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Review of the Theory of Infinite Nuclear Matter

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Given a two-body force, there seems to be two distinct starting points in the many-body perturbation-theoretic problem of computing the energy per nucleon of infinite (as well as finite) nuclear matter: ***ordinary*** Hartree-Fock theory and the Brueckner theory. The former theory, treated almost exclusively with plane-wave solutions, has long-ago fallen into disuse, to yield to the latter, apparently more sophisticated, theory. After a brief outline of many-fermion diagramatic techniques, the Brueckner-Bethe-Goldstone series expansion in terms of the density is discussed as a ***low density,*** non-ideal Fermi gas theory, whose convergence is analyzed. A calculation based on particle-hole Green's function techniques shows that a nucleon gas condenses to the ***liquid*** phase at about 3% of the empirical nuclear matter saturation density. The analogy between the BBG expansion and the virial expansion for a classical or quantum gas is studied with special emphasis on the apparent impossibility of analytically-continuing the latter gas theory to densities in the liquid regime, as first elucidated by Lee and Yang. It is finally argued that ordinary HF theory may provide a good starting point for the eventual understanding of nuclear matter as it gives (in the *finite* nuclear problem, at any rate) not only the basic liquid properties of a ***definite density*** and a ***surface*** but also provides independent-particle aspects, avoiding at the same time the idea of n-body clusters appropriate only for dilute gases. This program has to date ***not*** been carried out for ***infinite*** nuclear matter, mainly because of insufficient knowledge regarding low-energy, non-plane-wave solutions of the HF equations, in the thermodynamic limit.

Dada uma força entre pares de nucleons, existem dois diferentes pontos de partida, assim parece, no tratamento perturbativo, a muitos corpos, do problema de se calcular a energia por nucleon da matéria nuclear infinita (ou fínita): a teoria ***usual*** de Hartree--Fock e a de Brueckner. A primeira dessa teorias, onde quasi exclusivamente são utilizadas ondas planas, já há bom tempo caiu em desuso, cedendo lugar à segunda, aparentemente mais sofisticada. Após breve esboço das técnicas diagramáticas para muitos fermions, discute-se e analisa-se a convergência da expansão em série de Brueckner-Bethe--Goldstone, na densidade, para um gás de Fermi não ideal, a ***baixas densidades.*** Um cálculo, baseado no uso de funções de Green de partícula-buraco, mostra que um gás de nucleons se condensa, passando a fase ***líquida,*** a cerca de 3% da densidade de saturação, empírica, da matéria nuclear. Estuda-se, ademais, a analogia entre a expansão de Brueckner-Bethe-Goldstone e a expansão do virial para um gás, clássico ou quântico, enfatizando-se, especialmente, a aparente impossibilidade de se proceder a uma conti-

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nuação analítica da teoria do gás nucleônico aquela do sistema condensado, o que, pela primeira vez, foi elucidado por Lee e Yang. Argumenta-se, finalmente, que a teoria usual de Hartree-Fock pode constituir-se em um bom ponto de partida para o eventual entendimento da matéria nuclear, por propiciar, pelo menos no caso nuclear *finito*, a inferência não somente das propriedades básicas do líquido nucleônico, a uma densidade *definida*, e de sua superfcie, como também aqueles atributos de partícula independente, e possibilitando, ao mesmo tempo, por-se de lado a idéia de conglomerados a n nucleons, propícia tão somente para gases diluidos. Abordagem desse tipo não foi, até o presente, realizada para a matéria nuclear infinita e isso, principalmente, devido ao desconhecimento das soluções de Hartree-Fock, a baixas energias, distintas das ondas planas, no limite termodinâmico.

1. Introduction

We consider a system of A nucleons interacting through pairwise forces and confined in some volume R. The Hamiltonian is

$$H = T + v \equiv -\frac{\hbar^2}{2m}\sum_{i=1}^{A}\nabla_i^2 + \sum_{i>j}^{A} v_{ij}, \tag{1-1}$$

where v_{ij} is an arbitrary N − N interaction operator dependent on position, momentum, spin and isospin variables of the pair (i, j) and subject to the usual invariance principles[1] under translations, rotations, galilean transformations, etc.. The object is to develop an appropriate starting point from which one may proceed toward obtaining good approximate wave functions and eigenvalues of the A-body Schrodinger equation $(E, - H)\Psi_n = 0$, starting, with a given Hamiltonian, Eq. (1-1).

Physically, if the A particles are confined in the volume R, we must require that the wave functions Ψ_n vanish outside this volume. For mathematical convenience, one substitutes this true boundary condition by the familiar periodic boundary, whereby the total wave function must obey

$$\Psi_n(\ldots \quad \ldots \equiv \Psi_n(\ldots, r_{si} + \mathrm{L}, \ldots \quad \mathrm{i} = 1,2,\ldots,A;\ s = x, y, z, \tag{1-2}$$

where Lis the length of a cubic volume $R = L^3$. This relatively simple substitution might be justified in the thermodynamic limit ($A \longrightarrow \infty$ and $R \longrightarrow \infty$ but such that $\rho \equiv N/\Omega = \text{constant}$), thus providing a useful means of studying only the so-called volume effects which, for interacting systems, are independent of the specific characteristics of the surface, regardless of whether this surface is an extraneous container, as in "a gas enclosed in a vessel" or, a *proper* containing surface, as in the case pf a "liquid droplet". The condition, Eq. (1-2) (which

would certainly be satisfied if each particle were described by a plane wave), is obeyed if one expands the eigenfunctions

$$\Psi_n = \sum_m C_{nm} \Phi_m,$$

in terms of the complete, orthonormal, totally antisymmetric, basis set, $\{\Phi_m \mid m = 0, 1, 2, \ldots \infty\}$, defined by

$$\left(-\frac{\hbar^2}{2m} \sum_{i=1}^{A} \nabla_i^2 - W_m \right) \Phi_m = 0; \quad \Phi_m = (A!)^{-1/2} \det [\varphi_k(x)],$$

$$\begin{aligned} &\mathrm{k} \equiv (\mathrm{k}, \mathrm{s}, t), \; \mathrm{x} \equiv (\mathrm{r}, \sigma_z, \tau_z), \\ &\varphi_{\mathbf{k}}(x) \equiv \Omega^{-1/2} e^{i\mathbf{k}\cdot\mathbf{r}} \chi_s(\sigma_z) \chi_t(\tau_z), \end{aligned} \tag{1-3}$$

where $\mathrm{k} = (2\pi/L)\mathbf{n}; \; (\mathrm{n}_x, \mathrm{n}_y, \mathrm{n}_z) = 0, +1, \pm 2, \ldots.$

We note that the propagation vector k takes on discrete values consisting of points in a simple cubic lattice structure in momentum space, such that the volume of each "primitive cell" is just $(2\pi/L)^3$. Hence, in the limit A, $\Omega \longrightarrow \infty$, with p constant, the number of points equals the number of cells, or one may replace sums by integrals according to

$$\sum_{\mathbf{k}} \xrightarrow[\Omega \to \infty]{} [\Omega/(2\pi)^3] \int d^3k. \tag{1-4}$$

For the ground state of an ideal Fermi gas, one has a sharp step-function distribution up to the Fermi momentum $\hbar k_F$ so that **Eq.** (1-4) then gives, on allowing for the possibility of four nucleons per k-state,

$$\rho = A/\Omega = (2/3\pi^2)\, k_F^3. \tag{1-5}$$

The unperturbed problem of Eq. (1-3) can be written in occupation--number representation by making the correspondences

$$\varphi_k(x) \longrightarrow a_k^+ \mid 0\rangle \qquad \Phi_m \longrightarrow \prod_k (a_k^+)^{n_k} \mid 0\rangle \; ; \qquad k \equiv (\mathbf{k}, s, t), \tag{1-6}$$

where a_k^+ are fermion creation operators; the N-body state index, m, designates the set $\{\ldots, n_k, \ldots\}$ of single-particle occupation numbers (0 or 1). One may, further, characterize a Fermi sea, or unperturbed ground state ("vacuum state") by

$$\Phi_0 \text{ (with } n_k = 1 \text{ for all } |\mathrm{k}| \leq k_F; \; n_k = 0 \text{ for all } |\mathrm{k}| > k_F), \tag{1-7}$$

so that, since for any m one has $W_m = \sum_k (\hbar^2 k^2/2m)\, n_k$, one may order all the eigenvalues as $W_0 < W_1 \leq W_2 \leq \ldots$. In particular, the total energy of the unperturbed Fermi sea will be, using Eqs. (1-4) and (1-5),

$$W_0 = \sum_k (\mathrm{h}^2\mathrm{k}^2/2\mathrm{m})\,\theta\,(k_F - \mathrm{k}) = \frac{4\Omega}{(2\pi)^3} \int_{k \leq k_F} \mathrm{d}^3\mathrm{k}\,(\hbar^2 k^2/2m)$$

$$= \frac{3}{5}\,\frac{\hbar^2 k^2}{2m}\,A. \tag{1-8}$$

Finally, the Hamiltonian, Eq. (1-1), may be expressed as

$$H = \sum_k (\hbar^2 k^2/2m)\, a_k^+ a_k + \frac{1}{2} \sum_{\substack{k_1' k_2' \\ k_1 k_2}} \langle k_1' k_2' \,|\, v_{12} \,|\, k_1 k_2 \rangle\, a_{k_1'}^+ a_{k_2'}^+ a_{k_2} a_{k_1}, \tag{1-9a}$$

$$\langle k_1' k_2' \,|\, v_{12} \,|\, k_1 k_2 \rangle = \Omega^{-2} \int\int \mathrm{d}^3\mathrm{r}_1\, \mathrm{d}^3\mathrm{r}_2\, \exp\,[-i(\mathbf{k}_1' \cdot \mathbf{r}_1 + \mathbf{k}_2' \cdot \mathbf{r}_2)]\ \mathrm{x}$$

$$\mathrm{x}\ \langle s_1' t_1' s_2' t_2' \,|\, v_{12} \,|\, s_1 t_1 s_2 t_2 \rangle\ \exp\,[i(\mathbf{k}_1 \cdot \mathbf{r}_1 + \mathbf{k}_2 \cdot \mathbf{r}_2)]. \tag{1-9b}$$

