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Asymptotic Behaviour of Regge Trajectories

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An improvement in the demonstration of a lower bound for rising bosonic trajectories is presented.

Aperfeiçoa-se a demonstração de um limite inferior para trajetórias bosônicas crescentes.

1. The physics of strong interactions speaks the language of analytic functions. It is not a matter of using complex variables as a trick just to recover real variables at the end of the calculations: in strong interaction physics the singularities of the amplitudes that are located at finite points have'a direct physical interpretation. In order to get definite results, one needs still information about the behaviour of the amplitudes at infinity, that is, about the eventual singularities at the point at infinity. These are connected to the short-distance behaviour of the interactions, that is, to the unknown structure of the "elementary" particles. The trend has been to make hypotheses about the asymptotic behaviour, in energy,of tlie amplitudes. The weaker, the better. Recent studies of short-distance behaviour in quantum field theories[1] may be important in this connection.

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In some limited domains of hadron physics, simple, and, hopefully, general results concerning singularities at infinity have been found : among them the asymptotic behaviour of a rising bosonic Regge trajectory[2]. The problem is important, though the interest in Regge pole theory seems to be dwindling these days - a Regge trajectory, describing the rotational excitation spectrum of hadrons, speaks about their very structure, something that cannot be old-fashioned.

The possible asymptotic behaviours of rising Regge trajectories of the bosonic type can be found in a reasonably clean way[3]. At the moment, these results interest mainly people working on dual models[4], but insofar as predictions are made for the decay of very heavy bosons, less formal applications can be devised.

In a recent paper, Trushevsky[5] makes a detailed analysis of the problem and presents sound criticism to some previous papers. We intend here to follow his lead into putting the main results on a safer basis.

In Section 2, our approach to the problem is reviewed, together and in comparison with Trushevsky's. In Section 3, we present our main result: a proof of a lower bound that is simpler and avoids most criticisms . Trushevsky's ideas are reviewed in the Appendix.

2. To summarize our results, let us write the Regge trajectory as

$$\alpha(s) = - A(-s)^{\varepsilon} . \tag{1}$$

It is then possible to show that

$$(1/2) < \varepsilon < 1, \tag{2}$$

the extreme values being allowed provided we include logarithmic factors in (1), Ref. 6. The proof assumes that $\alpha(s)$ is a real analytic function with a branch point at some positive s_0, that $\mathrm{Im}\,\alpha(s) > 0$ for $s>s_0$ and that the width $\Gamma(s)$ of the resonances interpolated by the trajectories is given by

$$\Gamma(s) = \frac{\mathrm{Im}\ \alpha(s)}{s^{1/2}\mathrm{Re}\ \alpha'(s)} > 0 \ , \qquad (3)$$

the prime indicating differentiation with respect to s, Ref.7. Expression (3) is used only to get the lower bound at (2); there is no need of it to get the upper bound.

Trushevsky[5] made an important progress along these lines, replacing a somewhat formal step of our proof by a physical consideration. For the reader's convenience, Trushevsky's paper being not yet published, we reproduce its relevant parts in the Appendix. He derives a different expression for the width:

$$\frac{\Gamma}{M} = 2\ \tan\left[\frac{\pi}{2}\left(\frac{1}{\varepsilon} - 1\right)\right], \qquad (4)$$

(cf. the Appendix),
which follows from analyticity and the explicit time dependence of the wave function of an unstable state. Here M is the mass of the resonance. Both expressions (3) and (4) tend to the same limit $\varepsilon \to 1$, but they are otherwise different. Expression (3) depends strongly on the resonance being narrow, whereas, for expression (4) this requirement is only necessary insofar as the position of the resonance must be somehow identifiable. As a consequence, expression (3) should only be used for ε near 1 (Γ/M small), while, Trushevsky claims, expression (4), can be used for any ε. In our papers, (3) is used, as mentioned above, only to get the lower bound at (2), so that the criticism of Trushevsky (concerning the use of (3) for ε far from 1) does not affect our proof of the upper bound; as for the lower bound, we concede that we offered no rigorous a proof.

