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Relativistic Phase Space for Bosons and Fermions*

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A formal direct derivation of the quantum phase space for an ideal relativistic gas satisfying Bose-Einstein or Fermi-Dirac statistics is given. The result is a cluster decomposition in Boltzmannian phase spaces. The fundamental role of the symmetric group is stressed.

É feita uma derivação formal direta do espaço de fase para um gás ideal relativístico satisfazendo seja a estatística de Bose-Einstein ou a de Fermi-Dirac. Daí resulta uma decomposição a maneira de Mayer em espaços de fase do tipo Boltzmann. Enfatiza-se o papel fundamental do grupo simétrico.

1. Introduction

In high energy physics literature, the expression ***phase space*** is usually employed to indicate the phase space factor, that is, the momentum part of the whole phase space to which also the configuration space contributes. In this sense, it is commonly presented under two forms[1,2], the invariant and non-invariant ones. Both are useful in the calculation of transition probabilities once convenient normalizations for the states are used, but only the second one is acceptable in Relativistic Statistical Mechanics[3]. The reason for that is the non-invariance of the configuration space.

Yet, another difficulty arises concerning the phase spaces commonly used in Fermi-like models and in other multiple-production calculations: they do not account for the syrnmetrization of states imposed by quantum mechanics[4]. They correspond to the classical limit of quantum statistics.

Questions of statistics are best visualized in the partition function. In

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the canonical ensemble, it is, in the relativistic case, the four-dimensional Laplace transform of the whole phase space:

$$Q_N(\beta, V) = \int d^4P e^{-\beta P} R_N(P, V). \tag{1-1}$$

Here, β is the inverse-temperature four-vector. It is shown, in section 2, that the commonly used phase space leads to Boltzmann statistics.

The aim of this paper is to obtain in a direct, constructive, way the correct N-particle phase space for bosons and fermions and to exhibit the fundamental role played by the symmetric group. The same result was recently obtained by Chaichian, Hagedorn and Hayashi[5] by an inverse transformation procedure: $R_N(P, V)$ is given as a sum of the comrnonly used phase spaces. In our case, the coefficients in the sum are shown to be fixed by the symmetric group. This is the subject of section 3. In section 4, it is proved that the proposed R_N is indeed the correct one: it leads to the grand partition function for an ideal bosonic or fermionic gas. Section 5 is devoted to discuss further applications of our formal procedure.

2. Ordinary Invariant Phase Space and Statistics

We shall here be concerned with integrals of the type

$$I_N(P) = \int \prod_{i=1}^{N} \frac{d^3p_i}{2e_i} \delta^4 \left(P - \sum_{i=1}^{N} p_i\right) T. \tag{2-1}$$

They are related to cross-sections in the case that T is a squared transition matrix element and to the N-particle phase space when T is a convenient function of the volume. In particular, the whole invariant phase space currently used is

$$R_N^{(0)}(P, V) = \frac{1}{N!}\left(\frac{V}{h^3}\right)^N \int\left(\prod_{i=1}^{N} d^3p_i\right) \delta^4 \left(P - \sum_{i=1}^{N} p_i\right). \tag{2-2}$$

Of course $\mathbf{d^3p}$ is non-invariant, but so is V, in a way such as to make Vd^3p invariant. If we calculate the canonical partition function with the'suitable choice $(\beta, 0, 0, 0)$ for the inverse temperature four-vector,

$$Q_N(\beta, V) = \int d^3P\, e^{-\beta E}\, dE\, R_N^{(0)}(E, P, V), \tag{2-3}$$

we get the result

$$Q_N(\beta, V) = \frac{1}{N!}\left(\frac{V}{h^3}\right)^N \int \prod_{i=1}^{N} d^3p_i \exp\left[-\beta(\mathbf{p}_i^2 + \mu^2)^{1/2}\right]$$

$$= \frac{1}{N!}\left[Q_1(\beta, V)\right]^N. \tag{2-4}$$

This is the canonical partition function for an ideal Boltzmann gas. The expression (2-2) is the version of the phase-space commonly used in Fermi-like models, as well as in many calculations in multiparticle production. The meaning of the result (2-4) is clear: Eq. (2-2) corresponds to the classical limit of quantum statistics, where "indistinguishability" of the particles is only partially taken into account via the Gibbs factor $(N!)^{-1}$. In the next section, we shall find the expression for the phase space leading to the correct Bose or Fermi statistics.

