50) in the form

$$\vec{j}_s(\vec{k},\omega) = \frac{2\pi q\omega_p^2}{4\pi m\omega(\omega-\alpha\omega_p)} \int \frac{d^3k'}{(2\pi)^3} \int d^3v\, \vec{v}\, E_1(\vec{k}')E_1(\vec{k}.\vec{k}') .$$

$$\cdot \left(1 + \frac{\vec{k}.\vec{v}}{\omega}\right) \frac{\partial}{\partial\vec{v}}\left[\frac{\partial f_0(v)}{\partial\vec{v}} . (\hat{\vec{k}-\vec{k}'})\left(1 + \frac{(\vec{k}-\vec{k}').\vec{v}}{\omega-\alpha\omega_p}\right)\right] \hat{k}'\ \delta(\omega-2\alpha\omega_p) .$$

(54)

We now observe that $k \simeq \omega_p/c$ and $k' \simeq \omega_p/\langle v_z^2\rangle^{1/2}$ so that $k << k'$ and that $\hat{\vec{k}-\vec{k}'} \cong -\hat{k}'$. It is then clear that the two longitudinal waves must suffer a near head-on collision in order to produce emitted radiation.

We note that only the $\vec{k}'.\vec{v}/\omega$ term, in Eq. (54), is non-zero for a function $f_0(\vec{v})$ that is an even function. Since an odd function would imply a plasma with a bulk motion of one of the species with respect to the center of mass of the system (such as an electron beam in a plasma), we will only consider even f_0's. Without loss of generality, we take $\vec{k}'$, to be in the z-direction and thus $(\vec{k}-\vec{k}')$ is almost in that direction. We can, therefore, write Eq. (54) in the form

$$\vec{j}_s(\vec{k},\omega) = \frac{q\omega_p^2}{2m\omega} \int \frac{d^3k'k'}{(2\pi)^3} \int d^3v \; \vec{v} \; E_1(\vec{k}')E_1(-\vec{k}') .$$

$$\frac{v_z}{(\omega-\alpha\omega_p)^2} \, \delta(\omega-2\alpha\omega_p) \, \frac{\partial}{\partial v_z} \; \frac{\partial f_0(v)}{\partial v_z} \qquad , \qquad (55)$$

where we have taken $\hat{k}\cdot\vec{k}' = -\hat{z}$ and assumed the z-component of $\partial f_0/\partial\vec{v}$ is $\partial f_0/\partial v_z$ which is true for any $f_0(\vec{v})$ that is separable into the form $f_0(v_x)f_0(v_y)f_0(v_z)$. Clearly, only the z-component of $\vec{j}_s$ is non-zero so that

$$\vec{j}_s(\vec{k},\omega) = \frac{q\omega_p^2}{2m\omega(\omega-\alpha\omega_p^2)} \int \frac{d^3k'k'}{(2\pi)^3} E_1(\vec{k}')E_1(-\vec{k}') \; \delta(\omega-2\alpha\omega_p)\hat{z} \; .$$

$$\cdot \int dv_z \; v_z^2 \; \frac{\partial^2 f_0(v_z)}{\partial v_z^2} \qquad (56)$$

Equation (56) is now integrated by parts twice, to give the equation

$$\vec{j}_s(\vec{k},\omega) = \frac{q\omega_p^2}{2m\omega(\omega-\alpha\omega_p^2)} \int \frac{d^3k'}{(2\pi)^3} \vec{k}' \; E_1(\vec{k}')E_1(-\vec{k}')\delta(\omega-2\alpha\omega_p) . \qquad (57)$$

Clearly Eq. (57) is odd in $\vec{k}$ so that the integral goes to zero. In order to get a finite term, we must then include second order terms in $\vec{k}.\vec{v}/\omega$, which then gives the equation

$$\vec{j}_s(\vec{k},\omega) = \frac{q\omega_p^2 \; \delta(\omega-2\alpha\omega_p)}{2m\omega(\omega-\alpha\omega_p^2)^3} \int \frac{d^3k'}{(2\pi)^3} \vec{k}'^2 \; E_1(\vec{k})E_1(-\vec{k}) \int d^3v \;\; \vec{v} \; f_0 \qquad (58)$$

If $E_1(\vec{k})$ is sharply peaked near $k' = k_L$, where k_L is given by the dispersion relationship

$$k_L^2 = \frac{(\alpha^2-1)\omega_p^2}{<v_z^2>} \quad , \qquad (59)$$

then

$$\vec{J}_s(\vec{k},\omega) \cong \frac{2\pi q(\alpha^2-1)\langle\vec{v}\rangle\delta(\omega-2\alpha\omega_p)u_T}{m\,\alpha^4\,\langle v_z^2\rangle} \,. \tag{60}$$

From Eqs. (35) and (59), we can write the emission rate, W_ω, in the form

$$W_\omega = -\frac{2\alpha\omega_p^2(4\alpha^2-1)^{1/2}}{8\,\pi^2\,c^3}\int\frac{d\Omega_k}{2\pi}\sin^2\theta\; u_T^2\;\delta(\omega-2\alpha\omega_p)T\left\{\frac{4\pi q^2(\alpha^2-1)^2\langle\vec{v}\rangle^2}{m^2\,\alpha^8\,\langle v_z^2\rangle^2}\right\} \,. \tag{61}$$

The time average power emitted from the plasma is then given by the expression

$$P = \frac{1}{T}\int d\omega\; W_\omega = \frac{\omega_p^8(4\alpha^2-1)^{1/2}q^2(\alpha^2-1)^5\langle\vec{v}\rangle^2L^6m}{729\,\alpha^7\,c^3\,\langle v_z^2\rangle^2\,\pi^4\,kT} \,, \tag{62}$$

where L is a characteristic length of the plasma.

5. DISCUSSION

In the above Sections, we have derived an expression for the power emitted from an isotropic, non-relativistic, non-degenerate plasma in a narrow range of frequencies, near 2.6 ω_p. Due to the nature of the approximations made, we cannot make any precise statements about the shape of the spectrum; however, it is clear that it will be reasonably sharply peaked, near 2.6 ω_p. It should be noted that our expression is very different from those given in other papers on this problem of two plasma oscillations scattering into an electromagnetic wave. This difference is due to the fact that we kept all terrns to second order in $\vec{v}.\vec{k}/\omega$, and all other work done on this problem has been from a Poisson equation, neglecting thermal effects, which is equivalent to only keeping O'th order terms in our formulation. The results of this work are easily applied to many astrophysical problems which will be discussed in later papers.

REPERENCES

1. T. Birmingham, J. Dawson and C. Oberman, Phys. Fluids 8, 297 (1965) ; P.A. Storrock, J. Nucl. Energy, PT C2, 158 (1961); R. Aamodt and W. Drummond, J. Nucl. Energy, PT C6, 147 (1964); T.J.M. Boyd, Phys. Fluids 7, 59 (1964).
2. Opere citato in the first of Refs.1.
3. G. Bekefi, *Radiation* Processes *in Plasmas*, John Wiley & Sons, New York (1966).