Revista Brasileira de Física, Vol. 6, Nº 1, 1976

Bound State Models and Duality in e+–e- Annihilation

M. KRAMMER* and P. LEAL FERREIRA
Instituto de Física Teórica, São Paulo SP

Recebido em 7 de Dezembro de 1975

The duality properties in e^+-e^- annihilation of Schrödinger bound state models are discussed. For any confining potential, regular at the origin, it is found that duality arises if the eigenvalue E is linearly related to the squared mass M^2 of the bound state. The nonconfining Coulomb case is also discussed.

Discutem-se as propriedades de dualidade na aniquilação e+-e- em modelos descritos por estados ligados de Schrödinger. Para qualquer potencial confinante, regular na origem, obter-se-á dualidade quando o autovalor E for linearmente relacionado ao quadrado da massa, M^2, do estado ligado. O caso não confinante, do tipo coulombiano, é também discutido.

1. DUALITY IN e+-e- ANNIHILATION

Electron-positron annihilation into hadrons[1] seem to have two regions:

i) the "resonance" region, dominated by prominent peaks like ρ^0, ω; ϕ and $J/\psi, \psi'$ (q^2 around 0.78^2; 1.02^2 and 3.1^2, 3.7^2 GeV^2 ,respectively),

* Permanent address: Deutsches Elektronensynchroton, DESY, Hamburg.

ii) the "scaling" region, in which the total cross section is smooth and decreasing as $1/q^2$ ($(1.7\ \mathrm{GeV})^2 \leq q^2 \leq (3.0\ \mathrm{GeV})^2$ and from $(4.5\ \mathrm{GeV})^2$ up to the end of present e^+-e^- storage ring energies).

In the resonance region the cross section is given by[2]

$$\sigma_{tot}(e^+e^-\to \text{hadrons}) = \frac{4\pi^2\alpha^2}{(q^2)^2} \sum_V \frac{M_V^4}{\gamma_V^2} \frac{M_V \Gamma_V^{tot}}{(q^2 - M_V^2)^2 + M_V^2 (\Gamma_V^{tot})^2} \tag{1}$$

M_V and Γ_V^{tot} denote the mass and total width of the resonance, $M_V^2/2\gamma_V$ is the vector-meson-photon coupling and is related to the leptonic decay width via

$$\Gamma(V \to \ell^+\ell^-) = \frac{\alpha^2}{12} \cdot \frac{1}{\gamma_V^2/4\pi} . \tag{1'}$$

The scaling region can be described by the quark-parton model[3], yielding

$$\sigma_{tot}(e^+ e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{3q^2} \sum_{\text{quarks}} Q_i^2 , \tag{2}$$

where Q_i is the charge of the i^{th} quark

or

$$R \equiv \frac{\sigma_{tot}(e^+e^- \to \text{hadrons})}{\sigma_{tot}(e^+e^- \to \mu^+ \mu^-)} = \sum_{\text{quarks}} Q_i^2 . \tag{2'}$$

If there is "duality", then the parton model cross section and the local average over resonances should be equal to each other, not only in some transition region but also at large q^2 and in the threshold region. This can be formulated as

$$\sum_V \frac{9\pi}{\alpha^2} \cdot \frac{M_V\ \Gamma(V\to\ell^+\ell^-)}{\Delta M_V^2} \equiv R(M_V^2) = \sum_i Q_i^2 , \tag{3}$$

where $V=\rho^0,\omega;\ \phi,\ J/\psi,...$type and i = p, r; λ, c,. ..type; while ΔM_V^2 is the difference of squared masses of two neigbouring resonances.

Under the assumption that ΔM_V^2 is a constant, the restriction on the M_V-dependence of Γ_V, following from Eq. (3), was first conjectured by Bramon, Etim and Greco[4]. It was shown by Böhm, Joos and Krammer[5] that in a field theoretical dynamical quark model with strongly bound heavy quarks $R(M_V^2)$ is a constant for large M_V^2. Phenomenological studies of the threshold regions have been made by Sakurai[6], Schildknecht and Steiner[7].