Any approximate approach to the problem of solving the Schrodinger equation (in either coordinate or occupation-number representations) can be analyzed systematically within the framework of perturbation theory. For the time being at least, let us restrict the discussion to the exact ground state of the interacting system Eq. (1-1). As the unperturbed ground state can be assumed non-degenerate, one may apply ordinary Rayleigh-Schrödinger perturbation theory, in which the total energy of the exact ground state is given by the series, up to third order,

$$E = W_0 + \langle \Phi_0 \,|\, v \,|\, \Phi_0 \rangle + \langle \Phi_0 \,|\, v\, \frac{Q}{e}\, v \,|\, \Phi_0 \rangle$$

$$+ \langle \Phi_0 \,|\, v \frac{Q}{e} v \frac{Q}{e} v \,|\, \Phi_0 \rangle - \langle \Phi_0 \,|\, v \,|\, \Phi_0 \rangle\, \langle \Phi_0 \,|\, v \frac{Q}{e^2} v \,|\, \Phi_0 \rangle + \ldots, \tag{1-10a}$$

where

$$\frac{Q}{e} \equiv \sum_{m \neq 0}^{\infty} \frac{|\,\Phi_m\rangle\,\langle\Phi_m\,|}{W_0 - W_m},$$

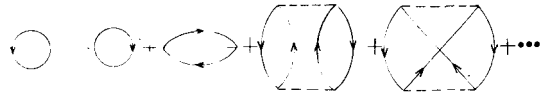
with W_0 given by **Eq.** (1-8). This series has been reinterpreted by **Brueckner**[2] and Goldstone[3], in terms of a "linked cluster" theorem, as

$$E_0 = W_0 + \sum_{n=1}^{\infty} \langle \Phi_0 | v \left(\frac{Q}{e} v \right)^{n-1} | \Phi_0 \rangle_{\text{linked}}$$

$$= W_0 + \sum_{n=1}^{m} \sum_{D_n} (\text{contributions of D,}), \qquad (1\text{-}10b)$$

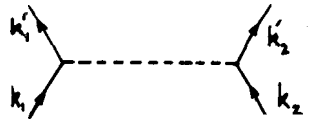
the last summation being only over linked Feynman diagrams D_n of order n, which is the degree of v. The first few diagrams, up to second order, are

$$\Delta E_0 \equiv E_0 - W_0 = \qquad (1\text{-}11)$$



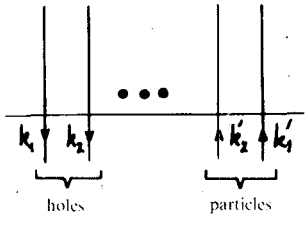
We briefly state the rules for calculating the contribution from a given linked diagram:

a) to each interaction line is associated the factor



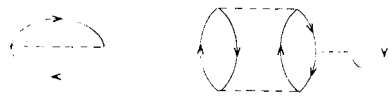
$$\longrightarrow \langle k_1' k_2' | v_{12} | k_1 k_2 \rangle$$

b) between two successive interaction lines one introduces the energy denominator, with $T_k \equiv h^2k^2/2m$,



$$: \left(\sum_{\text{holes}} T_k - \sum_{\text{particles}} T_{k'} \right)^{-1}$$

c) each diagram contributes with the factor $(-)^{l+h}/g$, where l is the number of closed loops, h the number of independent hole lines, and g equals 2 (or 1) if a given diagram has (or does not have) left-right symmetry. Examples:



$l = 1,\ h = 1,\ g = 2$ $\qquad$ $l = 3,\ h = 3,\ g = 1$

d) each diagram particle line must be summed over *all* possible momenta aboue the Fermi value $|\,\mathrm{k}\,| > k_F$, while each hole line over all possible momenta below, $|\,\mathrm{k}\,| \leq k_F$. Furthermore, if a nucleon line begins and ends at the same vertes of a diagram, it should be accounted as a hole line [viz., the first two diagrams on the right-hand side of Eq. (1-11)].

2. Brueckner Theory as a Low-Density Approximation to a Non-Ideal Fermi Gas

In this section, we discuss the highlights of the original Brueckner theory and subsequent modifications: a derivation is sketched in the Appendix. In order to compute the successive contributions in Eq. (1-10b), one must evaluate the integrals (1-9b). If Ω is considered very large, these integrals simplify to

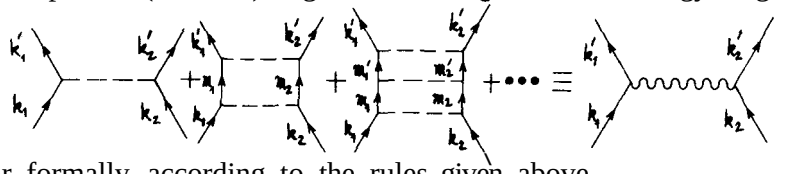
$$\langle k_1' k_2' \,|\, v_{12} \,|\, k_1 k_2 \rangle = \Omega^{-1}\, \delta(\mathbf{k}_1' + \mathbf{k}_2',\ \mathbf{k}_1 + \mathbf{k}_2) \int \mathrm{d}^3 r\, \langle s_1' t_1' s_2' t_2' \,|\, v_{12} \,|\, s_1 t_1 s_2 t_2 \rangle \exp\left[i(\mathbf{k}_1 - \mathbf{k}_1') \cdot \mathrm{r} \right], \tag{2-1}$$

where $\mathrm{r} = \mathbf{r}_1 - \mathbf{r}_2$; the notation $\delta(\mathbf{k}, \mathbf{k}')$ for the Kronecker delta has been adopted for graphical convenience. For the interaction operator v_{12}, in Eq. (2-1), one should prefer to insert a "realistic" N–N potential, i.e., as given by two-nucleon experiments. Since around 1950, after a suggestion made by Jastrow[4], the momentum dependence of the N–N potential v_{12} has very frequently been simulated by a local, velocity independent, infinitely repulsive (hard-cored), potential,

$$v_{12}(r) = +\infty \qquad (r \leq r_c), \tag{2-2}$$

the hard-core being surrounded ($r > r_c$) by a finite, short-ranged, attractive part of some kind. Basic theory[5] has to date determined unambiguously only the tail ($r \geq 2$fm) of the potential as the one-pion-exchange contribution. The well-known "realistic" potentials[6] of Hamada – Johnston and of the Yale group, as well as that of Reid[7], for example, are of this general type. Such potentials would, of course, make the matrix elements (2-1) infinite, thus invalidating ab initio any perturbational approach based on the kinetic energy operator as the unperturbed Hamiltonian.

To circumvent this difiiculty, Brueckner[8] suggested a rearrangement of the perturbation series so as to eliminate the bare, singular, N–N interaction, Eq. (2-2), in favor of a renormalized nonsingular ("effective") two-particle interaction operator, K_{12}. This was accomplished by effecting a partial summation to infínite order of the following *par*-ticle-particle ("ladder") fragments of our groundstate energy diagrams:



(2-3)

or formally, according to the rules given above,

$$\langle k_1' k_2' \,|\, v_{12} \,|\, k_1 k_2 \rangle + \sum_{m_1, m_2 > k_F} \frac{\langle k_1' k_2' \,|\, v_{12} \,|\, m_1 m_2 \rangle \, \langle m_1 m_2 \,|\, v_{12} \,|\, k_1 k_2 \rangle}{T_{k_1} + T_{k_2} - T_{m_1} - T_{m_2} + \delta}$$

$$\equiv \langle k_1' k_2' \,|\, K_{12}(\delta) \,|\, k_1 k_2 \rangle, \qquad (2\text{-}4)$$

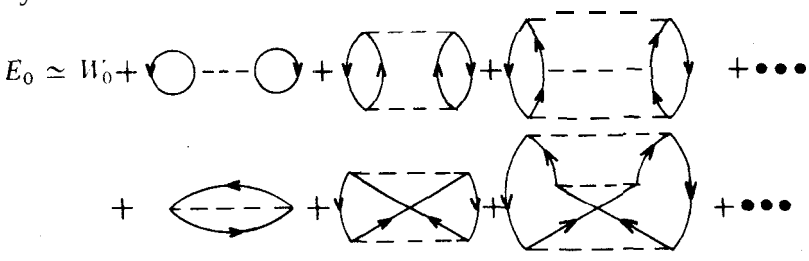
in which the unspecifíed quantity 6 will depend, in accordance with rule **(b)** above, on the particular structure of the remaining part of the diagram for which the "ladder summation" is being considered. The K-matrix element on the right-hand side of (2-4) is said to be "on-the--energy-shell" if $6 = 0$ and "off-energy-shell" if $\delta \neq 0$. The ladder summation (2-4) leads to a second-order integro-differential equation, the Bethe-Goldstone equation[9], for a "correlated two-particle wave function" $\psi_{k_1 k_2}(\mathbf{r}_1, \mathbf{r}_2)$ defined as

$$v_{12}\psi_{k_1 k_2}(\mathbf{r}_1, \mathbf{r}_2) \equiv K_{12} \,|\, k_1 k_2 \rangle \equiv \Omega^{-1} K_{12} \exp\left[i(\mathbf{k}_1 \cdot \mathbf{r}_1 + \mathbf{k}_2 \cdot \mathbf{r}_2)\right] \chi_{s_1}(\sigma_{z1}) \, \chi_{t_1}(\tau_{z1}) \, \chi_{s_2}(\sigma_{z2}) \, \chi_{t_2}(\tau_{z2}).$$

Thus, in view of Eq. (2-2), it becomes possible to require of $\psi_{k_1 k_2}$ that it satisfies the boundary condition

$$\psi_{k_1k_2}(\mathbf{r}_1, \mathbf{r}_2) \equiv 0 \text{ for all } r = |\mathbf{r}, - \mathbf{r}_2| \leq r_c$$

so that the reaction matrix elements on the right-hand side of Eq. (2-4) are then clearly nonsingular and the perturbation approach has recovered its original meaningfulness. On the basis of this particle-particle ladder summation, Brueckner at first proposed the following starting point for determining the total energy of the exact ground state of the system:



(2-5)

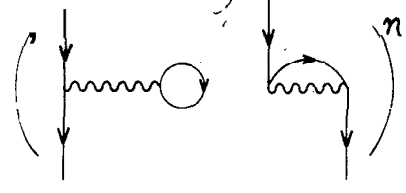
or, formally, applying the rules once more:

$$E_0 \simeq W_0 + \frac{1}{2} \sum_{|\mathbf{k}_1|, |\mathbf{k}_2| \leq k_F} \langle k_1 k_2 | \tilde{K}_{12}(\delta = 0) | k_1 k_2 \rangle, \qquad (2\text{-}5a)$$

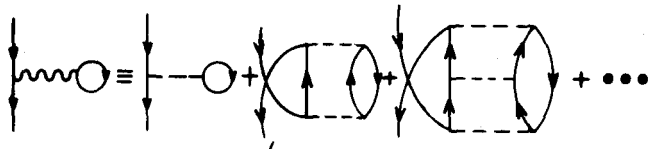
$$\langle k_1' k_2' | \tilde{K}_{12} | k_1 k_2 \rangle \equiv \langle k_1' k_2' | K_{12} | k_1 k_2 \rangle - \langle k_1' k_2' | K_{12} | k_2 k_1 \rangle, \qquad (2\text{-}5b)$$

where the required K-matrix elements all result on the energy shell, as is obvious from the general structure of the diagrams being summed. In Eqs. (2-5), as the particle and hole lines are "bare" lines, the corresponding energy denominators contain *pure* kinetic energies.