3. The Correct Invariant Phase Space

The canonical partition function of an ideal non-relativistic gas may be definecl as[6]

$$Q_N(\beta, V) = \mathrm{Tr}_N\, e^{-\beta H_0}, \tag{3-1}$$

where H_0 is the free Hamiltonian. The operator $\exp[-\beta H_0]$ can be written in terms of the resolvent of H_0 by the Cauchy formula

$$e^{-\beta H_0} = \frac{1}{2\pi i}\oint e^{-\beta z}\, dz\, \frac{1}{z - H_0}, \tag{3-2}$$

where the integration contour encircles the spectrum of H_0. It is easy to find that

$$e^{-\beta H_0} = \int_0^\infty dE\, e^{-\beta E}\, \delta(E - H_0), \tag{3-3}$$

so that

$$Q_N(\beta, V) = \int dE\, e^{-\beta E}\, \mathrm{Tr}_N\, \delta(E - H_0). \tag{3-4}$$

We shall assume that this formal procedure remains valid in the relativistic case. We use the momentum eigenstates $|p_1 p_2 \dots p_N\rangle$ properly normalized and symmetrized and calculate the trace with the constraint $\sum_{i=1}^{N} \mathbf{p}_i = \mathbf{P}$; that is,

$$Q_N(\beta, V) = \int d^4P\, e^{-\beta E}\, \mathrm{Tr}_N^{\mathbf{P}}\, \delta(E - H_0). \tag{3-5}$$

Comparison with (2-3) shows that

$$R_N(P, V) = \mathrm{Tr}_N^{\mathbf{P}}\, \delta(E - H_0). \tag{3-6}$$

The states $|p_1 p_2 \dots p_N\rangle$ are such that[7]

$$\langle p_1' p_2' \cdots p_N' \,|\, p_1 p_2 \cdots p_N\rangle =$$

$$= \left(\prod_{i=1}^{N} 2e_i\right)\left[\sum_{\{a_i\}} (\pm)^{\alpha}\, \delta^3(\mathbf{p}_1 - \mathbf{p}_{a_1})\, \delta^3(\mathbf{p}_2 - \mathbf{p}_{a_2}) \dots \delta^3(\mathbf{p}_N - \mathbf{p}_{a_N})\right]. \tag{3-7}$$

The sum is over all possible ways of pairing two momenta and the factor $(\pm)^{\alpha}$ comes from the statistics: $(+)$ for bosons, $(-)$ for fermions. The exponent, α, is the order of permutation of the set $\{a_1\, a_2 \dots a_N\}$.

Explicitly, Eq. (3-6) reads

$$R_N(P, V) = \frac{1}{N!} \int \left(\prod_{i=1}^{N} d^3p_i\right) \delta^3\left(\mathbf{P} - \sum_{i=1}^{N} \mathbf{p}_i\right) \times$$

$$\times \left(\prod_{i=1}^{N} 2e_i\right)^{-1} \langle p_1 p_2 \dots p_N \,|\delta(E - H_0)|\, p_1 p_2 \dots p_N\rangle. \tag{3-8}$$

The factor $\left(\prod_{i=1}^{N} 2e_i\right)^{-1}$ is due to the normalization (3-7).

