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Nonrelativistic Ouantization of the Sine-Gordon Theory

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We perform the "first quantization" of the sine-Gordon Theory. We obtain the classical potential that describes the long distance interaction between solitons. The "first quantization" is achieved by inserting that potential in the Schrddinger equation. The nonrelativistic region of the DHN spectrum is easily reproduced by using our procedure. We also compute scattering amplitudes for non relativistic soliton-antisoliton and soliton-soliton scattering. We improve Faddeev's quantization rule. That improved version leads, in the non relativistic region, to the scattering amplitudes obtained in our approach.

Realizamos, neste trabalho, a "primeira quantização" da teoria sinusoidal de Gordon. Obtemos, assim o potencial clássico que descreve a interação, de longo alcance, entre solitons. O procedimento de "primeira quantização" é processado por inserção do potencial clássico na equação de Schrddinger. A região não relativística, do espectro de Dashen, Hasslacher e Neveu, é facilmente reobtida fazendo-se uso de nosso método. Calculamos, também, amplitudes de espalhamento para espalhamento soliton-antisoliton e soliton-soliton. Mais ainda, melhoramos a regra de quantização de Fadeev. Essa nova versão conduz, na região não relativística, às amplitudes de espalhamento obtidas utilizando nosso método.

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1. INTRODUCTION

Recently, much attention has been drawn to semiclassical methods in Quantum Field Theory[1-6]. That is the case for the W.K.B. approach. In this context, we would like to mention the pioneering work of Dashen, Hasslacher and Neveu[2] (DHN) which, by extending the W.K.B. method to Q.F.T., succeeded in getting many features of the quantized sine-Gordon[7] theory (SGT). Other interesting features of the quantized version of the SGT has been studied by other authors[3-6].

Although the technique employed by DHN was an approximate one, there exists a region in the coupling constants space ($\lambda/m^2 << 1$) where we would expect the method to be reliable. We will refer to that region as the DHN region.

When applied to the SGT, the approach described in Ref.2 might allow us to answer many questions concerning the spectrum of the theory, and scattering of particles, without going over the intermediate step of a "first quantization". The "first quantization" or quantization à la Schrödinger, of the SGT, is what we study in this paper. This is the kind of quantization which, as we recall, allows us to get sensible results concerning energy levels of many bound state systems - e.g., the positronium - by using the Schrödinger equation.

First of all, we show that, asymptotically (see below what we mean by it), the classical interaction between a nonrelativistic soliton-antisoliton (SA) pair, in the S.G.T. is well described by a classical potential of the form

$$V_{SA}(x) = -g\, M_c\, e^{-m|x|}, \tag{1.1}$$

where $|x|$ is the distance between soliton and the antisoliton; m, the mass of the elementary meson, whereas M_c is the classical soliton mass; and g, a dimensionless constant of order of magnitude one.

When we mentioned that the potential description works asymptotically, we meant that expression (1.1) describes the classical interaction between SA pairs, whenever

$$|x| \gg 1/m \tag{1.2a}$$

or

$$|t| \gg 1/m\, v_\infty \tag{1.2b}$$

v_∞ being the modulus of the interacting soliton velocity for much earlier and later times $(t \to \pm\infty)$. We say that we have the nonrelativistic domain of a given process when $v_\infty^i \ll 1$, for all particles involved (v_∞^i stands for the asymptotic velocity of the i-th particle).

It can be verified that the soliton-soliton (SS) potential $V_{SS}(x)$ is, in the Asymptotic Region, given by

$$V_{SS}(x) = -\, V_{SA}(x). \tag{1.3}$$

Such a property confirms what one would expect from an intuitive reasoning.

The classical "size" of the soliton is of the order $(1/m)$. Then, in the Asymptotic Region, we can consider the soliton as a point-like particle. We point out that a similar approximation is also employed in the treatment of the Hydrogen Atom by a Coulomb potential, where we consider the proton as point-like. We justify this by arguing that the proton radius is much smaller than the Bohr radius.

It is known[2] that, in the DHN Region, the meson, of mass m, has a twofold role: it is the fundamental meson of SGT and the lower bound state of the SA system, as well. We note that the potential given by (1-1) is a one-dimensional Yukawa potential associated with that meson. In this way, we conclude that this particle becomes the one which mediates the interaction between solitons at large distances.

Once we have established that the classical interaction between solitons is determined, in the Asymptotic Region, by potential (1.1), we proceed to the non relativistic quantization. That quantization amounts to inserting the potential into the Schrödinger Equation. Fortunately, the Schrödinger equation, for the potential (1.1), is soluble, and as a consequence it is straightforward to get the binding energies, and scattering amplitudes, of the SA and SS systems.

Before proceeding, we would like to discuss the conditions under which we shall expect the method employed here to be a reliable one. Conserning the non relativistic approximation, we feel tempted to say that this approach is valid whenever

$$|E| << M, \tag{1.4a}$$

and

$$|V(x)| << M , \tag{1.4b}$$

where E is the energy of the state under consideration. Condition (1.4b) seems to be much too strong. If it were always necessary, we would not be able to understand the successful nonrelativistic description of the Hidrogen Atom. That is why we will addopt a weaker and more pragmatic condition

$$|<V(x)>| << M. \tag{1.5}$$

Another aspect of our approach which we would like to comment about refers to taking only the long range tail of the potential. As is well known, states corresponding to large wave lengths are not sensitive to the behavior of the potential at short distances. Such a wavelength can be obtained once we know the wave function of each state. Then, by computing the wave function, the energy and $<V>$, we will be able to check a posteriori the validity of the simplifications introduced in our scheme.