We want to investigate in this paper the scaling and duality properties of Schrödinger type (three-dimensional) quark-antiquark bound state models. In such a model $\Gamma(V\to\ell^+\ell^-)$ is expressed by

$$\Gamma(V\to\ell^+\ell^-) = \frac{16\pi\alpha^2}{3} <Q_V>^2 \cdot \frac{|\psi(0)|^2}{M_V^2} , \tag{4}$$

where $\psi(0)$ is the coordinate space Schrödinger wave function at zero distance and $<Q_V> \equiv \mathrm{Trace}\,(Q.V)$, with V being the internal symmetry part of the vector meson wave function. Since

$$\sum_{\text{quarks}} Q_i^2 = \sum_{\substack{\text{vector}\\ \text{mesons}}} <Q_V>^2 , \tag{5}$$

Eq. (3) can be put into the purely dynamical form

$$\frac{12\pi \cdot 4\pi|\psi_r(0)|^2}{M_r \cdot (M_{r+1}^2 - M_r^2)} \equiv R_{\text{red}}(r) = 1, \tag{6}$$

$r = 0, 1, 2, \ldots$.

Since $\psi(0)$ vanishes for angular momenta $\ell = 1,2,\ldots,$ only the S-wave bound states contribute to e+-e- annihilation. The index r denotes the radial quantum number, counting the nodes in the wave function.

We first discuss the "linear confinement" potential. which has been advocated very recently in connection with the new particles[8].

2. SCALING VIOLATION OF THE STANDARD LINEAR QUARK CONFINEMENT MODEL

In the nonrelativistic description of quark-antiquark bound states, one usually[8] starts from the Schrödinger equation

$$\Delta\psi + m(E-V)\psi = 0 \qquad (7)$$

and relates the eigenvalues E to the bound state masses M by

$$M = 2m + E \ . \qquad (8)$$

For a linear potential $V=V_0 + \lambda.R$ $(R^2 = \vec{x}^2)$, the Schrödinger equation has analytical solutions for S waves, which can be expressed in terms of an Airy function and its derivative[9]:

$$\psi_{\ell=0,r}(\vec{x}) = \frac{1}{\sqrt{4\pi}} \cdot \frac{\beta^{3/2}}{Ai'(a_{r+1})} \cdot \frac{Ai(a_{r+1} + \beta.R)}{\beta.R} \ , \qquad (9)$$

where $\beta=(m.\lambda)^{1/3}$. The a_n denote the position of the n^{th} zero of $Ai(x)$. An approximation to the a_n is given by[10]

$$a_n = -[\frac{3\pi}{8}(4n-1)]^{2/3} . (1 + o\ [(\frac{3\pi(4n-1)}{8})^{-2}]) \qquad (10)$$

which even reproduces the value for a_1 within 1%.

The eigenvalues E are given by

$$E_r = V_0 - (\frac{\lambda^2}{m})^{1/3} a_{r+1} \ . \qquad (11)$$

Note that $|\psi(0)|^2$ is independent of r :

$$4\pi\ |\psi_r(0)|^2 = \beta^3 \ . \qquad (12)$$

We now insert Eqs. (S), (11) and (12) into Eq.(6), obtaining

$$R_{\text{red}}(r) = 12\pi\, m\lambda\,(\lambda^2/m)^{-1/3}\,(1/2)\,\left[a_{r+1} - a_{r+2}\right]^{-1} \cdot$$

$$\cdot \left[2m+V_0 - (\lambda^2/m)^{1/3} a_{r+1}\right]^{-1} \left[2m+V_0 - (\lambda^2/m)^{1/3} \cdot (1/2)\,(a_{r+1}+a_{r+2})\right]^{-1} . \tag{13}$$

When we consider large r, we may neglect the constant $2m + V_0$, and from the asymptotic expression (10) we find

$$R_{\text{red}}(r) \xrightarrow[\text{large } r]{} \frac{\text{const}}{[\text{illegible}]} . \tag{14}$$

Thus, the linear confining potential with the identification $M = 2m + E$ does not scale for large bound state masses and, consequently, it cannot satisfy duality in the large q^2 region.