The N-particle phase space becomes

$$R_N(P, V) = \frac{1}{N!} \int \left(\prod_{i=1}^{N} d^3p_i\right) \delta^3\left(\mathbf{P} - \sum_{i=1}^{N} \mathbf{p}_i\right) \delta\left(E - \sum_{i=1}^{N} e_i\right) \times$$

$$\times \left[\sum_{\{a_i\}} (\pm)^{\alpha}\, \delta^3(\mathbf{p}_1 - \mathbf{p}_{a_1})\, \delta^3(\mathbf{p}_2 - \mathbf{p}_{a_2}) \dots \delta^3(\mathbf{p}_N - \mathbf{p}_{a_N})\right]. \tag{3-9}$$

Some insight about the summation inside the square bracket may may come from the consideration of the first few cases. Putting $\delta_{kl} = \delta^3(\mathbf{p}_k - \mathbf{p}_l)$, we have:

$$N = 1\colon \sum_{\{a_i\}} = \delta_{11}\,,$$

$$N = 2\colon \sum_{\{a_i\}} = \delta_{11}\,\delta_{22} \pm \delta_{12}\,\delta_{21}\,,$$

$$N = 3\colon \sum_{\{a_i\}} = \delta_{11}\,\delta_{22}\,\delta_{33} \pm \delta_{11}\,\delta_{23}\,\delta_{32} \pm \delta_{13}\,\delta_{22}\,\delta_{31} \pm$$

$$\pm\, \delta_{12}\,\delta_{21}\,\delta_{33} + \delta_{12}\,\delta_{23}\,\delta_{31} + \delta_{13}\,\delta_{21}\,\delta_{32}\,.$$

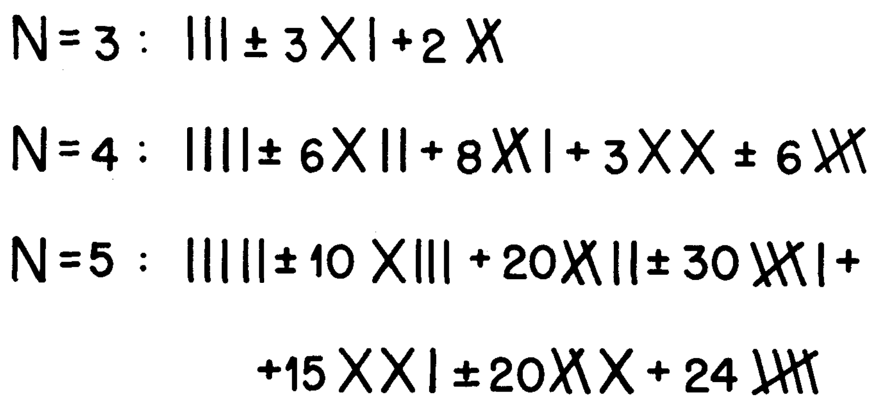

Fig. 1 - Graphical representation of the cluster decomposition of $\sum_{\{a_i\}}$ for $N = 3, 4$ and 5.

The integrations over the momenta put them on an equal footing. For instance, in the case N = 3, the first term represents each particle isolated; the following three terms exchange particles two-by-two and give the same contribution; the last two terms give also identical contributions and represent complete (anti-) symmetrization. So,

$$N = 3 : \sum_{\{a_i\}} = \delta_{11}\,\delta_{22}\,\delta_{33} \pm 3\delta_{12}\,\delta_{21}\,\delta_{33} + 2\delta_{12}\,\delta_{23}\,\delta_{31}\,.$$

It is known since long that quantum symmetrization may be viewed as an "interaction" (exchange interaction) imposed on an ideal Boltzmann gas. What has been shown above is, exactly, a cluster decomposition for that case (cases for N up to 5 are illustrated in Fig. 1).

The first term, for each N, is simply $[\delta^3(0)]^N = (V/h^3)^N$. It represents the case of N different momenta, where no (anti-) symmetrization is done and gives, if taken alone, precisely the Boltzmann case (Eq. 2-2).