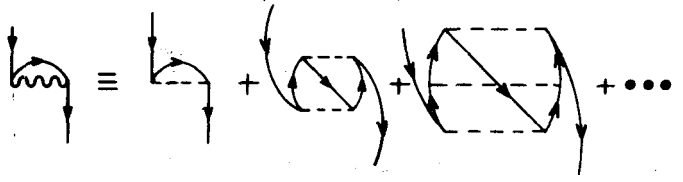
Were these lines "dressed", with additional (negative) potential energy, one would get an unperturbed Hamiltonian which is more than just the kinetic energy operator of Eqs. (1-3) and the addition would be expected to provide improved convergence properties for the series of Eq. (1-10a). This "dressing" is accomplished by considering an *additional class* of diagrams. These are obtained by applying the following procedure to both particle and hole lines; we discuss first what happens to hole lines: a) replace every "bare" hole line in the energy diagram of Eq. (2-5) by the sum of all possible K-matrix "bubble" plus exchange "half-oyster" insertions, namely,

replaced by $\sum_{n=0}^{j}$  (2-6a)

where each new insertion is. according to Eq. (2-3). explicitly



(2-6b)



(2-6c)

b) each hole line on the r.h.s. of (2-6b) is then in turn replaced by the possible K-matrix "bubble", plus exchange, insertions (2-6a); c) and so on indefinitely. However, it then happens that the inserted K-interaction lines are off-the-shell but can be put on-the-energy shell, as was shown by Bethe, Brandow and Petschek[10], by adding a further class of diagrams with K-interaction lines (see Appendix). This new class has the same structure as the class described in the above procedure except that successive K-interaction lines are arranged differently. Now, as regards particle lines: if the "bubble" plus exchange insertions of Eqs. (2-6) are also used to dress bare *particle* lines, it turns out that these inserted K-interaction lines are always *off*-the-energy-shell and cannot be put on-the-shell by the BBP "factorization-theorem" technique. The net result of this new "dressed" theory is that Eqs. (2-4) and (2-5) are replaced by the Brueckner-Hartree-Fock equations (as interpreted by Bethe-Brandow and Petschek):

$$E_0 \simeq W_0 + \frac{1}{2} \sum_{|k_1|,\ |k_2|>k_F} \langle k_1 k_2 \,|\, \tilde{K}_{12}(\delta = 0) \,|\, k_1 k_2 \rangle, \tag{2-7a}$$

$$\langle k_1' k_2' \,|\, K_{12}(\delta) \,|\, k_1 k_2 \rangle \equiv \langle k_1' k_2' \,|\, v_{12} \,|\, k_1 k_2 \rangle +$$

$$+ \sum_{|\mathbf{m}_1|,\ |\mathbf{m}_2|\ >\ k_F} \frac{\langle k_1' k_2' \,|\, v_{12} \,|\, m_1 m_2 \rangle \langle m_1 m_2 \,|\, K_{12}(\delta) \,|\, k_1 k_2 \rangle}{\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{m_1} - \varepsilon_{m_2} + \delta}\ ; \tag{2-7b}$$

$$\varepsilon_k \simeq T_k + U_k \equiv T_k + \sum_{|\mathbf{k}_1| \le k_F} \langle k k_1 \,|\, \tilde{K}_{12}(\delta = 0) \,|\, k k_1 \rangle, \text{ (for } |\mathbf{k}| \le k_F); \tag{2-7c}$$

$$\varepsilon_k \equiv T_k + U_k \equiv T_k + \underset{|\mathbf{k}_2|,\ |\mathbf{k}_3|,\cdots \le k_F}{\text{AVERAGE}} \sum_{|\mathbf{k}_1| \le k_F} \langle k k_1 \,|\, \tilde{K}_{12}(\delta \neq 0) \,|\, k k_1 ; k_2 k_3 \ldots \rangle \tag{2-7d}$$

for $|\mathbf{k}| > k_F$;

$$E_0 \simeq \sum_{|\mathbf{k}| \leq k_F} \left(T_k + \frac{1}{2} U_k \right). \tag{2-7e}$$

The "average" operation appearing in Eq. (2-7d) is over additional hole line indices $k_2, k_3, \ldots$, which are present and make the particle-particle ladder K-interaction "bubble" plus exchange insertions, for particle lines, *off*-the-energy-shell. Without this somewhat arbitrary procedure, the common potential U_k for *particles* would not depend, as it should, on only ***one*** orbital state index k. The BHF equations (2-7) seem as an improvement over the ordinary Brueckner equations (2-4) and *(2-5)*as one now has, through the dressed single-particle energies ε_k, an unperturbed Hamiltanian which is more than just pure kinetic energy as in Eqs. (1-3). One might thus expect an improved convergente of higher order corrections based on Eqs. (2-7) as starting point.

The BHF theory as sketched above corresponds to the partial summation to infinity of a selected class (or rather classes) of diagrams. The crucial question, of course, is: *why* are these particular classes selected to the exclusion of other classes? It appears that the only apparent explanation is to associate the resulting selected set of diagrams (the "ladders" with dressed particle and hole lines) with an assumed *small* density of the manybody system under consideration. This was shown some time ago by Hugenholtz[1], as well as others, and can be seen as follows. The contribution of a typical perturbation-theory diagram with n_h independent [recall the Kronecker-δ of Eq. (2-1)] hole lines and n_p independent particle lines ("bare" or "dressed") will be according to the rules stated above quite generally

$$\int_{k_F}^{\infty} d^3k'_1 \ldots \int_{k_F}^{\infty} d^3k'_{np} \int_0^{k_F} d^3k_1 \ldots \int_0^{k_F} d^3k_{nh}\, F(\mathbf{k}'_1, \ldots, \mathbf{k}'_{np};\ \mathbf{k}_1, \ldots, k_{nh}) \tag{2-8}$$

where the function F includes the two-particle interaction matrix elements as well as the energy denominators appropriate to the given diagram. As the energy denominators are never singular, and assuming a non-singular interaction v_{12}, the function F is then analytic and may be expanded in a Taylor series in any of its momentum-vector arguments. For low densities, $k_F \longrightarrow 0$, and every hole-integration thus brings in a factor

$$\int_0^{k_F} d^3k\, F(\ldots, \mathbf{k}, \ldots) \simeq F(\ldots, 0 \ldots)\, \frac{4\pi}{3} k_F^3 + O(k_F^4). \qquad (2\text{-}9)$$

In the same limit, every particle-integration becomes

$$\int_{k_F}^{\infty} d^3k'\, F(\ldots, \mathbf{k}', \ldots) = \int_0^{\infty} d^3k'\, F(\ldots, \mathbf{k}', \ldots) - \int_0^{k_F} d^3k'\, F(\ldots, k', \ldots)$$

$$\simeq \int_0^{\infty} d^3k'\, F(\ldots, \mathbf{k}', \ldots) + O(k_F^3). \qquad (2\text{-}10)$$

Consequently, the total contribution to the potential energy from any set of diagrams having at least n_h independent hole lines can be expressed, with the help of Eq. (1-5), as a power series, in the density,

$$\alpha\, \rho^{n_h} + \alpha_1\, \rho^{n_h + 1/3} + \alpha_2\, \rho^{n_h + 2/3} + \beta\, \rho^{n_h + 1} + \beta_1\, \rho^{n_h + 1 + 1/3} + \\ + \beta_2\, \rho^{n_h + 1 + 2/3} + \ldots . \qquad (2\text{-}11)$$

As there are no ground-state diagrams with none or only one hole line, the extreme low-density *limit* evidently dictates accounting only diagrams with two independent hole lines in first approximation. This is precisely the original "bare" Brueckner theory, Eq. (2-5), or the "dressed BHF theory, Eqs. (2-7). The next obvious correction would be to include all diagrams with three independent hole lines, etc..

The BHF equations (2-7), however, evidently contain contributions from only selected 3-hole-line diagrams, as well as from 4-hole, 5-hole, etc., diagrams. The inclusion of *all* three-hole diagrams (the so-called "three-body clusters") has been considered recently by Bethe[12] and others, using a three-body generalization, based on the Faddeev theory[13], of the two-body Bethe-Goldstone equation. Accepting the unavoidable approximations made in dealing with calculation of the "three-body clusters", one obtains, for the total energy per nucleon of nuclear matter, a series expansion in powers of the particle density p which, in view of Eqs. (2-8) and (2-11), would in principle be exact up to and including the $\rho^{2+2/3}$ term, namely,

$$E_0^{BBG}/A = a\, \rho^{2/3} + b\, \rho + b_1\, \rho^{1+1/3} + b_2\, \rho^{1+2/3} + c\, \rho^2 + c_1\, \rho^{2+1/3} + \\ + c_2\, \rho^{2+2/3} + O(\rho^3), \qquad (2\text{-}12)$$

where the first term is the kinetic energy contribution, Eq. (1-8). This procedure[12], based essentially on the BHF Eqs. (2-7), is the natural

extension of the original BHF theory to higher and higher density regimes and is now known as the Brueckner-Bethe-Goldstone theory of nuclear matter. Excellent reviews of this theory, which has also been extensively applied in recent years to finite nuclei[19], are listed in Ref. 14.

