Revista Brasileira de Física, Vol. 5, N.° 2, 1975

Discrete Convolution – Operators and Radioactive Disintegration

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Recebido em 10 de Abril de 1975

We first describe briefly the basic concepts of discrete convolution and discrete convolution-operators. Then, using the discrete convolution – operators, we solve numerically the differential equations associated with the process of radioactive disintegration. The importance of the method is emphasized to solve numerically, differential and integral equations.

Descrevemos inicialmente, de maneira breve, os conceitos básicos da convolução discreta e dos operadores de convolução. Depois, resolvemos numéricamente as equações diferenciais associadas com o processo de desintegração radioativa, usando os operadores de convolução discreta. Enfatiza-se a importância do método para resolver numéricamente equações diferenciais e integrais.

1. Introdution

Many, and perhaps most, mathematical problems encountered in Physics and Engineering are difficult to solve by analytical methods. It often happens that explicitly obtained exact solutions are too cumbersome for interpretation and numerical evaluation. Therefore, in these instances, it is either necessary or convenient to employ approximate methods which yield accurate numerical estimates of the solution. The recent development of high speed electronic digital computers has made practical the successful application of many of these methods to complex problems.

In 1969, Healy[1] explained very clearly the concept of discrete convolution of two sequences, which motivated the present authors[2] to develop a theory of discrete convolution – operators, which in some sense, may be regarded as the counterpart in discrete mathematics, of the theory of convolution quotients (generalized functions), due to Mikusinski[3] and Erdélyi[4]. In the following sections, first we briefly describe the basic concepts of discrete convolution and discrete operators, and then we apply them to solve numerically the classic equations of radioactive disintegration. However, with suitable and sometimes trivial modifications, the method presented here can often be applied to other situations.

2. Discrete Convolution

In computational mathematics, instead of working with the instantaneous values of the continuous function $\mathbf{f}(t)$, the values $f(t_k)$ of the function sampled at the discrete time intervals $t_1, t_2, \ldots, t_k, \ldots$ are employed. As a consequence, the function $f(t)$ is replaced by a sequence of values described as

$$f = [f_1, f_2, f_3, \ldots, f_k, \ldots]. \tag{1}$$

If we take two sequences, $A = [a_0, a_1, a_2]$ and $B = [b_0, b_1]$, then the sequence $C = A * B$, i.e. the convolution of the sequence A and the sequence B can be obtained as[1,6]

$$\begin{pmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \end{pmatrix} \times \begin{pmatrix} b_0 \\ b_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{2}$$

in which, the sequence A is represented by a square matrix of order $n_A + n_B - 1$, (n_A and n_B are the number of elements in the sequences A and B respectively) while B and C are column vectors of the same order. Once the elements a_i, b_i of the respective sequences A and B are exhausted, these matrices must be completed with zeros as indicated in (2).

Hence

$$\begin{aligned} A * B = C &= [c_0,\ c_1,\ c_2,\ c_3] \\ &= [a_0 b_0,\ a_1 b_0 + a_0 b_1,\ a_2 b_0 + a_1 b_1,\ a_2 b_1]. \end{aligned} \tag{3}$$

We call the sequence C as the open, complete or unlimited discrete convolution of sequences A and B. In physical applications, the process of convolution has limits, whereby this operation could be more properly named closed, limited or incomplete convolution. The current usage of the word convolution implies generally this concept. The criterion to establish the limits is given by the range of overlapping of the two sequences to be convolved. Thus. in the above case the closed convolution of A and B, $A * B = C = [c_0, c_1]$, that is, the number of ele-. ments of the result is equal to the number of elements of the shortest factor sequence (B, in the above example).