After elucidating these points, we will also understand in which region, in the DHN spectrum, we should look in order to compare with our results. That region will be the nonrelativistic limit of the DHN region. As expected, we achieved, in this part of the spectrum, a perfect agreement.

With regard to the scattering region, our results differs from those obtained by Jackiw and Woo[3]. One reason for such a discrepancy is that the approach used by those authors does not work in the neighborhood of the threshold, which is just the nonrelativistic region where our method is reliable.

Still concerning the scattering of solitons, we would like to mention that Faddeev's[5] rule for "quantizing" the S-matrix exhibits a small flaw. The S-matrix obtained by that procedure does not discriminate between even and odd parity states. In spite of that, such a rule can be improved in order to take into account parity. The scattering lengths, obtained by such an improved version, agree with the ones computed by us, in the DHN region.

This paper is organized as follows: the classical SGT is presented in Section 2, whereas some of the DHN results are presented in Section 3. In Section 4, we obtain the asymptotic potential. Section 5 is devoted to the calculation of bound state energies and scattering amplitudes. We conclude this paper with a section reserved to conclusions, and two appendices which complement some parts of the text.

2. CLASSICAL SINE-GORDON THEORY

We shall present in this Section a summary of the classical SGT[7]. This two dimensional model is described by a Lagrangian density

$$L(x,t) = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right] + \frac{m^4}{\lambda}\left[\cos\left(\frac{\lambda^{1/2}}{m}\phi\right) - 1\right] \quad . \tag{2.1}$$

By minimizing the action corresponding to (2.1), we obtain the sine-Gordon Equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi + \frac{m^3}{\lambda^{1/2}} \sin\left(\frac{\lambda^{1/2}}{m}\phi\right) = 0 \quad . \tag{2.2}$$

From (2.1) and (2.2), one can easily see that, when $\lambda=0$, the SGT is a field theory of a free scalar meson of mass m.

Before presenting the solutions of (2.2), relevant for our considerations, in this paper, it is convenient to change the variables x and t into dimensioneless ones. That can be achieved by defining

$$\begin{aligned} x' &= m\, x, \\ t' &= m\, t, \end{aligned} \tag{2.3}$$

and

$$\phi'(x',t') = \frac{\lambda^{1/2}}{m}\, \phi(x,t) \; . \tag{2.4}$$

Now we proceed exhibiting some solutions of (2.2). A whole set of solutions can be obtained by making use of the "Backlund Transformation" [*]. The procedure works as follows: suppose ψ_0 is a solution of the sine-Gordon equation written in terms of light cone variables $\sigma = (x'+t')/2$ and $\rho = (x'-t')/2$. Then, another solution ψ_1 can be generated by plugging ψ_0 into the "Backlund Transformation"

$$\begin{aligned} \frac{1}{2}\, \frac{\partial}{\partial\sigma}\, (\psi_1-\psi_0) &= a \sin \frac{\psi_1+\psi_0}{2} \, , \\ \frac{1}{2}\, \frac{\partial}{\partial\rho}\, (\psi_1+\psi_0) &= \frac{1}{a} \sin \frac{\psi_1-\psi_0}{2} \end{aligned} \tag{2.5}$$

The "vacuum" $\psi_0=0$ is an obvious solution of (2.2). From it, the procedure sketched above leads to the socalled soliton solution:

$$\phi_S^{u}(x',t') = 4 \tan^{-1} \exp\left[\frac{x' - ut'}{(1 - u^2)^{1/2}} \right], \tag{2.6}$$

where u stands for the soliton velocity. In Fig. (1.a), we sketch the function that represents the soliton. Other solutions, on which we will be interested, are the ones corresponding to the two soliton scattering. They can be generated by making $\psi_0 = \phi_S^u$. We shall get

$$\phi_{SS}^{u}(x',t') = 4 \tan^{-1} \left[\frac{\sinh\{ut'/(1 - u^2)^{1/2}\}}{u \cosh\{x'/(1 - u^2)^{1/2}\}} \right] \; . \tag{2.7}$$

We note that (2.7) describes the SS scattering in the frame of the center of mass of the pair, and u is the modulus of the velocity of each particle, when $|t| \to \infty$. The solution corresponding to SA scattering is

$$\phi_{SA}^{'u}(x',t') = 4 \tan^{-1}\left[\frac{u \sinh\{x'/(1-u^2)^{1/2}\}}{\cosh\{u\ t'/(1-u^2)^{1/2}\}}\right]. \qquad (2.8)$$

We shall also mention the "Breather" solutions. The simplest of thern i's the doublet one, which can be obtained from (2-8) by making the substitution $u \to i\ v$. The solutions so obtained correspond to an SA bound state.

An interesting feature exhibited by these classical solutions is that they describe extended objects. That property can be verified by looking at the energy density of some solutions. Following the usual classical treatment, we can associate, to each solution ϕ_α, an energy density given by

$$H_\alpha = \left(\frac{\partial\phi_\alpha}{\partial t}\right)^2 - L\ (\phi_\alpha). \qquad (2.9)$$

The soliton, for instance, is a block of energy which moves with velocity u, without deformation (see Fig. (1.b)).The total soliton energy, in its rest frame, will be

$$M_c = \frac{8\ m^3}{\lambda}, \qquad (2.10)$$

which is interpreted as the classical mass of the soliton.