Using the BBG theory, several authors[14], since Brueckner's initial work, have calculated the binding energy per particle of infinite nuclear matter for different values of k_F and different N–N potential models. The values of k_F are related to ρ, by Eq. (1-5), and E_0^{BBG}/A, plotted against ρ, shows a minimum at some saturation density $\rho_s > 0$. These results are compared, respectively, with the volume energy per nucleon of the semi-empirical mass formula and the experimental central density of heavy nuclei. It is entirely possible that these "saturation minima" might be completely spurious, since these calculations are based on an approximate function of density, like Eq. (2-12), so that genuine results can only be expected if the saturation density ρ_s lies in the domain of small ρ for which the approximate, mutilated (even though it contains an infinite number of terms) series acceptably converges to the left-hand side of Eq. (2-12). (A very simple illustration of "spurious minima" is seen by approximating the function $\exp(-x)$, for $x \geq 0$ say, successively by $1 - x$, $1 - x + (1/2)x^2$, etc. .

It has recently been recognized that the BBG series, (2-12), which to date[14] includes up to 4-body clusters, can in fact be rewritten[15] as the series

$$E_0^{BBG}/A = a\rho^{2/3} + B\kappa + B_1\kappa^{1+1/3} + B_2\kappa^{1+2/3} + C\kappa^2 + C_1\kappa^{2+1/3} + C_2\kappa^{2+2/3} + \ldots, \qquad (2\text{-}13)$$

where the new expansion parameter is

$$\kappa \equiv \rho \int d^3r \,|\, \chi(\mathbf{r}) \,|^2 \equiv \rho/\rho_{corr} \equiv v_{corr}/v, \qquad (2\text{-}14)$$

$$v \equiv 1/\rho,$$

in which $\chi(\mathbf{r})$ is the "wound" function[16] (non-perturbed minus perturbed two-body BG wave function). This function is basically determined by the short-ranged repulsion of the two-body force in the BG equation. The coefficients in the series (2-13) will depend on both strength and range of the assumed two-body force, as well as on the BG wave function. Because of its rapid "healing", $\chi(\mathbf{r})$ will differ appreciably from zero only for $r \leq 2$ to 4 fm and so, the "wound integral may be interpreted as the specific volume of correlation $v_{corr} = \rho_{corr}^{-1}$ of

the two particles. Calculations[17] show that $\kappa \simeq 0.16$ for a "realistic" soft-cored potential and this has been taken as evidence that, at last, a small expansion parameter has been found for the nuclear problem, an assertion which would undoubtedly be true provided that the radius of convergence of the series (2-13) were, say, unity, as occurs, e.g., with the geometric series representation of $(1-x)^{-1}$. The series (2-13), however, is not even a Taylor series, in κ, and there are no convincing arguments to exclude the possibility that its radius of convergence is not less than the above mentioned value of 0.16. The physical meaning of κ has been given as the "fractional time" a pair of particles are found in correlaied motion, i.e., not independent motion, an interpretation which obviously relies on the assumption that the radius of convergente is unity. The two-particle correlation, which determines the value of v_{corr} may not give the entire true correlation effect, since v_{corr} in Eq. (2.14), is obtained from a low-density two-hole-line theory with selected 3-hole, 4-hole, etc., correctiops, as discussed in connection with the BHF Eqs. (2-7). It is not clear, then, whether three-and more-hole terms will not drastically alter the correlation density ρ_{corr}, so that the value of κ quoted above seems highly uncertain in connection with real nuclear matter. Indeed, the fact that "healing" disappears, making v_{corr} and hence K infinite, upon inclusion of hole-hole ladders (see Appendix) in itself, constitutes a strong indictment against the concept of "healing" as a real physical phenomenon present in nuclear matter and consequently on the concept of κ as a small expansion-parameter.

But regardless of the exact value of $v_{\mathrm{corr}} = \rho_{\mathrm{corr}}^{-1}$, the higher-order terms in the BBG series, (2-13) or (2-12), will grow in importance as density is increased, i.e., as ρ increases so do the roles of the three-body, four-body, etc., cluster effects. The crucial question now is: what is low and high density for the nuclear matter system? A clarification of this point is attempted in the following two sections.

3. High-Density Fermi Gas RPA Instability

The purpose of this section is to carry out an estimation of the critical density at which a dense gas of fermions, interacting through a "realistic" purely-central N-N attractive plus repulsive potential, becomes unstable due to the appearance of spontaneous density fluctuations of the kind that set in, as a gas is compressed to the condensation point. A fuller account of this phase transtion problem appears elsewhere[23]. The procedure here is to solve a transcendental dispersion relation,

based on the so-called ring diagrams only, for the energy-spectrum of the elementary (collective) excitation associated with the density fluctuation. The spectrum appears as purely real for densities below a critical density, and partly imaginary for densities above that value, leading thus to an exponentially increasing amplitude in time which is interpreted as setting, in the gas-to-liquid transition point.

Consider the Fermi gas, particle-hole, Green's function

$$Q(q;\, t - t') \equiv \langle \Psi_0 \,|\, T\{\rho_q(t)\rho_q^+(t')\} \,|\, \Psi_0 \rangle, \tag{3-1}$$

where the expectation value is between the exact many-particle ground-state eigenfunction of the Hamiltonian, Eq. (1-1); T is the time-ordering operator and the collective variables are defined in Heisenberg representation by

$$\rho_q^+(t) = \exp\left(\frac{i}{\hbar} Ht\right) \rho_q^+ \exp\left(-\frac{i}{\hbar} Ht\right), \tag{3-2a}$$

$$\rho_q^+ \equiv \int d^3r \; e^{i\mathbf{q}\cdot\mathbf{r}} \, \rho(\mathbf{r}) = \sum_k a_k^+ \; a_{|\mathbf{k}-\mathbf{q}|}, \tag{3-2b}$$

where $\rho(\mathbf{r})$, the density operator in configuration space, is given by

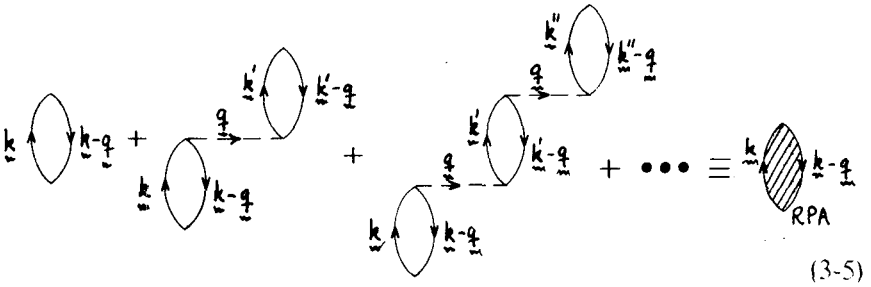
$$\rho(\mathbf{r}) = \psi^+(\mathbf{r})\psi(\mathbf{r}); \; \psi(\mathbf{r}) = \sum_k a_k \varphi_k(\mathbf{r}); \quad \varphi_k(\mathbf{r}) = \Omega^{-1/2} \, e^{i\mathbf{k}\cdot\mathbf{r}} \, \chi_s(\sigma_z) \, \chi_t(\tau_z), \tag{3-3}$$

where $a_k^+ \equiv a_{\mathbf{k},st}^+$ is the fermion creation and a_k the fermion annihilation operators. If the Fourier transform of Eq. (3-1) is defined as

$$Q(q;\, \omega) = \int_{-\infty}^{+\infty} d\tau \; e^{-i\omega\tau} \; Q(q;\, \tau), \tag{3-4}$$

then the elementary-excitation spectrum[20], $\omega = \omega(q)$, follows from the *poles* of $Q(q;\, \omega)$. This quantity is expressible as an infinite sum of so-called "polarization", or "density-fluctuation", Feynman diagrams[20].

At very high, but still gaseous, densities (and provided the interparticle interaction is long-ranged), the main contributions to Eq. (3-4) will come from the sa-called "ring" polarization diagrams, leading essentially to a form of the wellknown random-phase aproximation (RPA), namely,



(3-5)

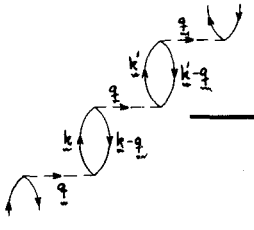
The salient feature of this particular class of diagrams is that each one contributes with the Fourier-transformed interaction,

$$v(q) = \int d^3r \; e^{-iq\cdot r} \, v_{12}(r),$$

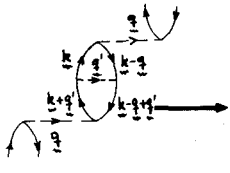
raised to the n^{th} power, n being the order of the diagram, i.e., the number of interacting lines. Any diagram of the same order but which does not belong to the ring type, (3-5), will contribute, in n^{th} order, with

$$v^{n_1}(q_1) \; v^{n_2}(q_2) \; \ldots \; v^{n_s}(q_s); \quad n = n_1 + n_2 + \ldots + n_s,$$

and its overall value is considerably smaller than the corresponding ring diagram, provided the $v(q)$'s are heavily concentrated around small q values (long-rangedness). To illustrate this, consider the two third-order cases:



$$\longrightarrow \int d^3q \; v^3(q) \int d^3k \; \phi_1(\mathbf{k}, \mathbf{q}) \int d^3k' \; \phi_2(\mathbf{k}', \mathbf{q}), \qquad (3\text{-}6)$$



$$\longrightarrow \int d^3q \; v^2(q) \int d^3q' \; v(q') \int d^3k \; \phi(\mathbf{k}, \mathbf{q}, \mathbf{q}'). \qquad (3\text{-}7).$$

In (3-6), $|\mathbf{k}| > k_F, |\mathbf{k}-\mathbf{q}| \leq k_F, |\mathbf{k}'| > k_F, |\mathbf{k}'-\mathbf{q}| \leq k_F$; in (3-7), $|\mathbf{k}| > \mathbf{k}_F$, $|\mathbf{k}-\mathbf{q}| \leq k_F, |\mathbf{k}+\mathbf{q}'| > k_F, |\mathbf{k}+\mathbf{q}'-\mathbf{q}| \leq k_F$. The ϕ's are the appropriate energy denominators dictated by rule (b) of Sec. 1. Thus, if the $v(q)$'s

are localized for small q values, the additional restrictions, on the k-integration of **Eq.** (3-7), will greatly suppress its contribution compared to that of **Eq.** (3-6) which has only two restrictions both on the k– and $\mathbf{k}'$-integrations.

