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Derivation of Bazánski's Lagrangian in a Lorentz Covariant Theory of Gravitation

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By using an approach similar to Thirring's (or equivalently, to Schwinger's source theory), in which gravitation with Lorentz covariance is introduced, Bazánski's Lagrangian, which corresponds at order c^{-2} to the Lagrangian of two particles with masses m_1, m_2, velocities $\vec{v}_1$, $\vec{v}_2$ and electric charges e_1, e_2, is obtained.

Usando-se um formalismo semelhante ao utilizado por Thirring (ou, equivalentemente, à teoria das fontes de Schwinger), que considera gravitação com covariância de Lorentz, obtém-se a Lagrangiana de Bazánski que corresponde, em ordem c^{-2}, à Lagrangiana de duas partículas com massas m_1, m_2, velocidades $\vec{v}_1$, $\vec{v}_2$ e cargas elétricas e_1, e_2.

INTRODUCTION

Some years ago, Bazánski[1] derived, from the equations of general relativity, up to order c^{-2}, the Lagrangian of two particles with masses m_1, m_2, velocities $\vec{v}_1$, $\vec{v}_2$, and electric charges e_1, e_2. It corresponds

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to the generalization of the Einstein, Infeld and Hoffmann Lagrangian to the case where one also takes into account the electromagnetic field.

Bazánski's Lagrangian is the sum of the following terms[2]:

i) T, the part which comes from the free Lagrangian, only;

ii) V_D, the potential term contribution to the Darwin Lagrangian of a system of two charges e_1, e_2;

iii) V_{EIH}, the potential term contribution to the Einstein, Infeld and Hoffmann Lagrangian of a system of two masses m_1, m_2;

iv) V_M, the mixed gravitation-electric potential.

In the present paper, we discuss the origin of these terms in the framework of Thirring's Lorentz covariant theory (Ref.3) (or, equivalently, in Schwinger's source theory of gravitation[4]).

Lorentz covariant theories of gravitation have been discussed by Fierz[5], and it was Gupta[6] who suggested, in this framework, how one can obtain the nonlinear term, in the Lagrangian, since the energy momentum tensor of the gravitational field must also act as a source of that field.

The deitaled calculation of the nonlinear term, up to order G^2, according to Gupta's programme, was made by Feynman[7], who solved the problem in a way which is both consistent, and mathematically general.

A more physical approach, to the nonlinear term, due to Thirring[3], is however, inconsistent. A consistent treatment, which makes use of Thirring's approach, was presented a few years ago[8]. It was shown that the problem of obtaining the nonlinear term, in the procedure of Feynman and Gupta, is very much simplified, in the static limit.

In Section 1, we write down Bazánski's Lagrangian, and justify the presence of the T term. In Section 2, we discuss the Darwin term V_D, and, in Section 3, the linear part of the Einstein, Infeld and Hoffmann term, V_{EIH}. We consider, in Section 4, the nonlinear part of V_{EIH}, and finally, in Section 5, we obtain the mixed term, V_M.

In discussing the linear part of V_{EIH}, we closely follow unpublished material of one of us[9]. A discussion in the same lines is presented in a more recent paper by Cho and Dass[10]. These last authors also discuss the nonlinear term, of V_{EIH}, according to the source theory[4], and, besides, in criticizing Schwinger's calculation[4], they calculate that term along the same lines. They assume, similarly to the case of the electromagnetic field, that the trace of the energy-momentum tensor, for the gravitational field, is null. Unfortunately, this is not correct in the context of the Feynman-Gupta procedure. The point is that the tensor is traceless for free gravitational fields, this not being the case for fields produced by sources. This is discussed in our Appendix, where we write down the gravitational energy-momentum tensor which is obtained in the Feynman-Gupta procedure.

In Section 4, we also mention the difficulty in obtaining, in Schwinger's approach[4,10], the nonlinear term of V_{EIH}. This term is obtained, here, in a consistent way by means of the equations of motion for a test particle in the field of another.

In Section 5, the mixed term, V_M, which measures the effect of the electromagnetic field on the gravitational interaction, is obtained in a very simple way. Apart from the discussion of the mixed term, a good deal of the material presented in this paper is found in the above mentioned references. We have chosen, however, to go over it in great detail, in order to give a unifield and self-contained presentation of the subject, free of interna1 inconsistencies.

1. THE T-TERM IN BAZANSKI'S LAGRANGIAN

Bazánski's Lagrangian, for two particles of masses m_1, m_2, charges e_1, e_2, velocities $\vec{v}_1$, $\vec{v}_2$, can be written as (Ref.2)

$$L = T - V_D - V_{EIH} - V_M , \tag{1}$$

where

$$T = \frac{1}{2} m_1 \vec{v}_1^2 + \frac{1}{2} m_2 \vec{v}_2^2 + \frac{1}{8c^2} (m_1 \vec{v}_1^4 + m_2 \vec{v}_2^4), \tag{2}$$

$$-V_D = -\frac{e_1 e_2}{r} + \frac{e_1 e_2}{2rc^2} \left[\vec{v}_1 . \vec{v}_2 + (\vec{v}_1 . \vec{n})(\vec{v}_2 . \vec{n})\right], \tag{3}$$

$$-V_{EIH} = \frac{G\ m_1 m_2}{r} - \frac{G\ m_1 m_2}{2rc^2} \left[\vec{v}_1 . \vec{v}_2 + (\vec{v}_1 . \vec{n})(\vec{v}_2 . \vec{n}) - 3(\vec{v}_1 - \vec{v}_2)^2\right] - \frac{G^2 m_1 m_2 (m_1 + m_2)}{2r^2c^2}, \tag{4}$$

$$-V_M = \frac{G}{2r^2c^2} \{2\ e_1 e_2 (m_1 + m_2) - (e_1^2 m_2 + e_2^2 m_1)\}, \tag{5}$$

where $r = |\vec{r}_1 - \vec{r}_2|$, and $\vec{n}$ is the unit vector along $\vec{r}_1 - \vec{r}_2$.

We recall that the free Lagrangian, for a particle of mass m and velocity $\vec{v}$, is given by

$$-mc^2 \left(\frac{1-\vec{v}^2}{c^2}\right)^{1/2} = \frac{1}{2} m \vec{v}^2 + \frac{1}{8} m \frac{\vec{v}^4}{c^2},$$

up to order c^{-2}. This justifies the expression for T given in Eq.(2).

2. THE DARWIN TERM

Due to the electrical nature of the particles involved, we have to add, to our free Lagrangian, the term:

$$L^e_{int}(x) = -c^{-1} j_\mu(x) A^\mu(x),$$

where j_μ is the electromagnetic current, and A_μ the electromagnetic potential, which satisfies the equation of motion

$$\Box A_\mu = -\frac{4\pi}{c} j_\mu \quad , \tag{6}$$

with $\Box = \partial_0^2 - \vec{\partial}^2$, and $j_\mu A^\mu = j_0 A_0 - \vec{j} \cdot \vec{A}$.