For a free solíton, we can determine position and velocity of its center of mass. No uncertainty at all arises from simultaneous measurements of physical quantities (as expected from a classical description]. The "first quantization", which is performed in Section 5, should implement the uncertainty principle.

3. THE DHN TREATMENT

Quantum corrections to a classical theory can be obtained, within Feynrnan's Path Integral framework, by expanding the action functional around classical solutions. In particular, when we take into account fluctuations up to quadratic terms, that procedure is equivalent to the usual W.K.B. approach [1,2].

Dashen et al.[2] succeeded in applying the W.K.B. rnethod to the SGT. We would like to exhibit some of their results.

The quantum correction to the soliton mass (represented, from now on, as M) is the following one:

$$\begin{aligned} M &= M_c - \frac{m}{\pi} + 0(\lambda) = \frac{8m^3}{\lambda} - \frac{m}{\pi} + 0(\lambda) \\ &= \frac{8m}{\gamma} + 0(\gamma) , \end{aligned} \tag{3.1}$$

where M_c is the classical soliton mass, and γ is given by

$$\gamma = \frac{\lambda/m^2}{1 - (\lambda/8\pi m^2)} \tag{3.2}$$

The spectrum of bound states of the SA system, that was obtained from a kind of "Bohr-Sommerfeld quantization rule" [2], is

$$E_N = \frac{16m}{\gamma} \sin\left(\frac{n\gamma}{16}\right) , \tag{3.3}$$

where $n = 1, 2, \ldots < \frac{8N}{\gamma}$. The region defined by

$$\frac{\lambda}{m^2} \simeq \frac{m}{M} \simeq \frac{\gamma}{8} << 1 , \tag{3.4}$$

we shall refer to as the DHN Region. In there, we should expect DHN results to be reliable. From now on, we shall assume that the parameters of the theory satisfy condition (3.4). From (3.3) and (3.4), we can see that the lowest bound state energy is [2,6]

$$E_1 \simeq m \, , \tag{3.5}$$

i.e., the meson of mass m is simultaneously the "elementary particle" of the theory, as well as the lower bound state of the SA system.

It will be convenient, for our purposes, to make an inverse ordering of the energy levels, i.e., to start ordering from the one at the top of the spectrum (3.3). That can be achieved in a very simple way. If $N_{\max}$ is the total number of bound states, there exists an ε satisfying

$$0 \leq \varepsilon < 1 \, , \tag{3.6}$$

such that

$$(N_{\max} + \varepsilon) = \frac{m}{M} = \pi \, . \tag{3.7}$$

Now we define, p, in the following way:

$$p = N_{\max} - n \, . \tag{3.8}$$

Then, if we obtain n in terms of p and ε (by means of (3.7) and (3.8)), after substituting it into (3.3), we get the following dependence, of the total energy, on p:

$$E^p_{\text{Total}} = 2M \cos\left[\frac{m}{M}(\varepsilon + p)\right] . \tag{3.9}$$

Now, by inspection of (3.9), we see that, by varying p, we have an ordering from the top of the spectrum to the bottom, or, in other words, $p=0$ corresponds to the highest binding energy, $p=1$ is the one just below, and so on. In the DHN Region, we can, for p small enough, $p \ll (\frac{M}{m})$, expand the right hand side of (3.9), obtaining

$$E^p_{\text{Total}} \simeq 2M - \frac{m^2}{4M}(p+\varepsilon)^2 \, . \tag{3.10}$$

Expression (3.10) is a very convenient one in order to compare with some of our results. The range of values of p, for which (3.10) is a good approximation for the spectrum, corresponds to the non relativistic domain.

4. THE POTENTIAL

In this Section, we shall study some aspects of the Soliton-Antisoliton classical interaction. We will compute the potential, $V_{SA}(x)$, which describes the asymptotic dynamics of the SA system for non relativistic processes. The $V_{SS}(x)$ potential, corresponding to the interaction between two solitons, can be computed by an analogous procedure but the calculations will not be presented here. We would like to report our finding which is expressed by (1.3).

We shall search for classical solutions, of the sine-Gordon equation, describing non relativistic SA scattering. In order to fix our conventions, we shall assume that the soliton is moving in the positive direction of the coordinate axis, and that the origin of the coordinate frame coincides with the position of the center of mass of the SA pair (see Fig.2).

The problem of determining the asymptotic potential can be solved if we know the velocity v of the center of mass of the soliton, in the asymptotic region. The argument goes as follows: we can always write

$$v = v_\infty + \Delta v \, , \tag{4.1}$$

and obviously, in the asymptotic region, Av is only a small correction to v_∞. On the other hand, energy conservation implies

$$M_c v_\infty^2 = M_c (v_\infty + \text{Av})^2 + V_{SA} \, . \tag{4.2}$$

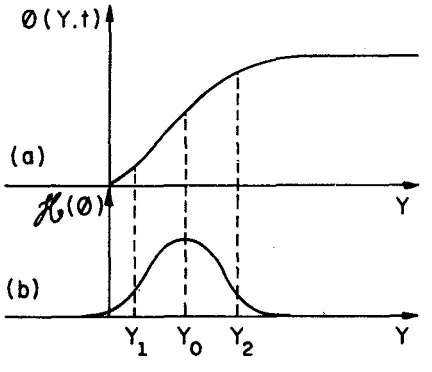

Fig. 1.(a) Free soliton; (b) Energy density associated to it. y_0 is the inflection point (which, here, is the center of mass); y_1 and y_2 are the points where the third derivative vanishes.