If, in Eqs. (3-1) to (3-4), the interaction potential is turned off (**H** is then pure kinetic energy), one may define the (ideal Fermi gas) Green's function, $Q_0(q;\mathrm{o})$, which turns out to be

$$Q_0(q;\omega) = \frac{2\hbar}{i} \sum_{\mathbf{k}}{}' \frac{T_k - T_{|\mathbf{k}-\mathbf{q}|}}{(T_k - T_{|\mathbf{k}-\mathbf{q}|})^2 - \omega^2 + i\eta}, \tag{3-8}$$

$T_l = \hbar^2 l^2/2m$. The dash. in the k-summation means that $|\mathrm{k}| > \mathrm{k}_F$ and $|\mathbf{k}-\mathbf{q}| \leq \mathrm{k}_F$.

Using relation (1-4) and carrying out the resulting integrations in Eq. (3-8) [see Ref. 22, Eqs. (6) to (8), for example], one is left with

$$\begin{aligned}\frac{i}{\hbar}\,\frac{1}{\Omega}\,Q_0(q;\omega) = -\frac{1}{8\pi^2}\,\frac{m^3}{\hbar^6}\,\frac{1}{q^3}\,\{&[(\hbar^2q^2/2m)^2 - (\hbar^2qk_F/m)^2 + \omega^2] \times \\ &\times \ln\frac{[(\hbar^2q^2/2m) + (\hbar^2qk_F/m)]^2 - \omega^2}{[(\hbar^2q^2/2m) - (\hbar^2qk_F/m)]^2 - \omega^2} + \\ &+ 2(\hbar^2q^2/2m)\omega \ln\frac{(\hbar^2q^2/2m)^2 - (\omega + \hbar^2qk_F/m)^2}{(\hbar^2q^2/2m)^2 - (\omega - \hbar^2qk_F/m)^2} \\ &- 4(\hbar^2q^2/2m)(\hbar^2qk_F/m)\}.\end{aligned} \tag{3-9}$$

Now, if v_{12} is, as before, the two-nucleon interaction, **Eq.** (2-1) gives (suppressing, by the moment, spin and isospin variables)

$$\langle k_1'k_2' \,|\, v_{12} \,|\, k_1k_2\rangle = \Omega^{-1}\delta(\mathbf{k}_1' + \mathbf{k}_2', \mathbf{k}_1 + \mathbf{k}_2)\; v(|\,\mathbf{k}_1 - \mathbf{k}_1'\,|),$$

where the 6 is the usual Kronecker delta and $v(q)$ is the Fourier transform defined after **Eq.** (3-5). The infinite sum, in (3-5), is clearly a geometric series and, therefore, the ring-diagrams Green's function is just

$$Q_{\mathrm{RPA}}(q;\omega) = Q_0(q;\omega)\,[1 + (i/\hbar\Omega)\,v(q)\,Q_0(q;\omega)]^{-1}, \tag{3-10}$$

a (boson) density-fluctuation propagator. Note that, since in the ring diagram summation, (3-5), one has an arbitrary number of hole-lines included, this approximation applies for high densities whereas the Brueckner ladder summation discussed before is for low densities. The poles of Eq. (3-10) are given by the solutions of

$$1 + i(\hbar\Omega)^{-1}\,v(q)\,Q_0(q;\omega) = 0, \tag{3-11}$$

with $Q_0(q;\omega)$ given by Eq. (3-9). The solutions lead to the following qualitative results: a) for densities $\rho = (2/3\pi^2)k_F^3$ below a certain critical value ρ_c one has two branches of stable elementary excitations with a phonon-like behaviour as $q \to 0$, namely, $\omega_{1,2} \simeq c_{1,2}q$, for small q, with $c_{1,2}$ the real propagation velocities of the corresponding excitations; b) for densities $\rho \geq \rho_c$, both c_1 and c_2 go through zero at $p = \rho_c$ and, for $\rho > \rho_c$, one of them becomes complex so that an instability of the gaseous phase has appeared. Hence, it suffices to take the simple special case $\omega = 0 = q$, in Eqs. (3-9) and (3-11), to obtain the value of the critical k_F^c associated with ρ_c by Eq. (1-5). For an attractive interaction, $v(q) = -|v(q)|$, one immediately obtains the result

$$k_F^c \, |v(0)| = 27c^2h^2/m.$$

Introducing spin and isospin one has, instead of Eq. (3-10), the equation

$$Q_{\mathrm{RPA}}^{\tilde{S}\tilde{T}}(q;\omega) = Q_0(q;\omega)\,[1 + (\hbar\Omega)^{-1}(-)^{\tilde{S}+\tilde{T}}\, v_{\tilde{S}\tilde{T}}(q) Q_0(q;\omega)]^{-1}, \qquad (3\text{-}12)$$

$$v_{\tilde{S}\tilde{T}}(q) \equiv \sum_{S,T} (2S+1)(2T+1)(-)^{S+T} \begin{Bmatrix} 1/2 & 1/2 & S \\ 1/2 & 1/2 & \tilde{S} \end{Bmatrix} \begin{Bmatrix} 1/2 & 1/2 & T \\ 1/2 & 1/2 & \tilde{T} \end{Bmatrix} \mu_{ST}(q), \qquad (3\text{-}13a)$$

$$\mu_{ST}(q) \equiv \int d^3r\, v_{12}^{ST}(r)\, e^{-i\mathbf{q}\cdot\mathbf{r}}, \qquad (3\text{-}13b)$$

where $v_{12}^{ST}(r)$ is the spin-isospin-dependent N-N potential which satisfies the Pauli principle through the simple requirement that $1 + S + T = \text{odd}$, 1 being the relative angular momentum, S and T the total spin and isospin of the two nucleons, and the 6-j symbols are the standard ones. The quantum numbers $\tilde{S}$, $\tilde{T}$ characterize the collective excitation boson. The poles of Eq. (3-12), for ω=O, are then given by

$$g_{\tilde{S}\tilde{T}}(y) \equiv (-)^{\tilde{S}+\tilde{T}+1}\, v_{\tilde{S}\tilde{T}}(yk_F)\, \frac{mk_F}{(2\pi\hbar)^2} \left[1 + \frac{4-y^2}{4y} \ln \left|\frac{y+2}{y-2}\right|\right] = 1, \qquad (3\text{-}14)$$

in which we define the dimensionless variable $y \equiv q/k_F$.

Simplicity dictates that for $v_{12}^{ST}(r)$, in Eq. (3-13), we use the purely central, even-state Tubingen potential[24] which is a superposition of three gaussians (one short-ranged repulsive plus two longer-ranged attractive) for each, singlet and triplet, states: the six adjustable parameters in each state are fitted[24] to N-N phase-shift and deuteron data. In order to maintain consistency with the long-range assumption cited before to justify predominance of ring diagrams, we suppress* the (soft)repulsive

*This procedure is not essential, since straighforward use of a Skyrme-Moszkowski "modified delta interaction"[39], which in addition saturates nuclear matter at the correct density, yields a critical density *smaller* by an order of magnitude.

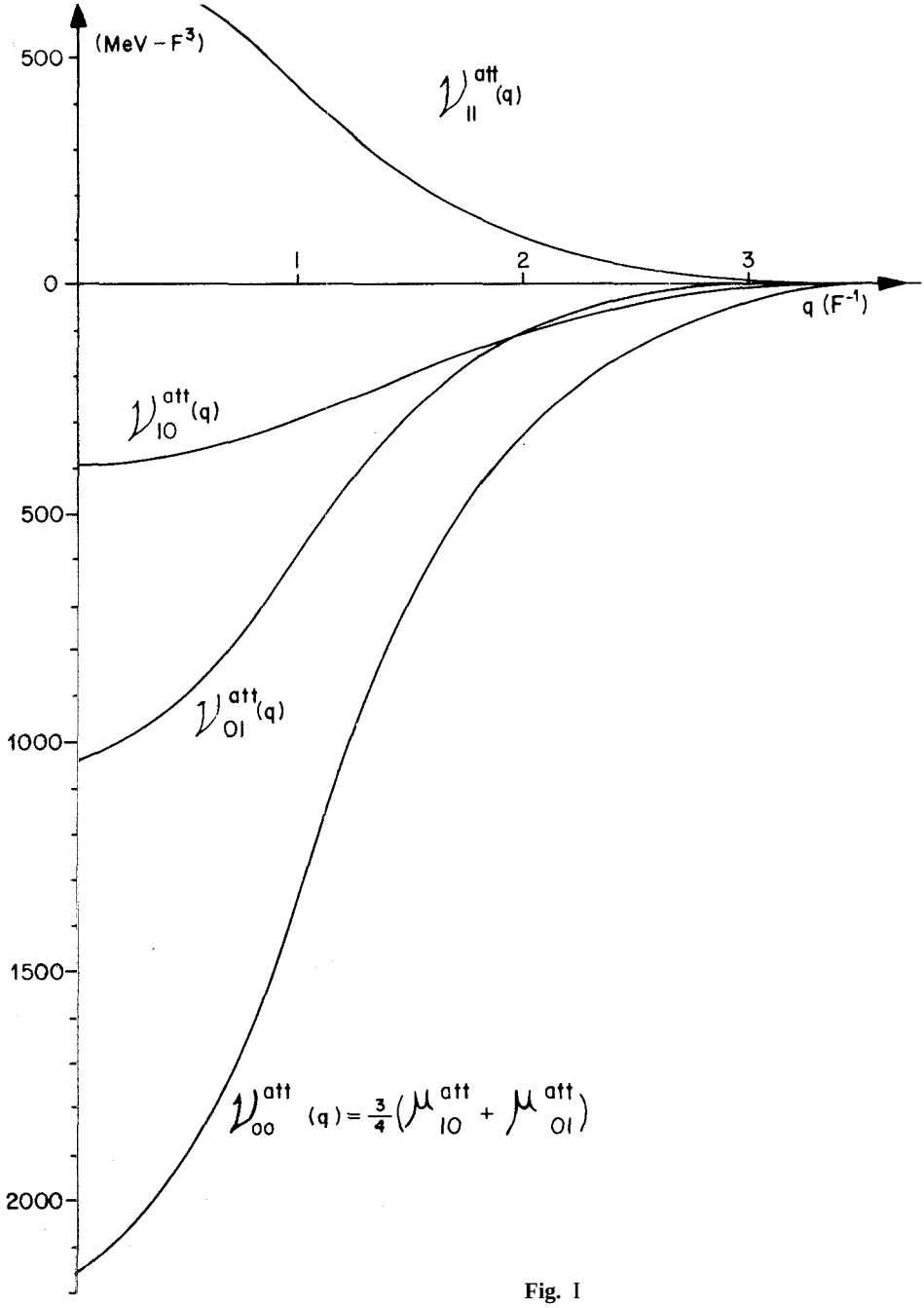

(MeV - F³)
500
0
1
2
3
q (F⁻¹)
$V_{11}^{att}(q)$
$V_{10}^{att}(q)$
500
$V_{01}^{att}(q)$
1000
1500
$V_{00}^{att}(q) = \frac{3}{4}(\mu_{10}^{att} + \mu_{01}^{att})$
2000