The "interaction" causes permutations among the particles and permutations among N particles are related to the symmetric group S_N. More precisely, both summations, in the examples above and in the graphs in Fig. 1, represent the decomposition of S_N into its conjugate classes (see Fig. 2). Each conjugate class accounts for permutations in sub-systems of the N particles. As we have seen, there are three ways of permuting 3 particles two-by-two. This illustrates

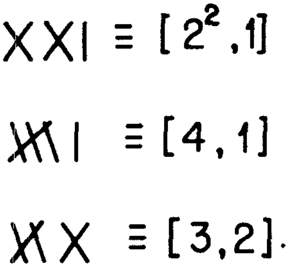

Fig. 2 - Typical graphs and its equivalence to terms of the decomposition of S_N into its conjugate classes.

the general fact that the coefficient before each graph in the decomposition is just the number of permutations of S_N in a given conjugate class, which is given by the well-known formula[8]:

$$c_{\{\nu_i\}} = N!\left[\prod_{j=1}^{N} j^{\nu_j}\nu^j\,!\right]^{-1}, \tag{3-10}$$

where ν_l is the number of l-cycles in the class. The ν_l's are subject to the important condition

$$\sum_{l=1}^{N} l\nu_l = N. \tag{3-11}$$

The nurnber

$$m = \sum_{l=1}^{N} \nu_l \tag{3-12}$$

counts the cycles in the class and will be useful below. The exponent of the sign factor (i—)is (N – m), as is easily seen. In this way we get, for the summation in Eq. **(3-9),**

$$\sum_{\{a_i\}} (\pm)^\alpha \delta^3(\mathbf{p}_1 - \mathbf{p}_{a_1})\, \delta^3(\mathbf{p}_2 - \mathbf{p}_{a_2})\, \delta^3(\mathbf{p}_N - \mathbf{p}_{a_N})$$

$$= \sum_{\{\nu_i\}}{}' (\pm)^{N-m} c_{\{\nu_i\}}\, \delta_1^{\nu_1}\, \delta_2^{\nu_2} \ldots\, \delta_N^{\nu_N}, \tag{3-13}$$

where the prime reminds us of condition (3-1**1).** The symbol δ_k **repre-** sents a product of δ^3-functions with suitable arguments, forming a k-cycle. Exemplifying:

$$\delta_3 = \delta^3(\mathbf{p}_l - \mathbf{p}_{l+1})\, \delta^3(\mathbf{p}_{l+1} - \mathbf{p}_{l+2})\, \delta^3(\mathbf{p}_{l+2} - \mathbf{p}_l).$$

It is convenient to impose Eq. **(3-12)** as an additional condition and sum over **m.** Eq. **(3-9)** becomes

$$R_N(P, V) = \frac{1}{N!} \int \left(\prod_{i=1}^{N} d^3p_i \right) \delta^4 \left(P - \sum_{i=1}^{N} p_i \right) \sum_{m=1}^{N} (\pm)^{N-m} \times$$

$$\times \sum_{\{\nu_i\}}{}'' c_{\{\nu_i\}}\, \delta_1^{\nu_1}\, \delta_2^{\nu_2} \ldots\, \delta_N^{\nu_N}.$$

The double primed summation is restricted by both Eqs. **(3-11)** and **(3-12).** We will now show that integration over **(N – m)** momenta yields a cluster decomposition of the correct phase space in terms of ordinary phase spaces with multiple masses. For N = 2, we have:

$$R_2(P, V) = \frac{1}{2!} \int d^3p_1\, d^3p_2\, \delta^4(P - p_1 - p_2)\, [\delta_1^2 \pm \delta_2].$$

However,

$$\delta_1 \equiv \delta^3(\mathbf{p}_1 - \mathbf{p}_1) = \delta^3(0) = V/h^3,$$

$$\delta_2 \equiv \delta^3(\mathbf{p}_1 - \mathbf{p}_2)\, \delta^3(\mathbf{p}_2 - \mathbf{p}_1),$$

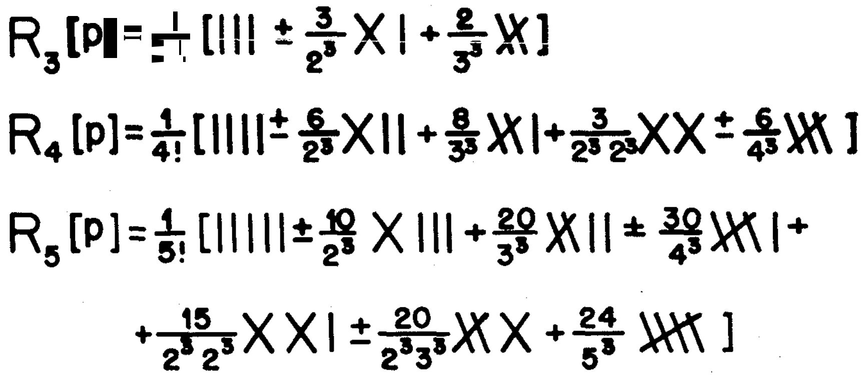

Fig. 3 - Graphical representation of the cluster decomposition of $R_N(P)$ for $N = 3$, 4 and 5.