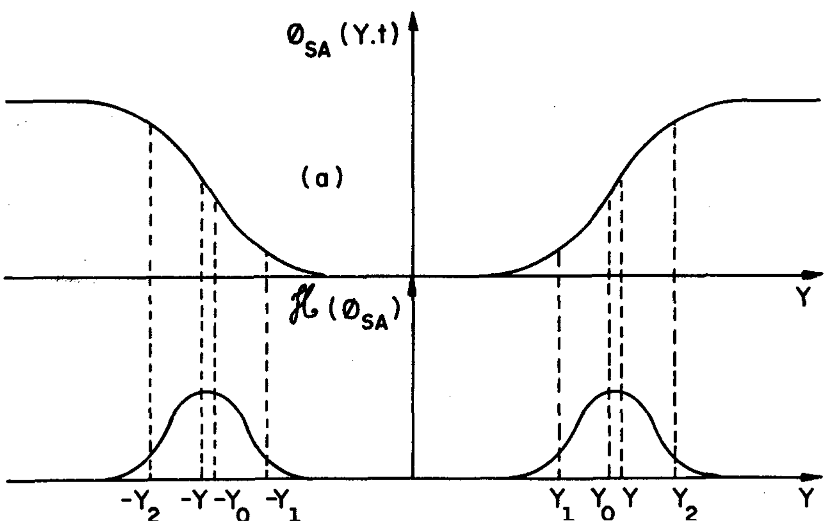

Fig.2. (a) Soliton-antisoliton pair interacting in the asymptotic region; (b) Energy density associated to them. $\pm\, y_0$ are the inflection points, $\pm y_1$ and $\pm y_2$ are the points where the third derivative vanishes; $\pm y_{cm}$ are tipical candidates to centers of mass of each particle.

From what has been said above, we conclude that, for $|x|>>1/m$, the potential is

$$V_{SA}(x) \simeq -2\, M_c v_\infty . \Delta v \ . \tag{4.3}$$

Expression (4.3) indicates how the knowledge of the asymptotic velocity leads to the asymptotic potential. With regard to that, we would like to make some comments. Although the position and velocity of the center of mass, of the free soliton, can be determined accurately, the same is not true when interacting. For example, when the separation distance $|x|$, between two solitons, is of the order of magnitude of the soliton size, i.e., $|x| \sim 1/m$, the two particles form a single block of energy, in such a way that it becomes impossible to say where is located the center of mass of each of them. Each particle loses, therefore, its identity.

On the other hand, if $|x|$ is large enough (when $t>>1/mv_\infty$), we shall observe two distinct blocks of energy - each of them representing a quasi-free particle (see Fig.2). Under these circumstances, we can determine, within a very good degree of accuracy, the position of the center of mass of each particle, and, therefore, to determine also its velocity. It is precisely for those values of $|x|$ that we calculate the potential.

A brief analysis of some features which characterize the free soliton, will shed some light on how to proceed in order to get information concerning the positions and velocities of the quasi-free solitons. Figure (l.a) represents a solution, $\phi(x,t)$, corresponding to a free soliton moving with velocity V. We shall be very much interested in three points of the soliton which somehow characterizes its position. These points labeled as y_0, y_1 and y_2 are defined as solutions to the equations

$$\frac{\partial^2}{\partial t^2}\, \phi(y,t)\bigg|_{y\,=\,y_0} = 0 \ , \tag{4.4}$$

and

$$\frac{\partial^3}{\partial t^3}\phi(y,t)\Big|_{y=y_1} = 0 = \frac{\partial^3}{\partial t^3}\phi(y,t)\Big]_{y=y_2} . \tag{4.5}$$

Fig. (1.b) exhibits the behavior of the energy density associated with the soliton. By comparison of Figs. (1.a) and (1.b), we conclude that y_0 (the inflection point) gives the position of the center of mass , while, loosely speaking, we can say that the soliton extends from y_1 to y_2. We shall note that, in the case of a free soliton, all its points move with the same speed, v.

In Fig.2, we represent an interacting SA pair, in the asymptotic region. From a close observation of that Figure, we can see that, when $|x|>>1/m$, the center of mass y_{cm} of the quasi-free soliton is somewhere between y_1 and y_2, and, more specifically, close to the inflection point y_0, Ref.8. From that, we should have

$$v \simeq v_0 , \tag{4.6-a}$$

and

$$v_1 \lesssim v \lesssim v_2 , \tag{4.6-b}$$

where v is the velocity of the center of mass of the soliton, and v_0, v_1 and v_2 the velocities of the points y_0, y_1 and y_2, respectively. The asymptotic velocities, namely, v_0, v_1 and v_2 are computed in Appendix I. The results we obtain are:

$$v_0 = v_\infty + \frac{1}{v_\infty} e^{-2m|y_0|} + 0\,(e^{-4m|y_0|}), \tag{4.7-a}$$

$$v_1 = v_\infty + \frac{(3-2\sqrt{2})}{v_\infty} e^{-2m|y_1|} + 0\,(e^{-4m|y_1|}), \tag{4.7-b}$$

and

$$v_2 = v_\infty + \frac{(3+2\sqrt{2})}{v_\infty} e^{-2m|y_2|} + 0\,(e^{-4m|y_2|}) . \tag{4.7-c}$$