Fig. I

cores but account for their presence in a rough way by reducing the total available volume by the corresponding "excluded volume" given by $A\frac{2\pi}{3}r_0^3$, with r_0 taken (roughly) as the interparticle distance at distance at which the Tübingen potential changes sign ($r_0 \simeq 0.67$ fm) so that the *effective* ***density*** becomes

$$\rho = A\left[\Omega - A\frac{2\pi}{3}r_0^3\right]^{-1} = \rho\left[1 - \frac{2\pi}{3}r_0^3\rho\right]^{-1}. \tag{3-15}$$

As a consequence of having no odd states, in the adopted interaction, one has only (ST)= (01) and (10) and this leads, in Eq. (3-13), to the collective excitation modes having $(\tilde{S}\tilde{T}) = (00)$, (01), (10) and (11). Figure 1 shows $v_{\tilde{S}\tilde{T}}^{\mathrm{att}}(q)$ for the ***attractive*** gaussians of the Tübingen potential in the four excitation modes mentioned. Figure 2 displays the function in the square brackets of Eq. (3-14), and is seen to be non-increasing in y. Thus, Eq. (3-14) can be satisfied only if $(-)^{\tilde{S}+\tilde{T}+1}v_{\tilde{S}\tilde{T}}^{\mathrm{att}}(q)$

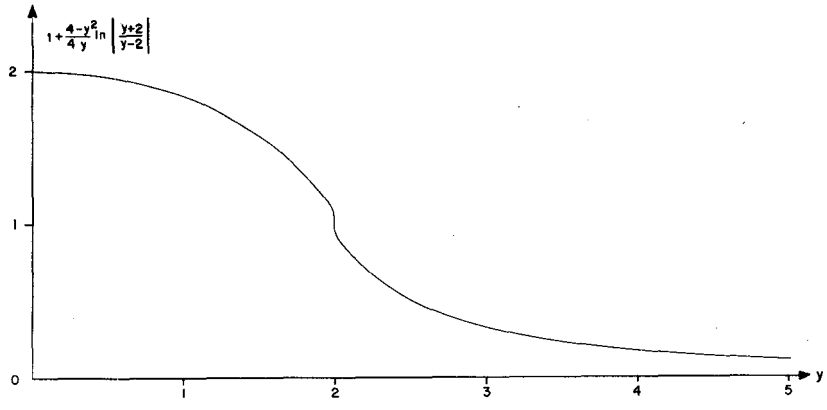


Fig. 2

is *positive;* Figure 1 shows that this occurs only for the $(\tilde{S}, \tilde{T}) = (0, 0)$ mode, i.e., for the ordinary (spacial) density-fluctuation. (The remaining modes correspond to various spin-isospin-density fluctuations but are relevant *only* if the interaction were *repulsive.)* A purely repulsive interparticle force will not, of course, produce *spacial* condensation as $-\nu_{00}(q)$ is then always negative and Eq. (3-14) has no solution. Finally, Figure 3 shows the graphical solution of Eq. (3-14) for several selected values of $y = q/k_F \geq 0$. Combining Eqs. (1-5) and (3-15) one may express the effective density $\tilde{\rho}$ in terms of k_F; the lowest value of k_F for which a solution exists indeed occurs for $q = 0$, i.e., when both the bracketed function in Eq. (3-14) as well as $-\nu_{00}^{\mathrm{att}}(k_F y)$ are maximized. One then has

$$k_F^c \left| \mu_{01}^{\mathrm{att}}(0) + \mu_{10}^{\mathrm{att}}(0) \right| = 8\pi^2\hbar^2/3m,$$

so that this value of k_F^c may be used in Eqs. (1-5) and (3-15) to determine α. Note that the onset of density fluctuations in the Fermi gas are here determined *exclusively* by the volume integral of the attractive parts of the N-N interaction: a *weaker* attractive portion makes for a smaller slope of the $y = 0$ line in Figure 3 and thus gives a *higher* critical density. *The striking result is that condensation appears at a density of only* $\tilde{\rho}_c \simeq 0.03\rho_s$, where ρ_s is the known saturation density of nuclear matter.

In determining the critical density at which the unstable density fluctuations appear, we have made the crucial assumption that, among *all* the polarization diagrarns which together would give the *exact* particle-hole Green's function, Eq. (3-1), the ring diagrams are the most important. The justification of this assumption *seemingly* relies on both large gas density and *long* interparticle force range. However, the resulting instability condition is very meaningful as it is related with the condition that a plane-waves determinantal zero-order state ceases to minimize the energy of the non-perturbed single-particle Hamiltonian, leading thus to *the breakdown of any HF perturbational approach* based on a plane-waves determinant. Detailed consideration, from this viewpoint, of the gas-to-liquid phase transition in a many--nucleon system is found in Refs[23], where the critical density found here is related to the *end* of the gaseous metastability branch.

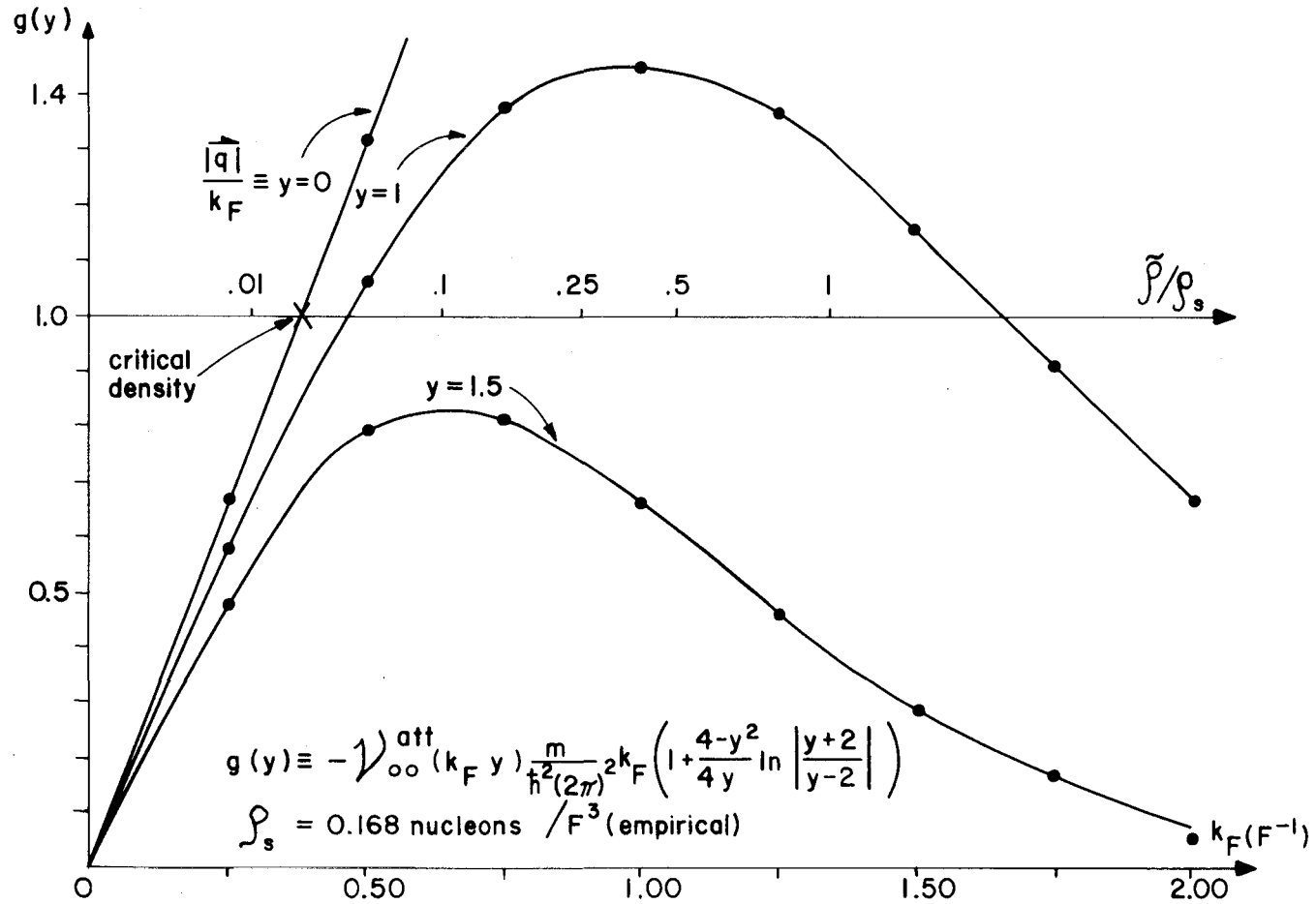

g(y)
1.4
1.0
0.5
0
0.50
1.00
1.50
2.00
$k_F(F^{-1})$
$\frac{|\vec{q}|}{k_F} \equiv y=0$
y=1
y=1.5
.01
.1
.25
.5
1
$\tilde{\rho}/\rho_s$
critical density
$g(y) \equiv -\nu_{oo}^{att}(k_F y)\frac{m}{\hbar^2(2\pi)^2}k_F\left(1+\frac{4-y^2}{4y}\ln\left|\frac{y+2}{y-2}\right|\right)$
ρ_s = 0.168 nucleons $/F^3$ (empirical)

Fig. 3

4. Analogy with the Virial Expansion for a Real Gas and Condensation Theory

It is well-known that the idea of expanding physical quantities of a many-body system as a series in the density, originated in the study[26] of a classical gas system under a given total potential energy function. In the simplest case, additivity is assumed for the potential energy, as a sum of pair potentials $v(r_{ij})$ taken to be, e.g., a Lennard-Jones shape having a short-ranged attraction ($\sim r_{ij}^{-6}$) plus a strong repulsion at shorter separations ($\sim r_{ij}^{-12}$) and a minimum value at an equilibrium distance r*. Starting from the partition function, which is just the phase-space integral of $\exp(-H/kT)$, where k is the Boltzmann's constant, T the absolute temperature and $H = \sum_i T_i + \sum_{i>j} v_{ij}$ the classical Hamiltonian function, Mayer[26] was able to *deduce* the heretofore semi-empirical equation of state for real gases

$$\frac{pv}{kT} = 1 + B_2(T)\,(\rho/\rho_0) + B_3(T)\,(\rho/\rho_0)^2 + \ldots,$$

$$\rho \equiv v^{-1};\ \rho_0 = [(4\pi/3)\,(r^*/2)^3]^{-1} \equiv v_0^{-1}. \tag{4-1}$$