The solution of Eq.(6), with the usual boundary conditions, in the Lorentr gauge, $\partial^\mu A_\mu = 0$, can be written as

$$A_\mu(x) = \frac{1}{c} \int d^4x' \; G_{ret}(x,x') j_\mu(x') \; , \tag{7}$$

where G_{ret} is given by

$$G_{ret}(x,x') = \frac{\delta\left(t'-t + \frac{|\vec{x}'-\vec{x}|}{c}\right)}{|\vec{x}-\vec{x}'|} \quad . \tag{8}$$

Let us now make a series expansion, in c^{-1}, for $G_{ret}(x,x')$:

$$G_{ret}(x,x') = \frac{1}{|\vec{x}' - \vec{x}|} \{\delta(t'-t) + \frac{|\vec{x}'-\vec{x}|}{c} \frac{\partial}{\partial t'} \delta(t'-t)$$

$$- \frac{1}{2} \frac{|x'-x|^2}{c^2} \frac{\partial^2 \delta(t'-t)}{\partial t \, \partial t'} + \ldots\} \; . \tag{9}$$

Substituting Eq. (7) into $L^e_{int}(x)$, we obtain the following contribution for the action:

$$S^e_{int} = \int L^e_{int}(x) d^4x = - \frac{1}{2c^2} \iint d^4x d^4x' \; G_{ret}(x,x') j_\mu(x) j^\mu(x') , \tag{10}$$

where we have divided by 2 because of the symmetry in x, x'

We now have:

$$j_\mu(x) = e_1 v_{1\mu}\delta(\vec{x} - \vec{r}_1(t)) + e_2 v_{2\mu}\delta(\vec{x}-\vec{r}_2(t)), \qquad (11)$$

where $v^\mu = (c,\vec{v})$.

Neglecting the self-energy terms, weobtain, by introducing Eq. (11) into Eq. (10),

$$S^e_{int} = -\frac{1}{c^2}\int dt dt'\, G_{ret}(t-t',\vec{r}_1(t)-\vec{r}_2(t'))e_1 e_2\, v_1 . v_2 , \qquad (12)$$

with

$$v_1 . v_2 = c^2 - \vec{v}_1 . \vec{v}_2 \quad . \qquad (13)$$

If we now introduce the first term of Eq. (9) into Eq. (12), we obtain the following contribution for S^e_{int}:

$$-\int \frac{dt}{|\vec{r}_1(t)-\vec{r}_2(t)|}\, e_1 e_2 \left(1 - \frac{\vec{v}_1 . \vec{v}_2}{c^2}\right) . \qquad (14)$$

The second term in Eq. (9), when substituted into Eq. (12), gives a contribution of the order c^{-3}, while the third term of Eq. (9), because of the following identity,

$$\iint dt dt' F(t,t')\, \frac{\partial^2 \delta(t'-t)}{\partial t \partial t'} = \int dt \left(\frac{\partial^2 F}{\partial t \partial t'}\right)_{t'=t} , \qquad (15)$$

gives for S^e_{int}, up to order c^{-2},

$$-\int \frac{e_1 e_2}{2c^2}\left(\vec{v}_1 . \vec{v}_2 - (\vec{v}_1 . \vec{n})(\vec{v}_2 . \vec{n})\right) \frac{dt}{|\vec{r}_2(t)-\vec{r}_1(t)|} , \qquad (16)$$

where use was made of the identity which follows:

$$\left[\frac{\partial^2}{\partial t\partial t'} \left|\vec{r}_2(t')-\vec{r}_1(t)\right|\right]_{t'=t} = - \frac{1}{\left|\vec{r}_2(t)-\vec{r}_1(t)\right|}\left[\vec{v}_1.\vec{v}_2-(\vec{v}_1.\vec{n})(\vec{v}_2.\vec{n})\right],$$

with

$$\vec{n} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

Adding Eqs.(14) and (16), we obtain

$$S^{e}_{int} = -\int V_D\, dt \quad .$$

In this may, we have justified the appearance of $(-V_D)$ given by Eq. (3).

3. THE LINEAR PART OF THE EINSTEIN-INFELD-HOFFMANN LAGRANGIAN

Let us now study the gravitational interaction of the particles[9]. The gravitational field will be described by a second order symrnetrical tensor $\psi_{\mu\nu}$, for which we assume the Hilbert gauge[3] to hold:

$$\partial^{\mu}(\psi_{\mu\nu}- \frac{\eta_{\mu\nu}}{2}\psi) = 0, \tag{17}$$

with $\eta_{00} = - \eta_{ii} = 1$, and $\psi = \eta_{\mu\nu}\psi^{\mu\nu}$.

The source of the gravitational field will be the energy-momentum tensor, $T_{\mu\nu}$, which is symmetrical, and with a four divergence equal to zero: $\partial^{\mu}T_{\mu\nu} = 0$.

For convenience, we will take the free Lagrangian density, of the gravitational field, of the form[8]:

$$L^{G}_{0} = \frac{1}{2} (\psi_{\mu\nu,\lambda}\psi^{\mu\nu,\lambda}-\frac{1}{2} \psi_{,\lambda}\psi^{,\lambda}), \tag{18}$$

while the interaction term will be given by

$$L^G_{int} = f\,\psi_{\mu\nu}\overset{p}{T}{}^{\mu\nu}, \tag{19}$$

where $\overset{p}{T}{}^{\mu\nu}$ is the energy-momentum tensor of the particles, which are the sources of the gravitational field. We have[3]:

$$\frac{f^2c^4}{8\pi} = G\,. \tag{20}$$

The equations of motion corresponding to $L^G + L^G_{int}$ are:

$$\Box\left(\psi_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}\psi\right) = f\,\overset{p}{T}_{\mu\nu}\,, \tag{21}$$

or

$$\Box\psi_{\mu\nu} = f\left(\overset{p}{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overset{p}{T}\right), \tag{22}$$

with

$$\overset{p}{T} = \eta^{\mu\nu}\overset{p}{T}_{\mu\nu}\,.$$

The solution of Eq. (22), with the usual boundary conditions, consistent with the Hilbert condition, Eq.(17), is:

$$\psi_{\mu\nu}(x) = \frac{f}{4\pi}\int d^4x'\,G_{ret}(x,x')\left[\overset{p}{T}_{\mu\nu}(x') - \frac{1}{2}\eta_{\mu\nu}\overset{p}{T}(x)\right], \tag{23}$$

where $G_{ret}(x,x')$ is given by Eq.(8). Introducing (23) into Eq.(19), we obtain the following contribution for the action, after use of Eq.(20):

$$S^G_{int} = f\int\psi_{\mu\nu}(x)\overset{p}{T}{}^{\mu\nu}(x)\,d^4x = Gc^{-4}\iint d^4x\,d^4x'\,G_{ret}(x,x')\;\times$$

$$\times\left[\overset{p}{T}_{\mu\nu}(x)\,\overset{p}{T}{}^{\mu\nu}(x') - \frac{1}{2}\overset{p}{T}(x)\overset{p}{T}(x')\right], \tag{24}$$

where

$$\overset{p}{T}_{\mu\nu} = \overset{p}{T}{}^{(1)}_{\mu\nu} + \overset{p}{T}{}^{(2)}_{\mu\nu}, \tag{25}$$

the indices 1 and 2 denoting the corresponding particles. The last term, of Eq. (24), was divided by 2 because of the x,x' symmetry.