From expressions (4.6) and (4.7), we conclude that the velocity of the center of mass, of the soliton, in the asymptotic region,can be written under the form

$$v \simeq v_\infty + \frac{g}{v_\infty} e^{-m|x|}, \tag{4.8}$$

where

$$g \sim 1 , \tag{4.9}$$

and we recall that $|x|$, in (4.8), is the distance between the centers of mass of soliton and antisoliton, i.e., $|x| = 2\, y_{cm}$.

By comparing (4.8) and (4.1), we infer that

$$\Delta v \simeq \frac{g}{v_\infty} e^{-m|x|} . \tag{4.10}$$

After substituting (4.10) into (4.3), we are led to the $V_{SA}(x)$ potential. It can be written under the form:

$$V_{SA}(x) = -2g\, M_c\, e^{-m|x|} . \tag{4.11}$$

This potential is responsible for the SA interaction, at long distances.

We shall add here that the relevant features, of the nonrelativistic bound state spectrum, is by no means dependent upon a specific value of g.

5. PHASE SHIFT AND ENERGY LEVELS

The quantization of the soliton-antisoliton system will be performed in this Section. As previously explained, that procedure is implemented, in our approach, by substituting the potential (4.11), in the Schrödinger equation

$$\frac{1}{M}\frac{d^2}{dx^2}\psi_E(x) + 2g\, M_c\, e^{-m|x|}\psi_E(x) = -E\, \psi_E(x) , \tag{5.1}$$

where $\psi_E(x)$ is the wave function describing a stationary state of energy E.

Our primary goal will be to compute the S-matrix elements associated with the potential (4.11). From $S(E)$, we shall compute the scattering amplitudes, whereas, by looking at the positions of its poles, the bound state energies will be determined.

Since the potential (4.11) is symmetric under the transformation $x \to -x$, thereexist solutions of equation (5.1) withwell defined parity. This means that, for a given energy E, we shall have an "even" matrix element $S^{\text{even}}(E)$, and an "odd" one, $S^{\text{odd}}(E)$, which are obtained from the asymptotic behavior $(x \to \pm \infty)$ of the wave functions:

$$\psi^{\text{even}}(x) \underset{x \to \infty}{\simeq} e^{-ikx} + S^{\text{even}}(E) . e^{ikx} ,$$

$$\psi^{\text{even}}(x) \underset{x \to -\infty}{\simeq} e^{ikx} + S^{\text{even}}(E) . e^{-ikx} , \tag{5.2-a}$$

and

$$\psi^{\text{odd}}(x) \underset{x \to \infty}{\simeq} e^{-ikx} - S^{\text{odd}}(E) . e^{ikx} ,$$

$$\psi^{\text{odd}}(x) \underset{x \to -\infty}{\simeq} -e^{-ikx} + S^{\text{odd}}(E) . e^{-ikx} , \tag{5.2.b}$$

where k is the magnitude of momentum of each particle, in the center-of--mass reference frame.

The SA forward scattering amplitude is

$$M_f(E) = \frac{1}{2} [S^{\text{even}}(E) + S^{\text{odd}}(E)] - 1 . \tag{5.3 }$$

We note that this is an invariant amplitude that can be analytically continued to the SS channel [4].

The SA backward scattering amplitude is given by

$$M_b(E) = \frac{}{2}\left[S^{\text{even}}(E) - S^{\text{odd}}(E)\right]. \qquad (5.4)$$

Denoting by $S_\nu(E)$ a generic matrix element, we define the. phase shift $\delta_\nu(E)$ as

$$\delta_\nu(E) = \frac{1}{2i}\,\ell n\, S_\nu(E)\ , \qquad (5.5)$$

while the scattering lengths a_ν are defined by the limit

$$a_\nu = \lim_{k \to 0_+} \frac{1}{k}\left[\delta_\nu(k) - \delta_\nu(0)\right]. \qquad (5.6)$$

For the discussion which will follow, we found convenient to define a function $\beta(E)$ as

$$\beta(E) = \frac{2}{m}\,(-ME)^{1/2}\ , \qquad (5.7)$$

and a constant A by

$$A = (8g)^{1/2}.\frac{(M_c\ M)^{1/2}}{m} \qquad (5.8)$$

The meaning of all constants which appear in (5.7) and (5.8) can be understood by taking notice of (5.1).

In Appendix II, we show that the matrix elements $S_\nu(E)$ are given by

$$S^{\text{even}}(E) = -\frac{J'_{-\beta}(A)\ \Gamma(1-\beta)}{J'_{\beta}(A)\ \Gamma(1+\beta)}\left(\frac{A}{2}\right)^{2\beta}\ , \qquad (5.9\text{-a})$$

and

$$S^{\text{odd}}(E) = \frac{J_{-\beta}(A)\ \Gamma(1-\beta)}{J_{\beta}(A)\ \Gamma(1+\beta)}\left(\frac{A}{2}\right)^{2\beta}\ , \qquad (5.9\text{-b})$$

where J_β is the Bessel function of order β, and J_β stands for its derivative.