The so-called virial coefficients $B_2(T)$, $B_3(T)$, ..., for the classical gas turn out to be universal functions of kT/ε, where $(-E)$ is the minimum value of the two-body potential, so that pv/kT is a universal function of kT/ε and ρ/ρ_0 (law of corresponding states[29]). (For a quantum gas[29], of either fermions or bosons, one still has an expansion of the type of Eq. (4-1) except that now the coefficients B_2, B_3, ..., depend in addition on the dimensionless quanturn parameter $\hbar/\sqrt{m\varepsilon}\,r^*$ where $\hbar/\sqrt{m\varepsilon}$ is essentially the de Broglie wavelength of a nearest-neighbor pair of molecules with relative kinetic energy E.) For the classical case at least[30], the coefficients B_2, B_3, ..., are fully determined by quadratures from a given interparticle potential function v_{ij}, viz.,

$$B_2 = -\left(\frac{\pi r^{*3}}{3}\Omega\right)^{-1} \int d^3r_1 \int d^3r_2\,[\exp(-v_{12}/kT) - 1],$$

etc.. Now, there evidently exists a close analogy between the virial expansion, Eq. (4-I), in ρ/ρ_0, and the Brueckner – Bethe – Goldstone series, Eq. (2-13), in $\kappa = \rho/\rho_{corr}$.

The virial expansion, Eq. (4-I), has turned out to be very useful[28] in predicting the thermodynamic properties of real but *dilute* gases

(under the assumption of pairwise additivity: deviations between calculated and experimentally-deduced higher-order virial coefficients, from B_3 on, have been attributed at least in part to this restrictive assumption). But even putting this latter discrepancy aside and taking the pairwise attractive-plus-repulsive force as a model, it is presently known[31] that the radius of convergence of Eq. (4-1) is considerably *less* than unity, that is, it converges only within the gas region. In the early days of the Mayer theory, attempts[26] were made to extend the virial expansion beyond its convergence circle in order to get a liquid eqiiation of state, as well as determine the condensation density. This extrapolation lead to the general result[32,34] that, in the pressure--volume diagram, all isotherms below the critical one were horizontal for all volumes below the predicted condensation volume, meaning that the theory was apparently incapable of producing a pure liquid phase (for which the isotherm must of course rise rapidly with decreasing volume). The failure of these attempts was not clearly understood until Lee and Yang[33] studied the analytical properties of the virial grand partition function with careful emphasis on the $\Omega \to \infty$ limit. They produced a very probable explanation of why the Mayer virial expansion theory failed in the pure liquid region: the pure liquid isotherm is not continuable analytically the pure vapor isotherm, as the two are in fact separated (below the critical temperature) by a flat segment along which both phases coexist in equilibrium. The difficulty with the Mayer theory in constructing a correct liquid theory is thus clear: it is an unjustified extrapolation from the low-density gaseous region. Attempts starting from the high-density crystalline side (viz., the so-called "hole", "cell" and "cell-cluster" theories) have since appeared[35] but to date have had only limited success.

5. Conclusions

Throughout the history of nuclear physics there has existed the tendency[36] to associate the gas phase with independent-particle behavior and the liquid phase with strong-interaction or collective behavior, both aspects being considered to a large extent self-contradictory. The former association is probably due to the notion[16] that "strong, short-ranged, forces are not compatible with independent-particle motion", so that one is somewhat obliged to adopt a low-density, Fermi gas theory, e.g., the "independent-pair model" or, basically, the

BHF theory. The natural step beyond this point was the BBG density--power-expansion theory. But, since no *a priori* assurance of the convergence of this theory exists, we feel that a meaningful treatment of the nuclear problem by successive approximations will be correctly initiated only if the basic physical properties of the *nuclear liquid state* are incorporated (viz., the *surface* and a *fixed density)* from the very beginning into the unperturbed, zero-order Hamiltonian. A "strong-interaction" model, like a quantum-liquid theory, based on a self--consistent, single-particle, potential well, deduced, e.g., by a Hartree--Fock approximation, with non-singular N-N forces, as carried out for finite systems[37], gives not only the surface property and a fixed density but, moreover, avoids all considerations related to clusterings as would be appropriate in dilute gases. The ordinary HF approach is furthermore capable in principle of reconciling both individual particle and collective aspects of real nuclei.

Great care, however, must be exercised in avoiding the use of plane--waves (or trivial) solutions to the HF equations, for an infinite system, as the existence of non-plane-wave (or non-trivial) solutions has been established[38] for a wide variety of both repulsive as well as attractive two-body interactions. Unfortunately, none of these are quadratically--integrable, over a very large but finite volume Ω, in such a way as to guarantee a *proper* surface, i.e., one that will prevent the orbitals from "leaking to infinity" if the "walls" of a normalization volume Ω are receded. This latter deficiency is most clearly manifested in the fact that, in every nuclear matter calculation, the energy per particle *increases* for densities below the predicted saturation density, instead of remaining constant[40].

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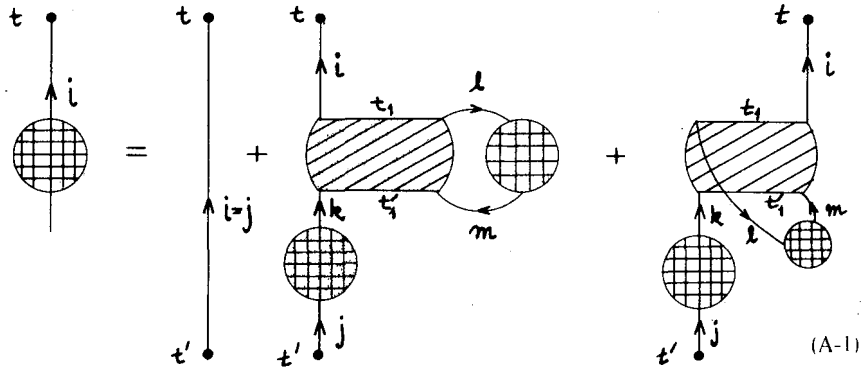
Appendix. Green's Function Derivation of Brueckner Theory[41]

The Brueckner-Hartree-Fock equations, (2-7), as interpreted by Bethe, Brandow and Petschek[10], correspond to a single-particle HF-like picture (consisting of *stable* quasi-particles) in which short-range correlations of the Brueckner ladder summation kind are included. To arrive at this scheme, several approximations must be made and these

are appreciated more lucidly by starting with the one-particle Green's function defined, diagramatically, by



(A-1)

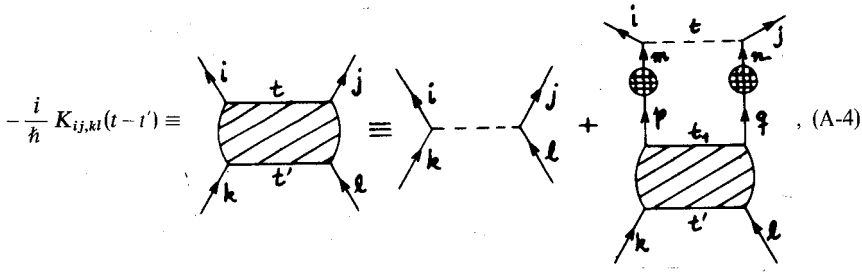
and, formally, by

$$G_{ij}(t-t') = G^0_{ij}(t-t')\,\delta_{ij} +$$

$$+\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt'_1 \sum_{klm} \tilde{K}_{il,km}(t_1-t'_1)\, G^0_i(t-t_1)\, G_{ml}(t'_1-t_1-0) G_{kj}(t'_1-t'), \tag{A-2}$$

$$\tilde{K}_{il,km}(\tau) \equiv K_{il,km}(\tau) - K_{il,mk}(\tau). \tag{A-3}$$

The shaded rectangle, in Eq. *(A-1),* is *a* generalized Brueckner reaction matrix defined as



(A-4)

the dashed line being the bare interaction v_{ij}. Formally, Eq. *(A-4)* is written as

$$K_{ij,kl}(t-t') = v_{ij,kl}\,\delta(t-t') -$$

$$-\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt_1 \sum_{pqmn} v_{ij,mn}\, G_{mp}(t-t_1)\, G_{nq}(t-t_1)\, K_{pq,kl}(t_1-t'). \tag{A-5}$$

This formulation is more general than the usual BHF theory in that: a) both particle--particle as well as hole-hole ladders are included since, in Eq. *(A-4),* both $t > t_1$ and $t < t_1$ are possible; *b)* ladder insertions are possible as well as particle lines, as is clear from Eqs. *(A-1)* to (A-3).