We are well aware of the criticism this statement may receive. We have used the model in the region of large excitations, where the naive Schrödinger description probably fails. However, from a mathematical standpoint, we may ask the question: how is scaling restored in this Schrödinger model ? Since we want to keep the probability interpretation of $\psi(\vec{x})$, it is obvious that we have to play with Eq. (8), i.e., the dependence of the bound state mass M on the eigenvalue E.

3. QUADRATIC MASS FORMULAE AND DUALITY

The requirement of scaling, at large q^2, of a Schrödinger bound state model implies that $R_{\text{red}}(r)$, defined in Eq. (6), should become a constant, which has to be equal to one, if duality is to be satisfied.

For the linear confining potential, discussed in the previous section, it is easily seen, that scaling is obtained, if we relate the eigenvalues E_r, Eq. (11), to the masses M_r via

$$M_r^2 = \text{const} + \text{const}' \cdot E_r . \tag{15}$$

So far this relation is *ad hoc*, and thus the entering of the quark mass and the potential parameters into Eq. (15) is not specified. We may, therefore, parametrize Eq. (15) as

$$M_r^2 = C_0^2 - \frac{c_1}{R_0^2} \cdot a_{r+1} \ , \qquad (15')$$

where C_0 has the dimension of a mass, and c_1 is a c-number ($\hbar = e = 1$) and R_0 denotes the ciiaracteristic range of the wave function. If we require duality to be satisfied for large r, we may determine c_1 from $R_{red}(r) = 1$. By simple algebra, one finds $c_1 = (12)^{2/3}$.

We have investigated other "confining" potentials which have analytical solutions, namely, the three-dimensional harmonic oscillator and the spherical bag. The surprising result is that *all these* models scale for large radial quantum numbers r, **iff** the identification Eq. (15) is made, i.e., if we linearly relate the Schrödinger eigenvalue and the bound state mass squared. A compilation of the normalized S-wave functions and the mass formulae is given in Table I.

Since we can hardly believe that this result is purely coincidental, we may look for a deeper reason and a generalization, independently of the specific form of the potential. In fact, the relation between M_r^2 and E_r arises because of the property

$$4\pi^2 |\psi_r(0)|^2 \underset{\text{large } r}{\simeq} (2\mu E_r/\hbar^2)^{1/2} \cdot \frac{d}{dr} (2\mu E_r/\hbar^2), \qquad (16)$$

which can be obtained[11] from the WKB solution

$$\psi_{\ell=0,r}(\vec{x}) = (4\pi)^{1/2} \frac{\chi_r(R)}{R} ,$$

$$\chi_r(R) = 2^{1/2} \left[\int_0^{R_{max}} d\tilde{R} \left[2\mu (E_r - V(\tilde{R}))\right]^{-1/2} \right]^{-1/2} \qquad (17)$$

$$\cdot \left[2\mu (E_r - V(R))\right]^{-1/4} \cdot \sin \frac{1}{\hbar} \int_0^R d\tilde{R} \left[2\mu (E_r - V(\tilde{R}))\right]^{1/2} ,$$