We have, at the order G^0:

$$\overset{p}{T}_{\mu\nu}(x) = m\gamma v_\mu v_\nu \delta(\vec{x}-\vec{r}(t)), \tag{26}$$

with

$$\gamma = (1 - \frac{\vec{v}^2}{c^2})^{-1/2}, \quad v^\mu = (c,\vec{v}). \tag{27}$$

Eq. (24) gives, therefore,

$$S^G_{int} = 2Gc^{-4}\iint dt dt' m_1 m_2 \gamma_1\gamma_2 \left[(v_1.v_2)^2 - \frac{1}{2} v_1^2 v_2^2\right] \times$$

$$\times G_{ret}(t-t', \vec{r}_1(t) - \vec{r}_2(t')). \tag{28}$$

It holds:

$$\gamma_1\gamma_2\left[(v_1.v_2)^2 - \frac{1}{2} v_1^2 v_2^2\right] = \frac{c^4}{2}\left[1 + \frac{3}{2}\frac{\vec{v}_1^2}{c^2} + \frac{3}{2}\frac{\vec{v}_2^2}{c^2} - \frac{4\vec{v}_1.\vec{v}_2}{c^2} + 0(c^{-4})\right]. \tag{29}$$

Introducing the first term of Eq.(9) into Eq.(28), and using Eq.(29), we obtain the following contribution for S^G_{int}, up to order c^{-2}:

$$G\int dt \frac{m_1 \, m_2}{|\vec{r}_1(t)-\vec{r}_2(t)|} \left[1 + \frac{3}{2}\frac{\vec{v}_1^2}{c^2} + \frac{3}{2}\frac{\vec{v}_2^2}{c^2} - \frac{4\vec{v}_1.\vec{v}_2}{c^2}\right]. \tag{30}$$

In a similar way as before, the second term of Eq. (9) when introduced into Eq. (28) gives a contribution of order c^{-3}, while the last term of Eq. (9), when substituted into Eq. (28), gives, by an argument similar to that used in the lart Section, the following confribution for S^G_{int}:

$$\frac{Gm_1m_2}{c^2}\int dt\left[-\frac{7}{2}\frac{\vec{v}_1.\vec{v}_2}{|\vec{r}_1-\vec{r}_2|}-\frac{(\vec{v}_1.\vec{n})(\vec{v}_2.\vec{n})}{|\vec{r}_1-\vec{r}_2|}\right] . \qquad (31)$$

Eqs. (30) and (31), together, reproduce the linear terms, in G, of V_{EIH}, in Eq. (4).

4. THE NONLINEAR PART OF THE EINSTEIN-INFELD-HOFFMANN LAGRANGIAN

Let us now discuss the last term of $(-V_{EIH})$, in Eq. (4), which is quadratic in G. In lowest order, the equation of the gravitational field produced by a particle was given by Eq. (21), namely,

$$\Box\left(\psi_{\mu\nu}-\frac{\eta_{\mu\nu}}{2}\psi\right)=f\overset{P}{T}_{\mu\nu} , \qquad (32)$$

where $\overset{P}{T}_{\mu\nu}$ is the energy-momentum tensor of the particle. We will now assume that the exact gravitational field equation has a form similar to (32), with $J_{\mu\nu}$ replacing $\overset{P}{T}_{\mu\nu}$, on the right hand side of Eq. (32). Here, $J_{\mu\nu}$ is the total energy-momentum tensor, particle plus gravitational field:

$$\Box\left(\psi_{\mu\nu}-\frac{\eta_{\mu\nu}}{2}\psi\right)=f J_{\mu\nu} , \qquad (33)$$

where the condition $\partial^\mu J_{\mu\nu}=0$ should be satisfied.

The zeroth order in f, for $J_{\mu\nu}$, is $\overset{P}{T}_{\mu\nu}$. In order to obtain, in a perturbative treatment, the next higher order term in f, for $J_{\mu\nu}$, we should add, to $\overset{P}{T}_{\mu\nu}$, the energy-momentum tensor of the gravitational field.

The canonical energy-momentum tensor which corresponds to Eq. (18) is given by:

$$\overset{c}{T}_{\mu\nu} = \frac{\partial L_0^G}{\partial\psi_{\alpha\beta,\mu}} \psi^{\alpha\beta}{}_{,\nu} - \eta_{\mu\nu} L_0^G = \psi_{\alpha\beta,\mu} \psi^{\alpha\beta}{}_{,\nu} - \frac{1}{2} \psi_{,\mu}\psi_{,\nu}$$

$$- \frac{1}{2} \eta_{\mu\nu}(\psi_{\alpha\beta,\lambda} \psi^{\alpha\beta,\lambda} - \frac{1}{2} \psi_{,\lambda}\psi^{,\lambda}) \ . \tag{34}$$

Both $\overset{P}{T}_{\mu\nu}$ and $\overset{C}{T}_{\mu\nu}$ are symmetric, but

$$(\overset{p}{T}{}^{\mu\nu} + \overset{c}{T}{}^{\mu\nu})_{,\mu} \neq 0 \tag{35}$$

In order that the corresponding divergence should vanish, we must add the "interaction tensor" [3]:

$$\overset{i}{T}_{\mu\nu} = - 2 f \overset{p}{T}_{\mu\alpha}\psi^{\alpha}{}_{\nu} , \tag{36}$$

where, up to order f^2, we have[8]

$$(\overset{p}{T}{}^{\mu\nu} + \overset{c}{T}{}^{\mu\nu} + \overset{i}{T}{}^{\mu\nu})_{,\mu} = 0 \ . \tag{37}$$

We can also add, to the above expression, the spin energy-mornentum tensor of the field, $\overset{s}{T}{}^{\mu\nu}$, which will symmetrize the expression inside the parenthesis of Eq. (37) (because $\overset{i}{T}{}^{\mu\nu}$ is not symmetric), and which is divergencelles. We add a symmetric term, $t^{\mu\nu}$, of the form[8]:

$$t^{\mu\nu} = M^{\mu\nu\rho\sigma}{}_{,\rho\sigma} , \tag{38}$$

which does not contribute neither to the energy-momentum tensor, nor to the total angular momentum, the following relation being satisfied:

$$t^{\mu\nu}{}_{,\mu} = 0 \ . \tag{39}$$

As $\overset{c}{T}^{\mu\nu}$, $\overset{s}{T}^{\mu\nu}$ and $\overset{i}{T}^{\mu\nu}$, to the order f^2, are quadratic functions of the fields $\psi_{\alpha\beta}$, with two derivative operations, we shall assume for $t^{\mu\nu}$ the same property and, therefore, $M^{\mu\nu\rho\sigma}$ will be constructed only with the help of quadratic expressions in $\psi_{\alpha\beta}$, without derivatives (general expressions are found in Ref.8).

We, therefore, take

$$J^{\mu\nu} = \overset{p}{T}^{\mu\nu} + \overset{c}{T}^{\mu\nu} + \overset{i}{T}^{\mu\nu} + \overset{s}{T}^{\mu\nu} + t^{\mu\nu}. \tag{40}$$

As we want to work in the lowest order in G^2, we shall consider a source particle of mass M, at rest at the origin. We, then, have:

$$\overset{p}{T}_{\mu\nu} = \begin{cases} Mc^2, & \text{for } \mu=\nu=0, \\ 0, & \text{otherwise}, \end{cases} \tag{41}$$

and the solution of Eq.(21) is[3] :

$$\psi_{\mu\nu}(\vec{x}) = \frac{f\,Mc^2}{8\pi|\vec{x}|}\,\delta_{\mu\nu} = \frac{1}{2}\,\Phi(\vec{x})\,\delta_{\mu\nu}, \tag{42}$$

with $\delta_{00} = \delta_{ii} = 1$.