In order to obtain the bound state energies, we will study the behavior of $S(E)$ for E < 0. In that region, $\beta(E)$ can be represented as [9]

$$\beta(E) \underset{E<0}{=} \alpha(E) = \frac{2}{m}(M|E|)^{1/2} \; . \tag{5.10}$$

From (5.9) and (5.10), we conclude that the binding energies will be given by the positions of the zeros of $J'_\alpha(A)$ and $J_\alpha(A)$. Since, in the DHN Region, A >> 1 (because there $(M/m)>>1$), we can make the following approximation:

$$J'_\alpha(A) \simeq \left(\frac{2}{\pi A}\right)^{1/2} \sin\theta(A,\alpha) \tag{5.11-a}$$

and

$$J_\alpha(A) \simeq -\left(\frac{2}{\pi A}\right)^{1/2} \cos\theta(A,\alpha) \; , \tag{5.11-b}$$

where $\theta(A,\alpha)$ is defined as

$$\theta(A,\alpha) = A - \frac{\alpha\pi}{2} - \frac{\pi}{4} \; . \tag{5.12}$$

From (5.9) and (5.11), we can see that the positions of the poles of $S^{\text{even}}(E)$, or $S^{\text{odd}}(E)$, are approximately given by the positions of the zeros of $\sin\theta$, or $\cos\theta$, respectively.

We can always write

$$A = \frac{\pi}{2}\left(N + \eta + \frac{1}{2}\right) \; , \tag{5.13}$$

where N is, by choice, an integer, and

$$0 \le \eta < 1 \tag{5.14}$$

By using (5.13) and (5.14), it is easy to see that, in the DHN Region, the binding energy spectrum will be given by

$$E_p = \frac{m^2}{4M}(p + \eta)^2 \; , \tag{5.15}$$

where p is an integer such that, if N is even (odd) the levels corresponding to $p = 0, 2, 4...$ will have even (odd) wave functions, and the levels corresponding to $p = 1, 3, 5...$ will have odd (even) wave functions.

By comparing (5.15) with (3.10), and identifying η with ϵ, (see also (5.14) and (3.6)), we verify that, in the nonrelativistic limit of the DHN Region, our spectrum coincides with that of DHN.

It can be explicitly verified that our results obey the criteria for the validity of our approximations [cf. (1.4-a) and (1.5)].

In the scattering region (E>0), $\beta(E)$ will be a pure imaginary number, i.e.[9]

$$\beta(E) = - ir , \tag{5.16}$$

where

$$r = \frac{2}{m} (ME)^{1/2} = \frac{2k}{m} \tag{5.17}$$

The S-matrix elements are

$$S^{\text{even}}(E) = - \frac{J'_{ir}(A)\ \Gamma(1+ir)}{J'_{-ir}(A)\Gamma(1-ir)} \left(\frac{A}{2}\right)^{-2ir} . \tag{5.18-a}$$

and

$$S^{\text{odd}}(E) = \frac{J_{ir}(A)\ \Gamma(1+ir)}{J_{-ir}(A)\Gamma(1-ir)} \left(\frac{A}{2}\right)^{-2ir} \tag{5.18-b}$$

By using (5.5) and (5.6), it is straightforward to compute phase shifts and scattering lengths. In the SA channel, we obtain

$$a_{\text{SA}}^{\text{even}} = -\frac{2}{m} \left[\ln\left(\frac{M}{m}\right) + \frac{\pi}{4}\text{cotg}\ \eta + \ln\left(2g\ \frac{M_c}{M}\right)^{1/2} + \gamma \right] \tag{5.19-a}$$

$$a_{SA}^{\text{odd}} = -\frac{2}{m}\left[\ell n\left(\frac{M}{m}\right) - \frac{\pi}{4}\,\mathrm{tg}\,\eta + \ell n\left(2g\frac{M_c}{M}\right)^{1/2} + \gamma\right] , \qquad (5.19\text{-b})$$

where γ is Euler's constant.

In an analogous way, we can compute the SS scattering lengths. If the solitons are fermions, as suggested in the literature, Refs.2, 5, 6, 12, Pauli's exclusion principle will imply that we need to take into account only states described by odd wave functions. Then, the SS scattering length will be

$$a_{SS}^{\text{odd}} = \frac{2}{m}\left[\ell n\left(\frac{M}{m}\right) + \ell n\left(2g\,\frac{M_c}{M}\right)^{1/2} + \gamma\right] . \qquad (5.20)$$

In the introduction, we have mentioned that Faddeev's[5] rule for "quantizing" the S-matrix [10] exhibits a little flaw. We will, now, clarify this point. It is known that parity is, tipically, a quantum concept, and, as a consequence, the classical S-matrix [10] does not discriminates between even and odd parity states. The flaw of Faddeev's rule lies in the fact that it extends this "blindness for parity" to the quantum S-matrix. Fortunately, it is not difficult to remedy this problem. Recalling that the parity, of the wave function of the n-th, (Ref.11), SA bound state, is given by $(-1)^{n+1}$, we can split Faddeev's quantum S-matrix into two parts: and S^{even}, that contains the poles of the even parity bound states, and an S^{odd} that contains the other set of poles .