3. Discrete Operators

We define[2] the impulse sequence, as

$$A = [1,\ o,\ o,\ \ldots], \tag{4}$$

which, when convolved with any sequence $A = [a_1\ a_2, \ldots]$, reproduces the same sequence A. Consequently **A** could be considered as the: unit operator. It can be easily verified that

$$\Delta * \Delta = \Delta^2 = [1,\ 0,\ 0,\ \ldots] = A. \tag{5}$$

Convolving A with A repeatedly shows that

$$\Delta = \Delta^2 = \Delta^3 = \cdots \tag{6}$$

Now, let us consider the sequence

$$U = [1,\ 1,\ 1,\ \ldots.], \tag{7}$$

and the result of its convolution with any sequence

$$A = [a_1,\ a_2,\ a_3,\ \ldots\ a,]. \tag{8}$$

By using the algorithm (2), it is observed that

$$U * A = [c_1,\ c_2,\ c_3,\ \ldots\ c_n], \tag{9}$$

where

$$\begin{aligned} c_1 &= a_1 \\ c_2 &= a_1 + a_2 \\ c_3 &= a_1 + a_2 + a3 \\ &-- \quad -- \quad -- \\ c, &= a_1 + a_2 + \ldots + a,. \end{aligned} \tag{10}$$

Each element c_i appears as the cummulative sum of the preceding elements; therefore as operator, U is equivalent to the process of integration.

The convolution of U with itself gives

$$U * U = U^2 = [1,\ 2,\ 3,\ \ldots\ n], \tag{11}$$

and similarly, we have

$$U * U^2 = U^3 = [1,\ 3,\ 6,\ \ldots,\ n(n+1)/2]. \tag{12}$$

We now pose the problem of solving the sequential convolution equation

$$U * S = \Delta, \tag{13}$$

where **A** and U are sequences as defined above in (4) and (7). This equation is easily solved by using the deconvolution (convolution quotient) algorithm described elsewhere[2,5], obtaining

$$S = [1,\ -1,\ o,\ o,\ \ldots]. \tag{14}$$

The sequence S, when applied to a sequence A gives

$$S * A = [a_1,\ a_2 - a_1,\ a_3 - a_2,\ \ldots], \tag{15}$$

indicating that this is a difference operator. We can consider S as the extended differential operator, which is related to the ordinary differentiation as

$$S * A = DA + A(0), \tag{16}$$

where D denotes the ordinary differentiation operator and $A(0)$ is the value of the function at the origin. The definition (16) is in complete agreement with the concept of extended derivative of generalized functions[4], in the continuous case. Repeated convolution with S gives

$$S^2 A = D^2 A + DA(0) + SA(0). \tag{17}$$

To summarize, we can say that the sequence A is equivalent to unity, the sequence U is equivalent to integration and the sequence S (convolution quotient $\Delta//U$) is equivalent to extended differentiation.

4. Radioactive Disintegration

The general theory of radioactive transformations, based on the assumption that radioactive atoms are unstable and disintegrate according to the law of chance, was given long ago by Rutherford and Soddy. **A** parent substance decays into a daughter substance which may be either stable or radioactive, and hence it is useful and interesting to know the variation, with time, of quantity or activity of a particular daughter substance under given initial conditions. Recently, this problem was attacked by Kalla, Battig and Luccioni[6], and they arrived at the solution by an appeal to the classical Laplace transform. In this section, we shall use the discrete convolution – operators to solve the problem of radioactive disintegration.

Let us denote the number of atoms of the parent and daughter substances at any time t, by $X(t)$ and $Y(t)$, respectively. We consider the case when the first daughter substance is stable, that is we have the differential equations

$$\frac{d}{dt} X(t) = - \mathrm{nx}, \tag{18}$$

$$\frac{d}{dt} Y(t) = \lambda X, \tag{19}$$

where λ is the disintegration constant of the parent substance.

We replace the functions $X(t)$ and $Y(t)$ in equations (18) and (19) by respective sequences, that is

$$X(t) = \mathrm{X} = [x_1,\ x_2,\ \ldots,\ x_n], \tag{20}$$

$$Y(t) = Y = [y_1,\ y_2,\ \ldots,\ y_n], \tag{21}$$

and apply the S operator, as defined by (16), to them. Thus we get

$$SX - x_0 = - \lambda X, \tag{22}$$

$$\mathrm{SY} - y_0 = \lambda X, \tag{23}$$

where x_0 and y_0 are the number of atoms of the parent and the daughter substance initially. By virtue of the relation (15), the equations (22) and (23) may be rewritten as

$$[x_1, x_2 - x_1, x_3 - x_2, \ldots x_n - x_n - 1] - x_0 = - \lambda[x_1, x_2, x_3, \ldots, x_n], \tag{24}$$

$$[y_1, y_2 - y_1, \mathrm{Y3} - y_2, \ldots y_n - y_{n-1}] = y_0 = \lambda[x_1, x_2, x_3, \ldots x_n]. \tag{25}$$