and therefore,

$$R_2(P, V) = \frac{1}{2!}\left[\left(\frac{V}{h^3}\right)^2 \int d^3p_1\, d^3p_2\, \delta^4(P - p_1 - p_2) \pm \frac{V}{h^3}\int d^3p_1\, \delta^4(P - 2p_1)\right].$$

The first term is just the ordinary phase space, $R_2^{(0)}(P, V; \mu, \mu)$; the second is

$$\frac{V}{h^3}\int d^3p_1\, \delta^3(\mathbf{P} - 2\mathbf{p}_1)\, \delta(E - 2e_1)$$

$$= \frac{1}{2^3}\frac{V}{h^3}\int d^3p_1'\, \delta^3(\mathbf{P} - \mathbf{p}_1')\, \delta[E - (p_1'^2 + 4\mu^2)^{1/2}] = \frac{1}{2^3} R_1^{(0)}(P, V; 2\mu),$$

so that

$$R_2(P, V) = \frac{1}{2!}\left\{R_2^{(0)}(P, V; \mu, \mu) \pm \frac{1}{2^3} R_1^{(0)}(P, V; 2\mu)\right\}. \tag{3-15}$$

For $N = 3$, we obtain

$$R_3(P, V) = \frac{1}{3!}\left\{R_3^{(0)}(P, V; \mu, \mu, \mu) \pm \frac{3}{2^3} R_2^{(0)}(P, V; \mu, 2\mu) + + \frac{2}{3^3} R_1^{(0)}(P, V; 3\mu)\right\}. \tag{3-16}$$

Fig. 3 shows graphical representations for N up to 5. Each conjugate class of S_N is determined by an ensemble $\{v_i\}$ which is a solution of Eqs. (3-11) and (3-12) and corresponds to an ordinary phase space for a system of v_1 particles of mass μ, v_2 particles of mass 2μ and so on. To arrive at these simple expressions, a trivial transformation of variables is needed and it produces, for each class, an extra factor

$$\left[\prod_{j=1}^{N} j^{3v_j}\right]^{-1}$$

The final result is

$$R_N(P, V) = \sum_{m=1}^{N} \sum_{\{v_i\}}'' (\pm)^{N-m}\left[\prod_{j=1}^{N} j^{4v_j}\, v_j!\right]^{-1} \times$$

$$\times\, R_m^{(0)}(P, V; , . \quad \mu, 2\mu, \ldots, 2 \quad . \quad . \tag{3-17}$$

with

$$R_m^{(0)}(P, V; \mu, \ldots, \mu, 2\mu, \ldots, 2\mu, \ldots) =$$

$$= (V/h^3)^m \int \prod_{i=1}^{m} d^3p_i\, \delta^3(\mathbf{P} - \sum_{i=1}^{m} \mathbf{p}_i)\, \delta(E - \sum_{i=1}^{m} e_i). \qquad (3\text{-}18)$$

Notice that no factor $(m!)^{-1}$ appears. The multiple masses will show themselves in $e_i = (\mathbf{p}_i^2 + \mu_i^2)^{1/2}$, with

$$\mu_1 = \mu_2 = \ldots = \mu_{\nu_1} = \mu,$$
$$\mu_{\nu_1+1} = \mu_{\nu_1+2} = \ldots = \mu_{\nu_1+\nu_2} = 2\mu,$$

$$\mu_{\nu_1+\nu_2+\cdots+\nu_{N-1}+1} = \ldots = \mu_m = N\mu.$$

These cluster expansions give the corrections to the ordinary phase space due to statistics. For the 3-particle case, they have been evaluated numerically by Chaichian, Hagedorn and Hayashi[5] and found to be very significant.