Faddeev's quantum S-matrix can be written in the following form[10]

$$S^{F} = \prod_{n=1}^{N} Q_n , \qquad (5.21)$$

where Q_n is the factor which contains the pole of the n-th bound state. Note that, in the scattering regions, each Q_n is <u>per se</u> explicitly unitary. The above rnentioned splitting consists in defining

$$S^{\text{even}} = \prod_{\substack{n=1 \\ \text{odd}}}^{N} Q_n , \qquad (5.22\text{-a})$$

and

$$S^{\text{odd}} = \prod_{\substack{n=2 \\ \text{even}}}^{N} Q_n \ . \qquad (5\text{-}22\text{-b})$$

One observes that the improved quantization rule leads to unitary S-matrix elements, whereas the spectrum remains the same as that of Ref.5. It also leads to scattering amplitudes which agree with those obtained by us in the nonrelativistic limit of the DHN Region. [Note that, in the DHN Region, $\ell n(g \frac{M_c}{M})$ and γ can be neglected when compared with $\ell n(M/m)$].

6. CONCLUSION

By using a quite different approach from that employed by DHN, we succeeded in giving a nonrelativistic quantum treatment to soliton interactions, within the sine-Gordon Theory. Our method is essentially the wellknown nonrelativistic quantum mechanical approach. In order to do this, we had to compute, first, the potential which is responsible for the interactions between solitons. After that, we studied the nonrelativistic motion by substituting the potential into the Schrödinger equation. In this way, we were able to compute the soliton-antisoliton bound states, reproducing part of the DHN spectrum, and the SA and SS scattering amplitudes in the nonrelativistic region.

Coleman[12] has shown that the sine-Gordon Theory is equivalent to the massive Thirring Model, strongly suggesting that the soliton is the fermion of this model. On the other hand, it is wellknown that the massive Thirring Model is equivalent to the two-dimensional massive Vector Gluon Model, in the limit

$$\mu \to \infty \ , \quad e \to \infty \ , \qquad (6.1)$$

whith $e/\mu = g$ fixed,
where μ is the mass of the Vector Gluon, and e is the Fermion - Vector Gluon Coupling constant. From that equivalence, we should naively expect the fermion-fermion interaction potential be given by[13]

$$V(x) = \lim_{\mu\to\infty} \left[g\mu e^{-\mu|x|} \right] = 2g\ \delta(x), \tag{6.2}$$

where $g\mu \exp(-\mu|x|)$, in (6.2), is the Yukawa potential associated with the Vector Gluon.

The naive argument, presented just above cannot, of course, be true because the correct SS potential, given by (4.11), is rather different from (6.2). We conclude with the remark that, here, the dynamics exhibits a very interesting feature, namely, the fact that the lighest fermion-antifermion bound state happens to be the one responsible for the interaction between fermions, at long distances.

APPENDIX I

We compute here the asymptotic velocities v_i. In the nonrelativistic domain, ($v_\infty^2 << 1$), the SA solution can be approximated by the expression:

$$\phi_{SA}(y,t) = \frac{4}{\sqrt{\lambda}} \tan^{-1}\left[\frac{\sinh(v_\infty mt)}{v_\infty \cosh(mx)}\right]. \tag{A.1}$$

The positions of the inflection points $y_o(t)$ and $-y_o(t)$ are defined implicitly as solutions of (4.4). By solving (4.4), we conclude that

$$\cosh(my_o) = f(t)\left[1 + \frac{2}{f^2(t)}\right]^{1/2}, \tag{A.2}$$

where

$$f(t) = \frac{1}{v_\infty} \sinh(v_\infty mt). \tag{A.3}$$

On the other hand, the third derivative of ϕ_{SA} vanishes at the origin and at the points which we call $y_1(-y_1)$ and $y_2(-y_2)$. Then $y_1(t)$ and $y_2(t)$ are the solutions of equations (4.5). After plugging, into

(4.5), $\phi_{SA}(y,t)$, given by (A.1), we will get

$$\cosh(my_1) = f(t)\left[3-2\sqrt{2}\left(1+\frac{2}{f^2}+\frac{9}{8f^4}\right)^{1/2}\right]^{1/2}, \qquad (A.4\text{-}a)$$

and

$$\cosh(my_2) = f(t)\left[3+2\sqrt{2}\left(1+\frac{2}{f^2}+\frac{9}{8f^4}\right)^{1/2}\right]^{1/2}, \qquad (A.4\text{-}b)$$

where $f(t)$ was defined in (A.3). When $t \gg 1/mv_\infty$, we shall have (remember that $v_\infty^2 \ll 1$) :

$$\cosh(my_i) = \frac{a_i}{2v_\infty} e^{v_\infty mt}\left[1 - e^{-2v_\infty mt} + 0(e^{-4v_\infty mt})\right], \qquad (4.5)$$

where the coefficients a_i are given by

$$a_0 = 1 , \qquad (A.6\text{-}a)$$

$$a_1 = (3 - 2\sqrt{2})^{1/2} , \qquad (A.6\text{-}b)$$

$$a_2 = (3 + 2\sqrt{2})^{1/2} . \qquad (A.6.c)$$

From (A.5), we conclude that the asymptotic velocities v_i of each point y_i will be given by equations (4.7).