In this case, it is convenient to introduce new fields[3] by

$$\phi_{\mu\nu} = \psi_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\psi, \tag{43}$$

which, because of Eq.(42), gives for $\phi_{\mu\nu}$, in lowest order in f,

$$\phi^{(1)}_{\mu\nu} = \frac{fMc^2}{4\pi|\vec{x}|}\,\delta_{0\nu}\delta_{0\mu} = \Phi(\vec{x})\,\delta_{0\nu}\delta_{0\mu}. \tag{44}$$

Eq.(33) can be rewritten in the form

$$\Box\phi_{\mu\nu} = f\,J_{\mu\nu} \tag{45}$$

or also as

$$\Box\left(\phi_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}\phi\right) = f\left(J_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}J\right) . \tag{46}$$

In first order in f, we have

$$\Box\left(\phi^{(1)}_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}\phi^{(1)}\right) = f\left(\overset{p}{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T\right), \tag{47}$$

and, writing

$$\phi_{\mu\nu} = \phi^{(1)}_{\mu\nu} + \phi^{(3)}_{\mu\nu} , \tag{48}$$

we have, because of Eqs. (46) and (47).

$$\Box\left(\phi^{(3)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\phi^{(3)}\right) = f\left(\overset{f}{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overset{f}{T}\right), \tag{49}$$

or

$$\Box\ \psi^{(3)}_{\mu\nu} = f\left(\overset{f}{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overset{f}{T}\right), \tag{50}$$

where

$$\overset{f}{T}_{\mu\nu} = J_{\mu\nu} - \overset{p}{T}_{\mu\nu} . \tag{51}$$

As $\overset{f}{T}_{\mu\nu}$ is quadratic in the field $\psi_{\alpha\beta}$ (and in the order we are working on, we can substitute, in Eq.(36), $f\overset{p}{T}_{\mu\alpha}$ by $\Box\left(\psi_{\mu\alpha} - \frac{\eta_{\mu\alpha}}{2}\psi\right)$), it so happens that if we add, to $L^G_0 + L^G_{int}$, a term $F^{(3)}$ (cubic in the $\psi_{\alpha\beta}$ fields, and which we assume to contain two derivative operations only), then the Eqs. (50) are the Euler equations for the total Lagrangian $L^G_0 + L^G_{int} + F^{(3)}$. (For more details, see Ref.8).

Therefore, the Euler equations read now as:

$$\Box\left(\psi_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}\psi\right) = f\,\overset{p}{T}_{\mu\nu} - \frac{\delta F^{(3)}}{\delta\psi^{\mu\nu}}, \tag{52}$$

and, hence,

$$\Box \phi^{(3)}_{\mu\nu} = - \frac{\delta F^{(3)}}{\delta\psi^{\mu\nu}} = - \left\{ \frac{\delta F^{(3)}}{\delta\phi^{\mu\nu}} - \frac{1}{2} \eta_{\mu\nu}\eta^{\alpha\beta} \frac{\delta F^{(3)}}{\delta\phi^{\alpha\beta}} \right\} , \tag{53}$$

or

$$\Box (\phi^{(3)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu}\phi^{(3)}) = - \frac{\delta F^{(3)}}{\delta \phi^{\mu\nu}} \ . \tag{54}$$

In a perturbative treatment, the most important term in $F^{(3)}$ has the form (with β a parameter to be determined):

$$F^{(3)} = f\beta \ \overset{(1)}{\phi}_{00} \ \overset{(1)}{\phi}_{00,k} \ \overset{(1)}{\phi}_{00,k} \ , \tag{55}$$

in the static approximation.

Then, Eq.(54) gives, for $\mu=\nu=0$,

$$\Box (\phi^{(3)}_{00} - \frac{1}{2} \phi^{(3)}) = - f\beta \left[\Phi_{,k}\Phi_{,k} + 2\Phi_{,kk}\Phi \right] , \tag{56}$$

after use of Eq. (44).

If we introduce, in $\overset{s}{T}_{\mu\nu}$ (as given in Ref.8), the static solution Eq. (42), we obtain:

$$\overset{s}{T}_{00} = \overset{s}{T}_{ii} = 0 , \tag{57}$$

while the use of Eqs. (34), (32) and (36) gives us

$$\overset{c}{T}_{00} + \overset{i}{T}_{00} = \Phi \ \Phi_{,kk} + \frac{1}{4} \Phi_{,k}\Phi_{,k} \ , \tag{58}$$

$$\overset{c}{T} + \overset{i}{T} = \Phi \ \Phi_{,kk} + \frac{1}{2} \Phi_{,k}\Phi_{,k} \ , \tag{59}$$

and, therefore,

$$(\overset{c}{T}_{00}+\overset{i}{T}_{00}) - \frac{1}{2}(\overset{c}{T}+\overset{i}{T}) = \frac{1}{2}\Phi\,\Phi_{,kk} \; . \qquad (60)$$

Let us note that the trace $\overset{c}{T}+\overset{i}{T} \neq 0$. It only vanishes for free gravitational fields.

Since $t_{00}-(1/2)t$ is quadratic in $\phi_{\alpha\beta}$, with second derivatives, and exhibiting rotational invariance, it has to be of the forrn

$$t_{00} - \frac{1}{2}t = \frac{1}{2}B\,\overset{(1)}{\phi}_{00}\overset{(1)}{\phi}_{00,kk} = B(\Phi\,\Phi_{,kk} + \Phi_{,k}\Phi_{,k}) \; , \qquad (61)$$

where B is a constant to be deterrnined.

Introducing Eqs.(60) + Eq.(61) into Eq.(49) (for $\mu=\nu=0$), we obtain

$$\Box\,(\overset{(3)}{\phi}_{00} - \frac{1}{2}\overset{(3)}{\phi}) = f\left[(B+\frac{1}{2})\Phi\,\Phi_{,kk} + B\,\Phi_{,k}\Phi_{,k}\right], \qquad (62)$$

which, cornpared with Eq. (56) , gives

$$B = -\beta = \frac{1}{2} \; . \qquad (63)$$

Therefore, Eq.(56) gives the solution

$$f\,\overset{(3)}{\psi}_{00}(\vec{x}) = \frac{G\delta M}{r} - \frac{G^2M^2}{c^4r^2} \; , \qquad (64)$$

with

$$\delta M = -\frac{2GM^2}{c^4\varepsilon} \; , \qquad (65)$$

where $\varepsilon\to 0+$. The first term of Eq. (64) can be absorbed in the Newtonian potential, and, therefore,

$$f\,\psi^{(3)}_{00}(\vec{x}) = -\frac{G^2M^2}{c^4r^2}, \qquad (66)$$

$r=|\vec{x}|$. If we take the corresponding line element in the static approximation $(\vec{v}=0)$, we have[3], then,

$$ds^2 = g_{00}dx_0^2 = c^2(1 - 2f\,\psi^{(1)}_{00} - 2f\,\psi^{(3)}_{00})dt^2, \qquad (67)$$

and, therefore, by using Eqs. (42) and (66), we obtain for the action corresponding to a particle of mass m, in the gravitational field produced by a mass M, at a distance r, the expression

$$S_m = -mc\int ds = -mc^2\int\left(1 - \frac{2GM}{c^2r} + \frac{2G^2M^2}{c^4r^2}\right)^{1/2}dt, \qquad (68)$$

or

$$S_m = -mc^2\int\left(1 - \frac{GM}{c^2r} + \frac{G^2M^2}{2c^4r^2} + \ldots\right)dt. \qquad (69)$$