Assume the K-operator, of Eq. *(A-4)*, to act *instantaneously:*

$$\tilde{K}_{il,km}(t-t') = \delta(t-t')\tilde{K}^{\mathrm{BHF}}_{il,km}. \tag{A-6}$$

Then, Eq. *(A-2)* becomes an ordinary HF propagator in which v_{12} is replaced by K^{BHF}_{12} and can be written as

$$G^{\mathrm{BHF}}_{ij}(t-t') = \sum_k c_{ik}\ c^*_{jk}\ \exp\left[-\frac{i}{\hbar}\varepsilon_k(t-t')\right]\{(1-n_k)\,\theta(t-t') - n_k\,\theta(t'-t)\}$$
$$= \theta(t-t')\ \tilde{G}^{\mathrm{BHF}}_{ij}(t-t') + \theta(t'-t)\ \tilde{G}^{\mathrm{BHF}}_{ij}(t-t'), \tag{A-7}$$

where n_k are occupation numbers, the ε's are single particle energies and, the c's, amplitude coefficients in the single-particle states

$$\varphi_i(x) = \sum_j c_{ji}\psi_j(x),$$

where

$$\psi_i(x) \equiv \Omega^{-1/2}\ \exp(i\mathbf{k}_i \cdot \mathbf{r})\ \chi_{si}(\sigma_z)\ \chi_{\tau i}(\tau_z).$$

The c's and ε's satisfy the BHF equations

$$(T_i - \varepsilon_j)\,c_{ij} + \sum_{klm}\tilde{K}^{\mathrm{BHF}}_{il,km}\Big(\sum_s c_{ms}\,c^*_{ls}\,n_s\Big)\,c_{kj} = 0. \tag{A-8}$$

To obtain the total ground state energy, in the BHF theory, we proceed as follows. Consider the exact retarded Green's function $\tilde{G}_{ij}(t-t') \equiv -\langle a^+_j(t')a_i(t)\rangle$, where the fermion operators are in Heisenberg representation, and the expectation value between exact ground state wave functions. If the total Hamiltonian is $H(g) = \mathrm{T} + v(g)$, then

$$\frac{\hbar}{i}\frac{\partial}{\partial t}\tilde{G}_{jj}(t,t'\,|\,g) = -\frac{\hbar}{i}\frac{\partial}{\partial t}\langle a^+_j(t')a_j(t)\rangle = -\langle a^+_j(t')\,[H, a_j(t)]\rangle, \tag{A-9}$$

where, now, expectation values are between ground state solutions of $H(g)$. Now, the commutator is just

$$[H, a_j(t)] = -T_j\,a_j(t) - g\sum_{klm} v_{jk,lm}\,a^+_k(t)\,a_m(t)\,a_l(t), \tag{A-10}$$

so that, from Eq. *(A-9)*, one can form the relation

$$\frac{1}{2}\sum_j\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial t} + T_j\right)\tilde{G}_{jj}(t,t'\,|\,g)\right]_{t'=t} = \frac{g}{2}\sum_{jklm} v_{jk,lm}\,\langle a^+_j(t)\,a^+_k(t)\,a_m(t)\,a_l(t)\rangle$$
$$\equiv g\langle\Psi_0(g)\,|\,v\,|\,\Psi_0(g)\rangle, \tag{A-11}$$

where $[E_0(g) - H(g)]\Psi_0(g) = 0$ and $E_0(1)$ is the exact ground state energy. But if W_0 is the unperturbed energy, then

$$\frac{d}{dg}\Delta E_0(g) = \frac{d}{dg}\,[W_0 + \Delta E_0(g)] = \frac{d}{dg}\,\langle\Psi_0(g)\,|\,T + gv\,|\,\Psi_0(g)\rangle$$

equals $\langle\Psi_0(g)\,|\,v\,|\,\Psi_0(g)\rangle$ and, thus, the energy shift is

$$\Delta E_0 = \int_0^1 d\Delta E_0(g) = \int_0^1 dg\ \langle\Psi_0(g)\,|\,v\,|\,\Psi_0(g)\rangle \tag{A-12}$$

and, combining it with (A-12), one has the formula

$$\Delta E_0 = \frac{1}{2} \int_0^1 \frac{dg}{g} \sum_j \left(\frac{\hbar}{i} \frac{\partial}{\partial t} + v_j \right) \tilde{G}_{jj}(t, t' | g)_{t'=t} , \tag{A-13}$$

giving the exact energy shift solely in terms of the exact one-particle retarded Green's function. This can, now, be used with Eqs. **(A-7)** and (A-8), to yield

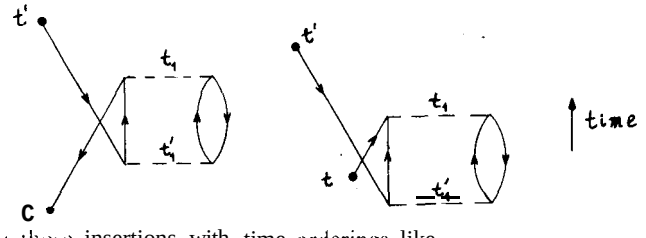
$$E_0^{\mathrm{BHF}} = \sum_i \varepsilon_i n_i - \frac{1}{2} \sum_{mnpq} \tilde{K}^{\mathrm{BHF}}_{mn,pq} \left(\sum_r c^*_{mr} c_{pr} n_r \right) \left(\sum_s c^*_{ns} c_{qs} n_s \right), \tag{A-14}$$

which, together with (A-8), constitutes the basic equations of BHF theory. Note that the basic assumption, Eq. **(A-6),** is, strictly speaking, ***inconsistent*** with Eq. **(A-5),** since Eq. **(A-7)** siibstituted into **(A-5),** gives

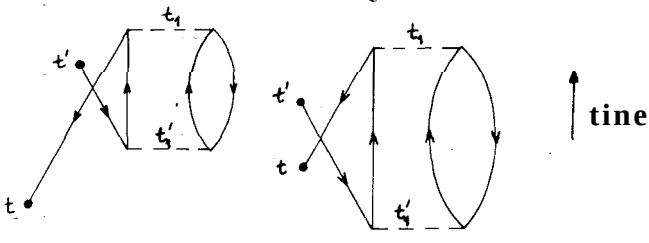
$$\begin{aligned} K_{ij,kl}(\omega) &\equiv \int_{-\infty}^{+\infty} dt \; e^{i\omega t} \; K_{ij,kl}(t) \\ &= v_{ij,kl} - \sum_{mnpq} v_{ij,mn} \sum_{rs} c_{ms} \, c^*_{ps} \, c_{nr} \, c^*_{qr} \times \\ &\times \left\{ \frac{(1-n_s)\,(1-n_r)}{\varepsilon_r + \varepsilon_s - \omega + i\eta} - \frac{n_r \, n_s}{\varepsilon_r + \varepsilon_s - \omega - i\eta} \right\} K_{pq,kl}(\omega), \end{aligned} \tag{A-15}$$

which is energy-dependent as opposed to Eq. **(A-6).** Standard BHF theory ignores the last term in Eq. **(A-15)** and assumes an "average" constant value of ω to reconcile Eqs. **(A-15)** and **(A-6).** The presence of the last term in Eq. (A-15), i.e., of hole-hole ladders, ***destroys the wellknown healing phenomenon,*** which occurs if only the particle-particle ladders are kept.

The Bethe-Brandow-Petschek theory is a less restrictive interpretation of Eqs. **(A-1)** to **(A-5)** than is the BHF theory and also leads to a conventional single-particle picture, i.e., ***stable*** quasiparticles. The assumptions made here are: a) to neglect completely, in Eq. **(A-1),** all hole-hole ladders; b) to replace all ***particle*** propagators by their zero--order propagator G^0; c) in dressing the bare ***hole*** lines, to consider only a ***certain class*** of insertions with time ordering, like, e.g., for second-order insertions:



but *neglect* those insertions with time orderings like



In other words, in the BBP theory only hole lines appear dressed and, then, only with a restricted class of particle–particle ladders. Eqs. (A-1) to (A-5) then become: *for* $t < t'$ (holes),

$$G_{ij}^{\mathrm{BBP}}(t-t') = G_i^0(t-t')\,\delta_{ij} +$$

$$+ \frac{i}{\hbar}\int_{-\infty}^{t'} dt_1 \int_{-\infty}^{t_1} dt_1' \sum_{klm} \tilde{K}_{il,km}^{\mathrm{BBP}}(t_1-t_1')\, G_i^0(t-t_1)\, G_{ml}^{\mathrm{BBP}}(t_1'-t_1-0) \times G_{kj}^{\mathrm{BBP}}(t_1'-t_1), \qquad (A\text{-}16)$$

while, for $t > t'$ *(particles),*

$$G_{ij}^{\mathrm{BBP}}(t-t') = G_i^0(t-t')\,\delta_{ij}, \qquad (A\text{-}17)$$

where

$$K_{ij,kl}^{\mathrm{BBP}}(t-t') = 0 \quad \textbf{for} \quad t < t', \qquad \textbf{(A-18)}$$

and, for $t > t'$,

$$K_{ij,kl}^{\mathrm{BBP}}(t-t') = v_{ij,kl}\,\delta(t-t'-0) -$$

$$- \frac{i}{\hbar}\int_{t'}^{t} dt_1 \sum_{pq} v_{ij,pq}\; G_p^0(t-t_1)\; G_q^0(t-t_1)\; K_{pq,kl}^{\mathrm{BBP}}(t-t_1). \qquad (A\text{-}18)$$

Eqs. (A-18) and (A-18') lead to

$$K_{ij,kl}^{\mathrm{BBP}}(\omega) = v_{ij,kl} - \sum_{pq} v_{ij,pq}\,\frac{(1-n_p)(1-n_q)}{T_p + T_q - \omega}\; K_{pq,kl}^{\mathrm{BBP}}(\omega). \qquad \textbf{(A-19)}$$

Eqs. (A-16) and (A-17) are now solved by writing

$$G_{ij}^{\mathrm{BBP}}(t-t') = \exp\left[-\frac{i}{\hbar}T_i(t-t')\right](1-n_i)\,\theta(t-t')$$

$$- \sum_s c_{is}\, c_{js}^*\, n_s\, \exp\left[-\frac{i}{\hbar}\varepsilon_s(t-t')\right]\theta(t'-t) \qquad (A\text{-}20)$$

$$\equiv \theta(t-t')\,\tilde{G}_{ij}^{\mathrm{BBP}}(t-t') + \theta(t'-t)\,\tilde{G}_{ij}^{\mathrm{BBP}}(t-t'),$$

the result being that,

for $n_s = n_i = 1$,

$$(T_i - \varepsilon_s)\,c_{is} + \sum_{klm}\sum_r \tilde{K}_{il,km}^{\mathrm{BBP}}(\varepsilon_r + \varepsilon_s)\,c_{mr}\,c_{kr}^*\,n_r\,c_{ks} = 0; \qquad (A\text{-}21)$$

for $n_s = n_i = 0$,

$$c_{is} = \delta_{is},\ \varepsilon_i = T_i \equiv \hbar^2 k_i^2/2m; \qquad (A\text{-}22)$$

for either $n_s = 0,\ n_i = 1$ *or* $n_s = 1,\ n_i = 0$,

$$c_{is} = 0. \qquad (A\text{-}23)$$

Finally, the total ground state energy becomes, with the use of Eq. (A-13),

$$E_0^{\mathrm{BBP}} = \sum_i{}' \varepsilon_i n_i - \frac{1}{2} \sum_{mnpq}{}' \sum_{rs}{}' \tilde{K}^{\mathrm{BBP}}_{mn,pq} (\varepsilon_r + \varepsilon_s) (c^*_{mr} c_{pr} n_r) \times (c^*_{ns} c_{qs} n_s). \tag{A-24}$$

For nuclear matter, describabie by a translational-invariant single-particle potential, all c_{α} become δ_{mr}.

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