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On the Spin Interaction in the Lorentz Covariant Theory of Gravitation*

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The "spin" transformation properties for a free particle, in the Moller and Weissenhoff forrnulations are studied. The results of a previous paper by the present authors' are reobtained.

Estudam-se as propriedades de transformação do "spin", para uma partícula livre, nas formulações de Moller e Weissenhoff. Reobtêm-se os resultados de trabalho anterior, dos presentes autores', na formulação de Moller.

1. INTRODUCTION

Years ago, we discussed[1] the effects of spin in Thirring's[2] Lorentz covariant approach to gravitation. For this purpose, we used Weissenhoff's[3] treatnient in order to study the dynamics of spinning particles in special relativity. Subsequently, Wald[4] has considered the effects of spin in the motion of particles in the framework of general relativity (in lowest order of C), by using Moller's[5] formulation instead of weissenhoff's[3].

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Wald[4] criticizes Weissenhoff condition (which he calls Pirani condition),

$$S_{\mu\nu}U_{\nu} = 0 \ , \tag{1}$$

by stating that this equation does not determine the motion of the particle in a unique way. Indeed, condition (1) is compatible with helicoidal motions, with non constant spin. The study of these solutions shows[3] that they depend on an arbitrary constant, namely,

$$M^2 = P_{\mu}\ P^{\mu} \ . \tag{2}$$

This ambiguity comes from the fact that Weissenhoff[3] takes "any point" z of the particle as its "representative point". Wald[4] shows that this reasoning is not correct since a body with spin cannot be reduced to a point, and, therefore, the choice of z_{μ} cannot be arbítrary.

The correct representative point that leads to a unique solution of the problem - constant spin, and momentum, for a free particle - is Mtlller's center of gravity, X_{μ} (Ref.5). With this point, we have the condition:

$$S_{\mu\nu}P_{\nu} = 0 \ , \tag{3}$$

where P_{μ} is the four momentum of the "particle", and $S_{\mu\nu}$ its spin with respect to the center of gravity, X_{μ}. $S_{\mu\nu}$ and P_{μ} are both conserved (for a free particle, which is the case under consideration).

The purpose of this paper is, first, to give another argument in favor of Mtlller's description of spin, and, second to obtain the results of our previous paper[1] in the present context.

In 51, we discuss the transformation properties of $S_{\mu\nu}$ under Lorentz transformations. We show that, for a free particle, $S_{\mu\nu}$, in Møller's formulation, is a true second order tensor, while in the Weissenhoff it is not. In §2, we obtain the results of our previous paper[1] in the present context, i.e., in Mtlller's formulation of spin.

§1. Following Weissenhoff[3], let us define $S_{\mu\nu}$ in the proper frame of the particle. In other Lorentz frames, we would obtain $S'_{\mu\nu}$ by the usual rule for transformation of tensors. Let us, then, consider the problem from the point of view of a fluid with a spin density tensor $S_{\mu\nu\lambda}(t,\vec{x})$. The total spin is defined, in any Lorentz frame, by

$$S_{\mu\nu}(t) = \int S_{\mu\nu 0}(t,\vec{x})d\vec{x} \; . \tag{4}$$

In another Lorentz frame, $(t',\vec{x}')$, we have instead

$$S'_{\mu\nu}(t') = \int S'_{\mu\nu 0}(t',\vec{x}')d\vec{x}' \tag{5}$$

Now as $x_\mu = a_{\mu\nu}x'_\nu$, from (4) and (5) it follows:

$$S_{\mu\nu}(t) = a_{\mu\rho}a_{\nu\sigma}S'_{\rho\sigma}(t') - \int S_{\mu\nu\lambda,\lambda}(x)d^4x \; , \tag{6}$$

and, therefore, $S_{\mu\nu}$ will be a second order tensor only if $S_{\mu\nu\lambda,\lambda} = 0$. This last condition does not hold in Weissenhoff's[3] theory, as then

$$S_{\mu\nu\lambda} = x_\mu \overset{S}{T}_{\nu\lambda} - x_\nu \overset{S}{T}_{\mu\lambda} \; ,$$

and, therefore,

$$S_{\mu\nu\lambda,\lambda} = \overset{S}{T}_{\nu\mu} - \overset{S}{T}_{\mu\nu} \neq 0 \; ,$$

with $\overset{S}{T}_{\mu\lambda,\lambda} = 0$.

In Møller's[5] theory, the spin is conserved:

$$\frac{d}{dt} S_{\alpha\beta} = 0 \; , \tag{7}$$

and, therefore, the corresponding spin fluid equation is

$$\partial_\mu S_{\alpha\beta\mu} = 0 \; , \tag{8}$$

with

$$S_{\alpha\beta\mu} = s_{\alpha\beta}U_{\mu} = - S_{\beta\alpha\mu} ,$$

where U_{μ} is the four velocity and $s_{\alpha\beta}$ is the spin density).

Therefore, in Møller's formulation, the spin $S_{\mu\nu}$ is a true second order tensor.

§2. Let us now introduce

$$A_{\beta\alpha\lambda} = \frac{1}{2} (S_{\alpha\beta\lambda} + S_{\alpha\lambda\beta} + S_{\beta\lambda\alpha}) = -A_{\beta\lambda\alpha} . \tag{9}$$

We have:

$$S_{\alpha\beta\lambda} = A_{\beta\alpha\lambda} - A_{\alpha\beta\lambda} \tag{10}$$

and, therefore,

$$S_{\alpha\beta} = \int_{\Sigma} S_{\alpha\beta\lambda} d\sigma_{\lambda} = \int_{\Sigma} \left[A_{\beta\alpha\lambda} - A_{\alpha\beta\lambda}\right] d\sigma_{\lambda} =$$

$$= \int_{\Sigma} \{ \left[A_{\beta\alpha\lambda} - A_{\alpha\beta\lambda}\right] + \left[x_{\alpha}A_{\beta\lambda\rho} - x_{\beta}A_{\alpha\lambda\rho}\right]_{,\rho} \} d\sigma_{\lambda}$$

$$= \int_{\Sigma} \left[x_{\alpha}\overset{S}{T}_{\lambda\beta} - x_{\beta}\overset{S}{T}_{\lambda\alpha}\right] d\sigma_{\lambda} , \tag{11}$$

with

$$\overset{S}{T}_{\alpha\beta} = A_{\beta\alpha\lambda,\lambda} = \frac{1}{2}\left[S_{\alpha\lambda\beta} + S_{\beta\lambda\alpha}\right]_{,\lambda} = \overset{S}{T}_{\beta\alpha} . \tag{12}$$

It also holds that

$$\overset{S}{T}_{\alpha\beta,\alpha} = A_{\beta\alpha\lambda,\lambda\alpha} = 0 .$$

and, therefore, $S_{\alpha\beta}$, given by Eq. (11), is conserved as we already knew (cf. Eq.7).

Eq. (12) can be rewritten as

$$\overset{S}{T}_{\alpha\beta} = \frac{1}{2}\,(s_{\alpha\lambda}U_{\beta}+s_{\beta\lambda}U_{\alpha})_{,\lambda}. \tag{13}$$

This equation reduces to Eq.(17), of Ref.1, in the order worked out in that paper. Therefore, the results of gravitational spin-spin interaction, spin-orbit, and scattering of a light beam by a spinning particle, remain the same as in Ref.1.

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