Similarly, the particle of mass M, in the field of a particle of mass m, is described by the action (in the static approximation):

$$S_M = -Mc^2\int\left(1 - \frac{Gm}{c^2r} + \frac{G^2m^2}{2c^4r^2} + \ldots\right)dt. \qquad (70)$$

Denoting, now, m and M by m_1 and m_2, respectively, the force which acts on the particle m_1, coming from the nonlinear term of Eq. (69), is

$$\vec{F}_1 = \frac{\partial}{\partial\vec{r}}\left(-\frac{G^2m_1m_2^2}{2c^2|\vec{r}-\vec{r}_2|^2}\right)_{\vec{r}=\vec{r}_1}, \qquad (70')$$

while the corresponding force, on particle m_2, is, because of Eq. (70), given by

$$\vec{F}_2 = \frac{\partial}{\partial \vec{r}} \left(- \frac{G^2 m_2^2 m_1}{2c^2 |\vec{r}_1 - \vec{r}|^2} \right)_{\vec{r}=\vec{r}_2} . \qquad (70'')$$

It is transparent that the last term of Eq. (4) gives rise to the same forces.

Let us observe that, it follows, from Eqs. (55) and (63),

$$\int d\vec{x}\, F^{(3)}(\vec{x}) = -\frac{f}{2} \int d\vec{x} \phi_{00}^{(1)} \vec{\nabla}\phi_{00}^{(1)} \cdot \vec{\nabla}\phi_{00}^{(1)} = -\frac{f}{4} \int d\vec{x}\, \vec{\nabla}(\phi_{00}^{(1)})^2 \cdot \vec{\nabla}\phi_{00}^{(1)} =$$

$$= \frac{f}{4} \int d\vec{x}\, (\phi_{00}^{(1)}(\vec{x}))^2 \; \nabla^2 \phi_{00}^{(1)}(\vec{x}) , \qquad (71)$$

which upon use of Eq. (44), for two particles with masses m_1 and m_2, a distance r apart, namely,

$$\phi_{00}^{(1)}(\vec{x}) = \frac{fc^2}{4\pi} \left(\frac{m_1}{|\vec{x}|} + \frac{m_2}{|\vec{x}-\vec{r}|} \right) , \qquad (72)$$

gives rise to

$$\int d\vec{x}\, F^{(3)}(\vec{x}) = - \frac{f^4 c^6}{4(4\pi)^2} \int d\vec{x} \left(\frac{m_1}{|\vec{x}|} + \frac{m_2}{|\vec{x}-\vec{r}|} \right)^2 \left(m_1 \delta(\vec{x}) + m_2 \delta(\vec{x}-\vec{r}) \right) ,$$

or

$$\int d\vec{x}\, F^{(3)}(\vec{x}) = - \frac{G^2 m_1 m_2 (m_1 + m_2)}{c^2 r^2} , \qquad (73)$$

which is exactly twice the last term in Eq. (4).

In a similar way as before, if we substitute, into $f\psi_{\mu\nu}J^{\mu\nu}$, the solution of Eq. (33), with $J^{\mu\nu}$ the total energy-momentum tensor, we obtain the action which follows

$$Gc^{-4} \iint d^4x d^4x' \ G_{ret}(x,x') \left[J_{\mu\nu}(x) \ J^{\mu\nu}(x') - \frac{1}{2} J(x) J(x') \right], \quad (74a)$$

where we have, again, divided by 2 because of the symmetry in x,x'.

Writing $J_{\mu\nu} = \overset{p}{T}_{\mu\nu} + \overset{f}{T}_{\mu\nu}$, where $\overset{p}{T}_{\mu\nu}$ is given by Eq.(25), we obtain, for the nonlinear term, in the static limiê,

$$2Gc^{-4} \quad d^4x d^4x' \ G_{ret}(x,x') \ \overset{p}{T}_{00}(x) \left(\overset{f}{T}^{00}(x) - 1 \ \overset{f}{T}(x') \right) \quad . \quad (74b)$$

Now, $\overset{f}{T}^{00}(x') - (1/2)\overset{f}{T}(x')$ is the sum of Eqs.(60) and (61), with $B=1/2$. Introducing it into Eq.(74), we obtain, if we use Eq.(72), a result which differs from the last term of Eq.(4).

The'conclusion is that a consistent way for obtaining the nonlinear term of V_{EIH} is through the equations of motion (70') and (70'').

5. THE MIXED TERM

Let us now discuss the last term V_M given by Eq. (5), of the Lagrangian(4). As a source of the gravitational field $\psi_{\mu\nu}$, we shall also consider the electromagnetic energy-momentum tensor contribution to the right hãnd side of Eq. (33). The corresponding contribution to the gravitational field is given by Eq.(19), where, to $\overset{p}{T}_{\mu\nu}$, $\overset{e}{T}_{\mu\nu}$, the pertinent electromagnetic energy-momentum tensor, has to be added.

Therefore, we obtain the following contribution for the relevant part of the action:

$$S^e_{int} = f \int \psi_{\mu\nu}(x) \overset{e}{T}^{\mu\nu}(x) d^4x = \frac{f^2}{4\pi} \iint d^4x d^4x' \ G_{ret}(x,x') \quad \times$$

$$\times \left(\overset{p}{T}_{\mu\nu}(x) \overset{e}{T}^{\mu\nu}(x') - \frac{1}{2} \overset{p}{T}(x) \overset{e}{T}(x') \right) \quad .$$

In the static approximation, only the component $_e\overset{P}{T}_{00}(x)$ does contribute to Eq. (75), and since $\underset{e}{T}_{00}(\vec{x})=(1/8\pi)\vec{E}^2(\vec{x})$ and $T(\vec{x})=0$, we have, using only the first term of Eq. (9):

$$S^e_{int} = \frac{f^2c^2}{2(4\pi)^2}\iint \frac{d^4x d^4x'}{|\vec{x}-\vec{x}'|}\left\{m_1\delta(\vec{x}-\vec{r}_1)m_2\delta(\vec{x}-\vec{r}_2)\right\} \times$$

$$\times\left\{\vec{\nabla}\left(\frac{e_1}{|\vec{x}'-\vec{r}_1|}+\frac{e_2}{|\vec{x}'-\vec{r}_2|}\right)^2\right\} . \qquad (75)$$

integrating in $\vec{x}'$, t' and using Eq. (20), we obtain:

$$S^e_{int} = \frac{G}{4\pi c^2}\int dt d\vec{x}\left\{\frac{m_1}{|\vec{x}-\vec{r}_1|}+\frac{m_2}{|\vec{x}-\vec{r}_2|}\right\}\left\{\vec{\nabla}\left(\frac{e_1}{|\vec{x}-\vec{r}_1|}+\frac{e_2}{|\vec{x}-\vec{r}_2|}\right)\right\} \cdot$$

$$\cdot\left\{\vec{\nabla}\left(\frac{e_1}{|\vec{x}-\vec{r}_1|}+\frac{e_2}{|\vec{x}-\vec{r}_2|}\right)\right\} . \qquad (76)$$