It remains for us to show that the proposed expression for $R_N(P, V)$ does lead to the correct statistical mechanics.

Before going into this, we emphasize that we have not cared about spins and isotopic spins. They can be properly taken into account by the inclusion of multiplicative factors $g = (2I + 1)(2S + 1)$, in Eq. (3-6).
Eq. (3-6) can easily be extended to the case of a mixture of N_1 noninteracting particles of type 1, N_2 of type 2, etc. The general formula is

$$R_{N_1N_2\cdots N_k}(P, V) = \left(\prod_{i=1}^{k} g_i^{N_i}\right) \mathrm{Tr}^{\mathbf{P}}_{N_1N_2\cdots N_k}\, \delta(E - H_0). \qquad (3\text{-}19)$$

4. Partition Functions

The canonical partition function may be simply evaluated by Laplace transforming Eq. (3-17) but we shall prefer to follow a lengthier procedure which shows better the meaning of each factor in the resulting series. We substitute Eq. (3-14) into Eq. (2-3). By integrating on the total momentum and energy, one easily arrives at the expression

$$Q_N(\beta) = \frac{1}{N!}\int\left(\prod_{i=1}^{N} d^3p_i\, e^{-\beta e_i}\right)\sum_{m=1}^{N}(\pm)^{N-m}\sum_{\{\nu_i\}}{}'' c_{\{\nu_i\}}\,\delta_1^{\nu_1}\delta_2^{2}\ldots\delta_N^{\nu_N}. \qquad (4\text{-}1)$$

(We shall omit the argument V from now on).

This can be rearranged into

$$Q_N(\beta) = \frac{1}{N!}\sum_{m=1}^{N}(\pm)^{N-m}\sum_{\{\nu_i\}}{}'' c_{\{\nu_i\}}\left[\int d^3p_1\, e^{-\beta e_1}\,\delta_1\right]\times$$

$$\times\left[\int d^3p_1\, d^3p_2\, e^{-\beta(e_1+e_2)}\,\delta_2\right]^{\nu_2}\ldots\left[\int\left(\prod_{i=1}^{N} d^3p_i\, e^{-\beta e_i}\right)\delta_N\right]^{\nu_N}. \qquad (4\text{-}2)$$

We now see that

$$\int\left(\prod_{i=1}^{k} d^3p_i\, e^{-\beta e_i}\right)\delta_k = \int d^3p_1\, d^3p_2\ldots d^3p_k \exp\left(-\beta\sum_{i=1}^{k} e_i\right)\times$$

$$\times\,\delta^3(\mathbf{p}_1-\mathbf{p}_2)\,\delta^3(\mathbf{p}_2-\mathbf{p}_3)\ldots\delta^3(\mathbf{p}_{k-1}-\mathbf{p}_k)\;\delta^3(\mathbf{p}_k-\mathbf{p}_1)$$

$$= \delta^3(0)\int d^3p\, e^{-k\beta e} = Q_1(k\beta), \qquad (4\text{-}3)$$

so that

$$Q_N(\beta) = \frac{1}{N!}\sum_{m=1}^{N}(\pm)^{N-m}\sum_{\{\nu_1\}}{}'' c_{\{\nu_i\}}\, Q_1^{\nu_1}(\beta)\, Q_1^{\nu_2}(2\beta)\ldots Q_N^{\nu_N}(N\beta). \qquad (4\text{-}4)$$

As Bose or Fermi statistics are more easily recognizable in the grand canonical partition function, we proceed to calculate it:

$$Z = 1 + \sum_{N=1}^{\infty} e^{\beta\lambda N}\, Q_N(\beta)$$

$$= 1 + \sum_{n=1}^{\infty}\frac{e^{\beta\lambda N}}{N!}\sum_{m=1}^{N}(\pm)^{N-m}\sum_{\{\nu_i\}}{}'' c_{\{\nu_i\}}\, Q_1^{\nu_1}(\beta)\, Q_1^{\nu_2}(2\beta)\ldots Q_1^{\nu_N}(N\beta). \qquad (4\text{-}5)$$