Trushevsky gets the lower bound imposing the condition that $M \geq 0$. Consider, however, his expression (4). As $\varepsilon \to (1/2)$, we have $\Gamma/M \to \infty$: the mass is totally undefined, and so its positiveness has no clear meaning. In fact, the very concept of resonance has no meaning in this limit.

3. In order to prove the lower bound at (2), for rising trajectories, without the use of quantities related to resonances we proceed as follows: assume $\alpha(s)$ to be real analytic on the s-plane, except for the cut starting at s_0, not to grow, at infinity, as fast as an exponential, and to be such that

$$\lim_{s\to-\infty} \frac{\alpha(s)}{(-s)^{\varepsilon}} = -A \, . \tag{5}$$

Then, using the Phragmén-Lindelöf[8] theorem, the limit at (5) is the same along every direction, and in particular,

$$\mathrm{Re}\,\alpha(s) \underset{s\to\infty}{\sim} -A\cos(\pi\,\varepsilon)s^{\varepsilon}, \tag{6}$$

$$\mathrm{Im}\,\alpha(s) \underset{s\to\infty}{\sim} A\sin(\pi\,\varepsilon)s^{\varepsilon} \, . \tag{7}$$

It is by requiring $\mathrm{Im}\,\alpha(s)$ to be positive (a consequence of unitarity) that we get $\varepsilon < 1$.

Now, if the trajectory is rising, that is, if $\mathrm{Re}\alpha(s)$ is a monotonic increasing function of s, as s grows along the real axis, and if at least one resonance seats on the trajectory, (so that we can say that, for some real value of the variable, $\mathrm{Re}\alpha(s)>0$), we must have $\mathrm{Re}\alpha(s)$ positive in (6). Combining both positivities it follows that

$$-\tan(\varepsilon\,\pi) > 0,$$

wherefrom the condition $\varepsilon > 1/2$ follows.

Some comments are in order. First, one could think that the hypothesis that, at least one resonance seats on the trajectory, makesthe result less general than the previous ones. As a matter of fact, this hipothesis was implicit in the previous treatment, as the positivity of the mass or of the width of a resonance, on the trajectory, was required. Second, rising trajectories are suggested by dual models[9], and by constructions such as Van Hove's model[10], so that this restriction, though strong, is not ad hoc.

APPENDIX

This is a brief summary of the parts of Trushevsky's work that are relevant to us. He writes the trajectory as

$$\alpha(s) = - A(-s)^{\varepsilon}, \tag{A1}$$

and, calling ϕ the argument of s, is led to

$$\alpha(s) = A|s|^{\varepsilon} \exp i \left[\pi-(\pi-\phi)\varepsilon\right] , \tag{A2}$$

the argument ϕ being zero at the upper edge of the cut. The trajectory can take positive integral values along the ray defined by the condition

$$\pi - (\pi-\phi) \varepsilon = 0 , \tag{A3}$$

that is, at

$$\phi = \pi (1-1/\varepsilon). \tag{A4}$$

The requirement that this ray be on the second sheet leads to $\varepsilon < 1$. Consider now a resonance located on the trajectory. The time dependente of its wave function is

$$\psi(t) \sim \exp-i\left[M-i(\Gamma/2)\right]t. \tag{A5}$$

It is natural to impose the condition $M>0$. From (A4), we have

$$\frac{\phi}{2} = \frac{\pi}{2} \left(1 - \frac{1}{\varepsilon} \right) ,$$

and this is the argument of $s^{1/2} = M-i(\Gamma/2)$. Hence, $M>0$ implies $\phi/2 > -\pi/2$, i.e.,

$$\varepsilon > \frac{1}{2} .$$

Also, from (A4) and (A5), it follows that

$$\Gamma/M = 2 \tan \left[\frac{\pi}{2} \left(\frac{1}{\varepsilon} - 1\right) \right],$$

Γ being, of course, the width of the resonance.

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