Performing a partial integration, we can write

$$S^e_{int} = -\frac{G}{4\pi c^2}\int dt d\vec{x}\ \left\{ \frac{1}{2}\vec{\nabla}\left(\frac{m_1}{|\vec{x}-\vec{r}_1|}+\frac{m_2}{|\vec{x}-\vec{r}_2|}\right)\vec{\nabla}\left(\frac{e_1}{|\vec{x}-\vec{r}_1|}+\frac{e_2}{|\vec{x}-\vec{r}_2|}\right)^2\right.$$

$$\left.+\left(\frac{m_1}{|\vec{x}-\vec{r}_1|}+\frac{m_2}{|\vec{x}-\vec{r}_2|}\right)\left(\frac{e_1}{|\vec{x}-\vec{r}_1|}+\frac{e_2}{|\vec{x}-\vec{r}_2|}\right)\nabla^2\left(\frac{e_1}{|\vec{x}-\vec{r}_1|}+\frac{e_2}{|\vec{x}-\vec{r}_2|}\right)\right\} , \qquad (77)$$

and integrating, again by parts, the first term inside the curly brackets of Eq. (77), it follows

$$S^{e}_{int} = -\frac{G}{2c^2}\iint dt d\vec{x} \left\{ \left(\frac{e_1}{|\vec{x}-\vec{r}_1|} \quad \frac{e_2}{|\vec{x}-\vec{r}_2|}\right)^2 (m_1\delta(\vec{x}-\vec{r}_1) + m_2\delta(\vec{x}-\vec{r}_2)) \right.$$

$$\left. - \left(\frac{m_1}{|\vec{x}-\vec{r}_1|} + \frac{m_2}{|\vec{x}-\vec{r}_2|}\right)\left(\frac{e_1}{|\vec{x}-\vec{r}_1|} + \frac{e_2}{|\vec{x}-\vec{r}_2|}\right)(e_1\delta(\vec{x}-\vec{r}_1)+e_2\delta(\vec{x}-\vec{r}_2))\right\} \cdot \tag{78}$$

After integrating in $\vec{x}$, we obtain, apart from renormalization effects to the total energy, or to the Coulomb and Newtonian interactions, exactly the result expressed by Eq. (5).

We have, in this way, reproduced all the terms present in Bazánski's Lagrangian.

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APPENDIX

In this appendix, we will discuss, in more detail, the Feynman - Gupta approach, in order to derive both the term, $F^{(3)}$, appearing in Eq. (52), and which has been added to our Lagrangian density, and the corresponding expression for the gravitational field energy-momentum tensor, the source of the field, in order f^3.

Let us recall that our Lagrangian density was written in the form:

$$L = L_p + L^G_{int} + L^G_0 \, , \qquad (A.1)$$

where L_P represents the free part for the particles; $L^G_{int} = f\psi_{\mu\nu}\overset{p}{T}^{\mu\nu}$, L^G_0 was taken of the form

$$L^G_0 = \frac{1}{2}\,(\psi_{\mu\nu,\lambda}\psi^{\mu\nu,\lambda} - \frac{1}{2}\,\psi_{,\lambda}\,\psi^{,\lambda}) \, . \qquad (A.2)$$

The conserved energy-momentum tensor of the system is

$$T_{\mu\nu} = \overset{p}{T}_{\mu\nu} + \overset{c}{T}_{\mu\nu} + \overset{i}{T}_{\mu\nu} + \overset{s}{T}_{\mu\nu} \, , \qquad (A.3)$$

where $\overset{P}{T}_{\mu\nu}$ represents the energy-momentum of the particles; $\overset{C}{T}_{\mu\nu}$, the canonical energy-mornentum of the gravitational field; $\overset{i}{T}_{\mu\nu}$ equals $-2\overset{p}{T}_{\mu\alpha}\psi_\nu^{\ \alpha}$ and arises frorn the interaction; finally, $\overset{s}{T}_{\mu\nu}$ is the "spin part" contribution which arises from the spin-2 character of the gravitational field. We have also assumed the Hilbert condition, namely,

$$\partial^\mu(\psi_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}\psi) = 0 \, . \qquad (A.4)$$

Expression (A.3) satisfies $\partial^\mu T_{\mu\nu} = 0$, and is symmetrical, being therefore a serious candidate for the source of the gravitational field, in order f^3:

$$\Box(\psi_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}\psi) = f\,J_{\mu\nu} \, . \qquad (A.5)$$

As we shall see, it is incorrect to identify $J_{\mu\nu}$, of Eq.(A.5), with $T_{\mu\nu}$ of Eq.(A.3). First, we shall define the expression

$$\overset{f}{T}{}^{(1)}_{\mu\nu} = \overset{c}{T}_{\mu\nu} + \overset{i}{T}_{\mu\nu} + \overset{s}{T}_{\mu\nu} \, , \qquad (A.6)$$

which is given by[8]

$$\overset{f}{T}{}^{(1)}_{\mu\nu} = I_{,,} \quad \psi^{,\beta}_{,\beta} - (\psi_{\alpha\mu}\,\psi^{\alpha}{}_{\nu,\beta}{}^{,\beta} + \psi_{\alpha\nu}\,\psi^{\alpha}{}_{\mu,\beta}{}^{,\beta})$$

$$+ \quad \psi_{\alpha\beta}{}^{,\beta}\,(\psi_{\mu}{}^{\alpha}{}_{,\nu} + \psi_{\nu}{}^{\alpha}{}_{,\mu}) + \psi_{\alpha\beta}(\psi_{\mu}{}^{\alpha,\beta}{}_{,\nu} + \psi_{\nu}{}^{\alpha,\beta}{}_{,\mu})$$

$$- \quad (\psi_{\mu\alpha,\beta}\,\psi^{\alpha\beta}{}_{,\nu} + \psi_{\alpha\nu,\beta}\,\psi^{\alpha\beta}{}_{,\mu}) - (\psi_{\alpha\mu}\,\psi^{\alpha\beta}{}_{,\nu\beta} + \psi_{\alpha\nu}\,\psi^{\alpha\beta}{}_{,\mu\beta})$$

$$+ \quad \psi_{\alpha\beta,\mu}\,\psi^{\alpha\beta}{}_{,\nu} - \frac{1}{2}\,\psi_{,\mu}\,\psi_{,\nu} \quad \frac{1}{2}\,\eta_{\mu\nu}(\psi_{\alpha\beta,\lambda}\psi^{\alpha\beta,\lambda} - \frac{1}{2}\,\psi_{,\lambda}\,\psi^{,\lambda}) \; .$$

(A.7)

In $\overset{I}{T}_{\mu\nu}$, we have substituted $f\,\overset{P}{T}_{\mu\alpha}$ by $\Box(\psi_{\mu\alpha} - \frac{\eta_{\mu\alpha}}{2}\psi)$ which is allowed at this order.