APPENDIX II

In this Appendix, we shall solve Schrödinger equation (5.1). Since β and A are given by expressions (5.7) and (5.8), respectively, that equation can be written as

$$\frac{1}{m^2}\frac{d^2}{dx^2}\psi + \frac{A^2}{4}e^{-m|x|}\psi = \frac{\beta^2}{4}\psi. \qquad (A.7)$$

We change the variable:

$$\xi = A\ e^{-\frac{m}{2}|x|} \tag{A.8}$$

In terms of ξ, Eq. (A.7) becomes

$$\xi^2 \frac{d^2}{d\xi^2}\psi + \xi \frac{d}{d\xi}\psi + (\xi^2 - \beta^2)\psi = 0\,. \tag{A.9}$$

Equation (A.9) can be easily recognized as the Bessel Equation. The even solutions (even under the change $x \to -x$) are given by

$$\psi_\beta^{\text{even}} = C\left[J'_{-\beta}(A)\ J_\beta(\xi) - J'_\beta(A) J_{-\beta}(\xi)\right]\ , \tag{A. 10-a}$$

where J_ρ is the Bessel function of order ρ, J'_ρ its derivative, and C a constant. The odd solutions can be written, for $x > 0$, as

$$\psi_\beta^{\text{odd}} = C'\left[J_{-\beta}(A)\ J_\beta(\xi) - J_\beta(A)\ J_{-\beta}(\xi)\right]\ . \tag{A. 10-b}$$

Since, in the limit $x \to \infty$, ξ goes to zero, we can make use, in this limit, of the following approximation for Bessel functions[14]:

$$J_\rho(\xi) \underset{\xi\to 0}{\sim} \frac{1}{\Gamma(1+\rho)} \left(\frac{1}{2}\,\xi\right)^\rho\ . \tag{A. 11}$$

From (A.10) and (A.11), it follows that, when $x \to \infty$, we have

$$\psi_\beta^{\text{even}} \underset{x\to\infty}{\simeq} C\left[\frac{J'_\beta(A)}{\Gamma(1-\beta)}\left(\frac{A}{2}\right)^{-\beta} e^{\frac{\beta m x}{2}} - \frac{J'_{-\beta}(A)}{\Gamma(1+\beta)}\left(\frac{A}{2}\right)^{\beta} e^{-\beta\frac{m x}{2}}\right]\ , \tag{A. 12-a}$$

and

$$\psi_\beta^{\text{odd}} \underset{x\to\infty}{\simeq} C'\left[\frac{J_\beta(A)}{\Gamma(1-\beta)}\left(\frac{A}{2}\right)^{-\beta} e^{\frac{\beta m x}{2}} - \frac{J_{-\beta}(A)}{\Gamma(1+\beta)}\left(\frac{A}{2}\right)^{\beta} e^{-\beta\frac{m x}{2}}\right]\ . \tag{A. 12-b}$$

Recalling that, in the scattering region, $\beta(E)$ is represented by (5.16), and taking in account our definitions (5.2), we can deduce (5.9) simply by suitably adjusting C and C'.

REFERENCES AND NOTES

1. R. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D10, 4114 (1974); ibid. D10, 4130(1974); ibid. D10, 4138(1974). J.M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D10, 2428(1974). J. Goldstone and R.Jackiw, Phys. Rev. D11, 1486(1975).
2. R. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D11, 3424(1975).
3. R. Jackiw and G. Woo, Phys. Rev. D (to be published).
4. Sidney Coleman, Appendix to ref. (3).
5. L.D. Faddeev, IAS preprint (April 1975).
6. R. Rajaraman, IAS preprint (May 1975).
7. For a review on the classical sine-Gordon theory, see A.C. Scott, F. Y.F. Chu and D.W. Mc Laughlin, Proc. IEEE 61, 1443(1973).

8. In the case of the interacting soliton, the definitions of y_0, y_1 and y_2 are also given by (4.4) and (4.5).
9. The choice of the square roots signs, in (5.10) and (5.16), agree with the convention that the scattering amplitude is the one obtained when one approaches the right-hand cut from above.
10. Faddeev's classical S-matrix is the one obtained in the free approximation. It assumes the form

$$S(E) = \exp \left\{ \frac{N}{\pi} \int_0^\pi d\theta \, \frac{\xi(E) + e^{i\theta}}{\xi(E)e^{i\theta} + 1} \right\} ,$$

where $\xi(E)$ is defined in Ref.5. Faddeev's quantization is implemented in the following way:

$$\frac{N}{\pi} \int_0^\pi d\theta \, \ell n \left(\frac{\xi + e^{i\theta}}{\xi \, e^{i\theta} + 1} \right) \to \to \sum_n^N \ell n \left(\frac{\xi + e^{i\theta_n}}{\xi \, e^{i\theta_n} + 1} \right) ,$$

where $\theta_n = \frac{\pi}{N} n$, $n = 1, 2, \ldots N$.

11. Here we are ordering the spectrum, starting from the lower bound state which corresponds to $n=1$.

12. Sidney Coleman, Phys. Rev. D11, 2088(1975).

13. In the final discussions of Ref.12, Coleman had assumed that the fermion-fermion potential is a δ-function.

14. Milton Abramowitz and Irene A. Stegun, Handbook of Mathematical Functions; Dover (1965), U.S.A. .