This, as is easily verified, can be rewritten as

$$Z = 1 + \sum_{m=1}^{\infty}(\pm)^m\sum_{N=m}^{\infty}\frac{1}{N!}(\pm\, e^{\beta\lambda})^N\sum_{\{\nu_i\}}{}'' c_{\{\nu_i\{}\, Q_1^{\nu_1}(\beta)\ldots Q_1^{\nu_N}(N\beta).. \qquad (4\text{-}6)$$

We now recall one of the multinomial series[g], namely,

$$\left(\sum_{k=1}^{\infty}\frac{x_k}{k}\,t^k\right)^m = m!\sum_{N=m}^{\infty}\frac{t^N}{N!}\sum_{\{\nu_i\}}{}'' c_{\{\nu_i\}}x_1^{\nu_1}\, x_2^{\nu_2}\ldots x_N^{\nu_N}. \qquad (4\text{-}7)$$

Putting $t = \pm e^{\beta\lambda}$ and $x_k = Q_1(k\beta)$, Eq. (4-6) becomes

$$Z = 1 + \sum_{m=1}^{\infty} \frac{1}{m!}\left[\pm \sum_{k=1}^{\infty} \frac{(\pm e^{\beta\lambda})^k}{k} Q_1(k\beta)\right]^m$$

$$= \exp\left[\pm \sum_{k=1}^{\infty} \frac{(\pm e^{\beta\lambda})^k}{k} Q_1(k\beta)\right]\cdot \qquad (4\text{-}8)$$

By using Eq. (4-3), we can write

$$\ln Z = \pm \sum_{k=1}^{\infty} \frac{(\pm e^{\beta\lambda})^k}{k} \frac{V}{h^3} \int d^3p\, e^{-k\beta e}$$

$$= \frac{V}{h^3} \int d^3p \left\{\pm \sum_{k=1}^{\infty} \frac{(\pm)^k}{k} \exp\left[-k\beta(e-\lambda)\right]\right\}\cdot \qquad (4\text{-}9)$$

It is now enough to recognize the logarithmic series inside the curly bracket to write

$$\ln Z = \mp \frac{V}{h^3} \int d^3p \ln\{1 \mp \exp [\![-\beta\left[(\mathbf{p}_2 + \mu^2)^{1/2} - \lambda\right]]\!]\}, \qquad (4\text{-}10)$$

the correct relativistic grand canonical partition function for bosons (upper sign) or fermions (lower sign).

5. Final Comments

We have given a constructive derivation for the correct expression for the N-particle relativistic phase space. The fundamental role of the symmetric group was made explicit. The procedure is equivalent to a cluster expansion for the exchange interaction, the first term of which gives the ordinasy phase space comrnonly used in calculations in multiple production and in Fermi-like models. The higher order corrections have been shown by the authors of Ref. (5) to be very important and an obvious step forward would be to examine their consequences on Ferrni rnodels and general distributions. The forms obtained are compact and convenient for computation.

The cluster decomposition has an interest by itself, as it can be used as a model to guide the introduction of real interactions in a way alike to Lee-Yang binary method[6]. For binary interactions, one simply has to replace the two particle phase sub-spaces by

$$R_2(P) \to R_2(P) + \frac{1}{\pi} \mathrm{Tr}_2^{\mathbf{P}}\left[\frac{\partial\Delta}{\partial E}\right], \qquad (5\text{-}1)$$

where **A** is the phase operator (10). In the case of particles interacting through a zero-width resonance of "mass" M, the second term becomes $(V/h^3)\delta[E - (M^2 + \mathbf{P}^2)^{1/2}]$. By using the recursion formulae to decompose $R_N^{(0)}$, in Eq. (2-2), into its two-particle components and substituting **Eq.** (5-1) one easily finds the partition function for a mixture of two Boltzmann gases.

The extension to the case of interactions of bosons and/or fermions is more intricate and is still under study, but this procedure is possibly simpler than the one followed by Dashen and Rajaraman[11].

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