More generally, we shall write, for $J_{\mu\nu}$, which is thesource of Eq. (A.5),

$$J_{\mu\nu} = T_{\mu\nu} + t_{\mu\nu} \; , \tag{A.8}$$

where $t_{\mu\nu}$ should satisfy: $\partial^{\mu}t_{\mu\nu} = 0$, $t_{\mu\nu} = t_{\nu\mu}$, and would not change the total angular momentum, as well. Therefore, it is of the form[a] :

$$t_{\mu\nu} = M_{\mu\nu\sigma\lambda}{}^{,\sigma\lambda} \tag{A.9}$$

with

$$M_{\mu\nu\rho\sigma} = \overline{M}_{\mu\nu\rho\sigma} - \overline{M}_{\mu\rho\nu\sigma} \; , \tag{A.10}$$

and[8]

$$\overline{M}_{\mu\nu\rho\sigma} = a\,\psi_{\mu\nu}\psi_{\rho\sigma} + b(\eta_{\mu\nu}\psi_{\lambda\rho}\psi^{\lambda}_{\sigma} + \eta_{\rho\sigma}\psi_{\lambda\mu}\psi^{\lambda}_{\nu})$$

$$+ c(\eta_{\mu\nu}\psi_{\rho\sigma}\psi + \eta_{\rho\sigma}\psi_{\mu\nu}\psi) + \eta_{\mu\nu}\eta_{\rho\sigma}(d\psi_{\alpha\beta}\psi^{\alpha\beta} + e\psi\psi) \, . \tag{A.11}$$

Following the Feynman-Gupta procedure, we add to (A.1) a term, $F^{(3)}$, of third order in $\psi_{\alpha\beta}$, and fourth order in f, which possesses two first derivatives. We have, in this way,

$$\Box\,(\psi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\psi) = f\,\overset{p}{T}_{\mu\nu} - \frac{\delta F^{(3)}}{\delta\psi^{\mu\nu}} \,. \tag{A.12}$$

The general forn of $F^{(3)}$ can be taken as

$$F^{(3)} = - f \sum_{i=1}^{14} a_i X_i \,, \tag{A.13}$$

with

$$X_1 = \psi\,\psi_{,\rho}\psi^{\rho\sigma}{}_{,\sigma} \,, \quad X_2 = \psi\,\psi_{,\rho}\psi^{,\rho} \,, \quad X_3 = \psi\,\psi_{\alpha\rho}{}^{,\rho}\psi^{\alpha\sigma}{}_{,\sigma} \,,$$
$$X_4 = \psi_{\rho\lambda}\psi_{\beta\sigma}{}^{,\sigma}\psi^{\lambda\beta,\rho} \,, \quad X_5 = \psi\,\psi_{\alpha\beta,\rho}\psi^{\alpha\beta,\rho} \,, \quad X_6 = \psi\,\psi_{\alpha\sigma,\beta}\psi^{\alpha\beta,\sigma} \,,$$
$$X_7 = \psi_{\alpha\beta}\psi^{\alpha\beta,\rho}\psi_{\rho\sigma}{}^{,\sigma} \,, \quad X_8 = \psi_{\alpha\beta}\psi^{\alpha\beta,\rho}\psi_{,\rho} \,, \quad X_9 = \psi_{\rho\sigma}\psi_{\alpha\beta}{}^{,\rho}\psi^{\alpha\beta,\sigma} \,,$$
$$X_{10} = \psi_{\lambda\beta}\psi^{\lambda\rho}{}_{,\rho}\psi^{\beta\sigma}{}_{,\sigma} \,, \quad X_{11} = \psi_{\lambda\beta}\psi^{\lambda\gamma,\rho}\psi^{\beta}{}_{\gamma,\rho} \,, \quad X_{12} = \psi_{\lambda\beta}\psi^{\lambda\sigma,\rho}\psi^{\beta}{}_{\rho,\sigma} \,,$$
$$X_{13} = \psi_{\lambda\sigma}\psi^{\lambda\rho}{}_{,\rho}\psi^{,\sigma} \,, \quad X_{14} = \psi_{\rho\sigma}\psi^{,\rho}\psi^{,\sigma} \,. \tag{A.14}$$

We have, e.g.,

$$\psi_{\lambda\rho}\psi^{\lambda\sigma,\rho}\psi_{,\sigma} = X_3 - X_7 + X_{13} + \text{a four divergence,}$$

$$\psi_{\lambda\sigma}\psi^{\lambda\beta,\rho}\psi_{\beta\rho}{}^{,\sigma} = X_{10} - X_{12} + X_4 + \text{a four divergence.}$$

Comparison of Eqs.(A.5), (A.8) and (A.12) gives

$$- \frac{\delta F^{(3)}}{\delta\psi^{\mu\nu}} = f(J_{\mu\nu} - \tilde{T}_{\mu\nu}) = f(\overset{f}{\tau}{}^{(1)}_{\mu\nu} + t\mu\nu) \,, \tag{A.15}$$

with $\overset{f}{T}{}^{(1)}_{\mu\nu}$ given by Eq.(A.6), and $t_{\mu\nu}$ by Eqs.(A.9), (A.10), (A.11).

Using Eq.(A.4), Eqs.(A.15) can be written as linear cornbinations of 21 tensors $\chi^k_{\mu\nu}$:

$$\sum_{k=1}^{21} c_k(a_i)\chi^k_{\mu\nu} = \sum_{k=1}^{21} d_k(a,b,c,d,e)\chi^k_{\mu\nu}. \qquad \text{(A.16)}$$

The coefficients c_k and d_k are given in Table I. The 21 equations:

$$c_k(a_i) = d_k(a,b,c,d,e), \; k = 1,\ldots,21, \qquad \text{(A.17)}$$

can be solved, and give

$$\begin{aligned}
&a = -2, \quad b = -1, \quad c = +1, \quad d = +1, \quad e = -\frac{1}{2};\\
&a_4 = 4, \quad a_5 = \frac{1}{2}, \quad a_6 = +1, \quad a_7 = -2, \quad a_8 = 2; \qquad \text{(A.18)}\\
&a_9 = -1, \quad a_{11} = -2, \quad a_{12} = -2;
\end{aligned}$$

$$4a_2 - a_3 = 0, \quad a_1 + a_3 = -1; \qquad \text{(A.19)}$$

$$a_{10} + a_{13} = 0, \quad a_{13} + a_{14} = 0. \qquad \text{(A. 20)}$$

As

$$\overset{f}{T}_{\mu\nu} + t_{\mu\nu} = \sum_{k=1}^{21} d_k(a,b,c,d,e)\chi^k_{\mu\nu},$$

we obtain, making use of Eq. (A.18) and Table I:

$$\overset{f}{T}_{\mu\nu} = \overset{f(1)}{T}_{\mu\nu} + t_{\mu\nu} = -\,2\psi_{\mu\nu,\rho\sigma}\psi^{\rho\sigma} - \psi_{\alpha\beta,\mu}\psi^{\alpha\beta}{}_{,\nu}$$

$$+\ 2(\psi_{\mu\lambda,\nu}{}^{,\rho}\psi^{\lambda}_{\rho} + \psi_{\nu\lambda,\mu}{}^{,\rho}\psi^{\lambda}_{\rho}) + 2\psi_{\mu\alpha,\rho}\psi^{\rho,\alpha}_{\nu} + \psi_{\mu\nu}\psi^{,\sigma}_{,\sigma} - 2(\psi_{\mu\rho,\sigma}{}^{,\sigma}\psi^{\rho}_{\nu} + \psi_{\nu\rho,\sigma}{}^{,\sigma}\psi^{\rho}_{\mu})$$