		Linear	Harmonic Oscillator	Spherical Bag
Confining Potential	$V(R) =$	R/R_0	$(R/R_0)^2$	$0, \quad R < R_0$ $\infty, \quad R \geq R_0$
S-Wave Functions	$\psi_r(\vec{x}) =$	$(4\pi)^{-1/2} \cdot \frac{1}{R_0^{3/2}} \cdot \frac{1}{\mathrm{Ai}'(a_{r+1})} \cdot \frac{\mathrm{Ai}(a_{r+1} + R/R_0)}{R/R_0}$	$(4\pi)^{-1/2} \cdot \frac{1}{R_0^{3/2}} \cdot \left[\frac{2\,\Gamma(r+1)}{\Gamma(r + 3/2)}\right]^{1/2} \cdot L_r^{1/2}(R^2/R_0^2) . \exp(-R^2/2R_0^2)$	$(4\pi)^{-1/2} \cdot \frac{2^{1/2}}{R_0^{3/2}} \cdot \frac{\sin(r+1)\pi \cdot (R/R_0)}{R/R_0}$
	$4\pi\lvert\psi_r(0)\rvert^2 =$	$\frac{1}{R_0^3}$	$\frac{8}{\pi} \cdot \frac{1}{R_0^3} \cdot \frac{\Gamma(r + 3/2)}{\Gamma(r + 1)}$	$2\pi^2 \cdot \frac{1}{R_0^3} \cdot (r + 1)^2$
Mass Formula	$M_r^2 =$	$C_0^2 - \frac{(12)^{2/3}}{R_0^2} \cdot a_{r+1}$	$C_0^2 + \frac{(12)^{2/3}}{R_0^2} \cdot 2 \cdot (2r + 3/2)$	$C_0^2 + \frac{(12)^{2/3}}{R_0^2} \cdot \pi^2(r+1)^2$
		$a_n \approx - \left[\frac{3\pi}{8} (4n-1)\right]^{2/3}$	$\Gamma(r + 3/2)/\Gamma(r + 1) \approx (r + 1)^{1/2}$	
Duality		$\sum_V \frac{9\pi}{\alpha^2} \cdot \frac{M_r \cdot \Gamma(V_r \to \ell^+\ell^-)}{M_{r+1}^2 - M_r^2} \xrightarrow[\text{large } r]{} \sum_{\text{quarks}} Q_i^2$		

Table I - Schrtdinger bound s e models which are dual to the parton model.

In the summaticn appearing in e last row, V runs over the vector mesons ρ^0, ω; ϕ, J/ψ,... .

with the Bohr-Sommerfeld quantization condition

$$\int_0^{R_{\max}} d\tilde{R}\left[2\mu(E_r - V(\tilde{R}))\right]^{1/2} = \pi.\ h.\ r\ , \tag{17'}$$

provided $V(R)$ is smooth around the origin.

We also investigated a non confining potential which is singular at the origin, namely, the Coulomb potential. This potential "scales", even for a mass formula

$$M_r^{\alpha} = \text{const} + \text{const}'.\ E_r\ , \quad \alpha>0\ . \tag{18}$$

We should point out that in this case the limit of large r implies $M_r \to$ const, whereas for confining potentials $M_r \to$ const' . $(E_r)^{1/2}$. Amusingly enough, one could generalize Eq.(18) to an arbitrary mass formula

$$M_r = f(C_0 + E_r), \qquad E_r \equiv \frac{-1}{4(r+1)^2}\ . \tag{18'}$$

The only restrictions are that $f(x)$ increases monotonically with x and $f(C_0)$ is finite. To sketch the proof:

$$\begin{aligned} \frac{12\pi\ .\ 4\pi|\psi(0)|^2}{M\ .\ \Delta M^2} &\to \frac{12\pi}{2M_r^2\ .\ \frac{dM_r}{dE_r}}\ .\ \frac{4\pi|\psi_r(0)|^2}{\frac{dE_r}{dr}\ .\ \Delta r} = \\ &= \frac{12\pi}{2M_r^2\ .\ \frac{dM_r}{dE_r}}\ .\ \frac{8}{R_0^3}\ . \end{aligned} \tag{19}$$

Thus the Coulomb case is rather exceptional*.

---

* Another example[12] of a solvable nonconfining potential is $V(R) \sim -(\cosh R/R_0)^{-2}$, which we did not work out here. But it should be mentioned that, in this case, the number of bound states is finite.