Eq. (24) yields a system of n algebraic equations of the type

$$\begin{aligned} x_1 - x_0 &= - \lambda\ x_1, \\ x_2 - x_1 &= - \lambda\ x_2, \\ \text{---} \quad \text{---} &\quad \text{---} \\ x_n - x_{n-1} &= - \lambda\ x_n. \end{aligned} \tag{26}$$

Solving this set of ordinary algebraic equations, we get

$$\begin{aligned} x_1 &= \frac{x_0}{1+\lambda}, \\ x_2 &= \frac{x_1}{1+\lambda} = \frac{x_0}{(1+\lambda)^2}, \\ x_3 &= \frac{x_2}{(1+\lambda)} = \frac{x_0}{(\mathrm{i}+\lambda)^3}, \\ \text{---} &\quad \text{---} \quad \text{---} \\ x_n &= \frac{x_{n-1}}{(1+\lambda)} = \frac{x_0}{(1+\lambda)^n}. \end{aligned} \tag{27}$$

The sequence $X = [x_1, x_2, \ldots, x_n]$ is the discrete solution of the equation (18). Similarly from equation *(25)* we obtain the following n algebraic equations

$$
\begin{aligned}
y_1 - y_0 &= \lambda x_1 \\
y_2 - y_1 &= \lambda x_2 \\
\text{---} \quad &\text{---} \quad \text{---} \\
y_n - y_{n-1} &= \lambda x_n
\end{aligned}
\tag{28}
$$

which yields

$$
\begin{aligned}
y_1 &= y_0 + \lambda x_1 \\
y_2 &= y_1 + \lambda x_2 \\
\text{---} \quad &\text{---} \\
y_n &= y_{n-1} + \lambda x_n .
\end{aligned}
\tag{29}
$$

Substituting the values of $x_1, x_2, \ldots, x_n$ from (27), we obtain the discrete solution of the equation *(19)* as the sequence $[y_1, y_2, \ldots, y_n]$ where

$$
\begin{aligned}
y_1 &= y_0 + \frac{\lambda x_0}{1+\lambda}, \\
y_2 &= y_0 + \frac{\lambda x_0}{1+\lambda} + \frac{\lambda x_0}{(1+\lambda)^2}, \\
\text{---} \quad & \quad \text{---} \quad \quad - \\
y_n &= y_0 + \frac{\lambda x_0}{1+\lambda} + \frac{\lambda x_0}{(1+\lambda)^2} + \ldots + \frac{\lambda x_0}{(1+\lambda)^n},
\end{aligned}
$$

or

$$y_n = y_0 + \lambda x_0 \sum_{r=1}^{n} \frac{1}{(1+\lambda)^r}. \tag{30}$$

As one would expect, the sum of the number of atoms of the parent and the daughter substance at any instant t is equal to $(x_0 + y_0)$, that is the number of elements present initially.

Let us suppose that initially the number of atoms of the parent and the daughter substances are $x_0 = 100$ and $y_0 = 20$ respectively. Let $\lambda = 3.85 \times 10^{-3}$, which corresponds to the case of Radium A. Then, using the results (27) and (29), we can form the following short table indicating the number of atoms of the parent and the daughter substance at time t.

t	$X(t)$	$Y(t)$
0	100	20
1	99.62	20.38
2	99.24	20.75
3	98.87	21.13
4	98.49	21.50
5	98.12	21.87
6	97.75	22.24
7	97.38	22.61
8	97.01	22.98
9	96.69	23.35
10	96.28	23.72

5. Discussion

Due to its simplicity, here we have considered the case when the firsi: daughter substance is stable. But, the method of discrete convolution-. operators can be easily applied to the cases when the daughter subs-. tances are also radioactive.

Discrete convolution-operators show potentialities for the numerical solution of differential, integrals and integro-differential equations, as exemplified in the previous section. However, there are difficulties and limitations that need to be studied.

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