$$-\ 2\psi_{\alpha\mu,\beta}\psi_{\nu}{}^{\alpha,\beta} + \psi_{\mu\nu,\sigma}{}^{,\sigma}\psi + 2\eta_{\mu\nu}\psi_{\lambda\rho,\sigma}{}^{,\sigma}\psi^{\lambda\rho}$$

$$-\ \eta_{\mu\nu}\psi_{\alpha\beta,\lambda}\psi^{\lambda\beta,\alpha} + \frac{3}{2}\,\eta_{\mu\nu}\psi_{\alpha\beta,\lambda}\psi^{\alpha\beta,\lambda}$$

$$-(\psi_{\mu\lambda,\nu}\psi^{,\lambda} + \psi_{\nu\lambda,\mu}\psi^{,\lambda}) - \frac{1}{2}\,\eta_{\mu\nu}\psi^{,\sigma}_{,\sigma}\psi - 2\psi_{\alpha\beta}\psi^{\alpha\beta}{}_{,\mu\nu}\ . \qquad (A.21)$$

It is simple to show that the trace of $\overset{f}{T}_{\mu\nu} + t_{\mu\nu}$ is zero for free fields, while, for a static source, the corresponding trace does not vanish.

It is not clear, to us, what are the rigorous arguments one could use in order to obtain, from an expression like Eq. (74a), the correct non-linear term which appears in Eq. (4).

TABLE I

k	$\chi^{k}_{\mu\nu}$	d_k	c_k
1	$\psi_{\mu\nu,\rho\sigma}\psi^{\rho\sigma}$	a	$2a_9$
2	$\psi_{\alpha\beta,\mu}\psi^{\alpha\beta}{}_{,\nu}$	$-2d + 1$	$a_7 - a_9$
3	$\psi_{\mu\nu,\rho}\psi^{,\rho}$	$a + 2c$	$2a_5 + a_9$
4	$\psi_{\mu\lambda,\nu\rho}\psi^{\lambda\rho} + (\mu\leftrightarrow\nu)$	$-b + 1$	$\frac{1}{2}a_4$
5	$\psi_{\mu\alpha,\rho}\psi_{\nu}{}^{\rho,\alpha}$	$-a$	$-a_{12}$
6	$\psi_{\alpha\mu,\beta}\psi^{\alpha\beta}{}_{,\nu} + (\mu\leftrightarrow\nu)$	$-b - 1$	$\frac{1}{2}a_4 + a_{12}$
7	$\psi_{\mu\nu}\psi^{,\sigma}_{,\sigma}$	$\frac{1}{2}a + c + 1$	$\frac{1}{2}a_7 + a_8$
8	$\psi_{\mu\rho,\sigma}{}^{,\sigma}\psi^{\rho}_{\nu} + (\mu\leftrightarrow\nu)$	$b - 1$	a_{11}

TABLE I (cont.)

k	$\chi^k_{\mu\nu}$	d_k	c_k
9	$\psi_{\alpha\mu,\beta}\psi_\nu^{\alpha,\beta}$	$2b$	a_{11}
10	$\psi_{,\mu}\psi_{,\nu}$	$-\frac{1}{4}a - c - 2e - \frac{1}{2}$	$a_1 + a_3 + \frac{1}{4}a_4 - \frac{1}{4}a_{10} - \frac{1}{2}a_{13} - a_{14}$
11	$\psi_{,\mu\nu}\psi$	$-c - 2e$	$a_1 + a_3 + a_6$
12	$\psi_{\mu\nu,\sigma}{}^{,\sigma}\psi$	c	$2a_5$
13	$\eta_{\mu\nu}\psi_{\lambda\rho,\sigma}{}^{,\sigma}\psi^{\lambda\rho}$	$2d$	a_8
14	$\eta_{\mu\nu}\psi_{,\lambda\rho}\psi^{\lambda\rho}$	$b + c$	$\frac{1}{2}a_{13} + 2a_{14}$
15	$\eta_{\nu\mu}\psi_{,\lambda}\psi^{,\lambda}$	$\frac{1}{4}b + c + 2e + \frac{1}{4}$	$a_2 - \frac{1}{4}a_3 + \frac{1}{4}a_{13} + a_{14}$
16	$\eta_{\nu\mu}\psi_{\alpha\beta,\lambda}\psi^{\lambda\beta,\alpha}$	b	$-a_6$
17	$\eta_{\nu\mu}\psi_{\alpha\beta,\lambda}\alpha^{\alpha\beta,\lambda}$	$2d - \frac{1}{2}$	$-a_5 + a_8$
18	$\psi_{,\rho\mu}\psi^\rho_\nu + (\mu\leftrightarrow\nu)$	$-\frac{1}{2}a - \frac{1}{2}b - c - \frac{1}{2}$	$\frac{1}{4}a_4 + \frac{1}{2}a_{10} + \frac{1}{2}a_{12} + \frac{1}{2}a_{13}$
19	$\psi_{\mu\lambda,\nu}\psi^{,\lambda} + (\mu\leftrightarrow\nu)$	$-\frac{1}{2}b - c + \frac{1}{2}$	$-\frac{1}{4}a_4 + a_6 + \frac{1}{2}a_{10} + \frac{1}{2}a_{13}$
20	$\eta_{\mu\nu}\psi^{,\sigma}{}_{,\sigma}\psi$	$\frac{1}{2}c + 2e$	$\frac{1}{2}a_1 + 2a_2$
21	$\psi_{\alpha\beta}\psi^{\alpha\beta}{}_{,\mu\nu}$	$-2d$	a_7

REFERENCES AND NOTES

1. S. Bazánski, Acta Phys. Polon. 15, 363(1956), and 16, 423(1957).
2. A. R. Khan and R.Ç. O'Connell, " Gravitational Analogue of Magnetic Forces"; Internal Report, International Centre for Theoretical Physics, Trieste (January 1976). To be published.
3. W.E. Thirring, Ann. of Physics, 16, 96(1961). For an extension in

which the spin is included, within the formalism of Weissenhoff and Raabe, see H. Fagundes and A.H. Zimerman: Nuovo Cimento 3B, 60(1971). A similar approach, within Moller's formulation of spin, is given by H. Fagundes and A. H. Zimermn in this issue of the Rev. Brasil. Fis.

4. J. Schwinger, Phys. Rev., 173, 1264(1968).

5. M. Fierz, Helv. Phys. Acta, 12, 3 (1939).

6. S. Gupta, Rev. Modern Phys., 29, 334(1957).

7. R. P. Feynrnan, *Lectures on Gravitation*; California Institute of Tecnology, Pasadena (1962-1963), unpublished.

8. H. Fagundes and A. H. Zimermn, Lett. a1 Nuovo Cimento 3, 593 (1972)--Erratum Lett. al Nuovo Cimento 8, 600 (1973).

9. H. Fagundes, Ph.D. Thesis presented at the *Instituto de Fisica Teórica*, São Paulo, Brazil (1974).

10. C. F. Cho and N. D. Hari Dass, Ann. of Phys., 96, 406(1976).