4. SUMMARIZING REMARKS

We have investigated the scaling behaviour of quark-antiquark bound state models of the Schrödinger type. Under the assumption that this description makes sense even for large excitations, we have shown that, for the class of analytically solvable confining potentials, the local average of high mass resonances exhibits scaling behaviour and, therefore, can be made dual to the parton model, iff the eigenvalues of the Schrödinger equation are linearly related to the mass squared. We have given a proof that the quantum mechanical property, Eq. (16), underlying the relation between M^2 and E is rather independent of the specific form of the potential.

We have also shown that the Coulomb potential exhibits scaling for a rather general class of mass formulae. But, clearly, it does not confine the quarks.

The problem, which remains to be solved, is the dynamical justification of a Schrödinger type model with a quadratic mass formula. As a hint, we would like to mention that quadratic mass formulae arise more or less naturally in field theoretical bound state models based on the Bethe-Salpeter equation[5]. This analogy, however, may be misleading because of the relativistic nature of these models. Recently, Craigie and Preparata[13] have made an *Ansatz* which starts from a Bethe-Salpeter type model, in which the time component of the amplitude is eliminated. These authors solved the "bag" with the radii increasing with the bound state mass. We are not surprised that this model shows scaling, since the essential input seems not to be the "bag-potential" but rather the quadratic mass formula.

One of us (M.K.) would like to thank the Instituto de Física Teórica for inviting her, for the warm hospitality and the pleasant working conditions. She acknowledges the financial support, during her stay, from CNPq. (Rio de Janeiro) and DAAD (Bonn-Bad Godesberg).

REFERENCES AND NOTES

1. For a recent survey of the experimental data, we refer to the Proceedings of the " 1975 International Symposium on Lepton and Photon Interactions at High Energies " (Stanford, August 1975).

2. See, e. g. M. Gourdin in " Hadronic Interactions of Electrons and Photons" (J. Cumming and H. Osborn eds., Academic Press, London and New York, 1971).

3. R. P. Feynman, " Photon-Hadron Interactions " (Benjamin, Reading , Massachusetts, 1972).

4. A. Bramon, E. Etim and M. Greco, Phys. Lett. 41B, 609(1972).

5. M. Böhm, H. Joos and M. Krammer, Acta Physica Austriaca 38, 123(1973) and in " Recent Developments in Mathematical Physics ", pp.3-116 (P. Urban, ed., Springer Verlag, Wien and New York, 1973).

6. J. J. Sakurai, Phys. Lett. 46B, 207(1973).

7. D. Schildknecht and F. Steiner, Phys. Lett. 56B, 36(1975). For the Han-Nambu interpretation of the new particle see: M. Krammer, D. Schildknecht and F. Steiner, Phys.Rev. D12, 139(1975).

8. T. Appelquist, A. De Rújula, H. D. Politzer and S. L. Glashow, Phys. Rev. Lett. 34, 365(1975); E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K.D. Lane and T. M. Yan, Phys. Rev. Lett. 34, 369(1975); B. J. Harrington, S.Y. Park and A. Yildiz, Phys. Rev. Lett. 34, 706(1975) ; R. Barbieri, R. Kögerler, Z. Kunszt and R. Gatto, Ref.TH.2036-CERN; J. S. Kang and H. J. Schnitzer, Brandeis University preprint, June 1975.

9. For a discussion of the properties of the S-wave solutions of the linear potential we refer to: P.Leal Ferreira and J.A.Castilho Alcarás, IFT 200/75, Lett.N.Cim. 14, 500(1975).

10. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, p. 450 (Dover Publications, Inc., New York, 1965).

11. We would like to thank R. Haag and H. Joos for a very helpful discussion on this point.

12. We would like to thank J. Leal Ferreira for calling this to our attention.

13. G. Preparata and N.S. Craigie, Ref. TH. 2038-CERN; G. Preparata, Lecture at the " International Summer Institute in Theoretical Particle Physics" (Hamburg, September 1975).