Revista Brasileira de Física, Vol. 5, N.º 2, 1975

Magnetic Properties of Neutron-Star Matter*

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Recebido em 10 de Fevereiro de 1975

An array of qualitative and quantitative evidence is presented to the effect that neutron-star matter in its ground state is antiferromagnetic rather than ferromagnetic. The energy of pure neutron matter is evaluated as a function of spin polarization by a two-body Jastrow procedure, for densities up to five times that of ordinary nuclear matter. The anti-ferromagnetic state is energetically preferred to states with non-zero spin polarization, and lies considerably lower in energy than the ferromagnetic state. The magnetic susceptibility of the material is calculated as a function of density in the same approximation, with results which are in good agreement with independent estimates.

Apresenta-se um conjunto de evidências qualitativas e quantitativas de que a matéria das estrelas de neutrons, no seu estado fundamental, é antiferromagnética em vez de ferromagnética. Calcula-se a energia de matéria pura de neutrons em função da polarização do spin, por um procedimento de dois corpos tipo Jastrow, para densidades de até cinco vezes a da matéria nuclear comum. O estado antiferromagnético é energéticamente preferido a estados com polarização de spin não nula e está considerávelmente mais abaixo em energia que o estado ferromagnético. Calcula-se a susceptibilidade do material em função da densidade na mesma aproximação, obtendo-se resultados que concordam muito bem com outros cálculos independentes.

1. Introdução

The magnetic properties of neutron matter are of great interest for pulsar models, since strong magnetic fields, of the order of 10^{12} gauss, are necessary to explain the radio emission. These fields are extremely high compared to the magnetic fields in ordinary stars, which are of the order of 10^2 gauss. On a qualitative level, it is argued that during the birth of a neutron star, the magnetic flux in the original star is conserved

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*Research supported in part by the United States National Science Foundation under Grant No. GP-34598A1.

**Research supported in part by CNPq and BNDE (Brazilian Government).

in the supernova event, under compression into a volume about 10^{-15} of the original volume. The magnetic field, which scales inversely with the square of the radius, is thus increased by a factor 10^{10}. For a star with an initial surface field of 100 gauss the resulting neutron star would on this basis have a field, at its surface, of some 10^{12} gauss.

On a more detailed and quantitative level, it was suggested that a ferromagnetic state in the degenerate electron gas, present in white dwarfs and neutron stars, might arise from the LOFER (Landau orbital ferromagnetism) mechanism.[1] In this description, the sum of the magnetic moments associated with the system of electrons, all in their respective Landau levels, gives rise to a self-consistent macroscopic magnetization, the corresponding field in turn maintaining the Landau levels of the electrons. (A Landau level is a quantized orbit of a free electron in a crystal in a magnetic field.) The LOFER state was predicted to take over below a certain transition temperature ($\sim 10^{4\circ} K$) and to generate a field as high as 10^{12} gauss in a neutron star.

Both the above pictures yield magnetic fields for white dwarfs of about 10^6 to 10^7 gauss. This is puzzling because, observationally, most white dwarfs have magnetic fields smaller than 10^5 gauss [Ref.2]. In addition, it can be argued that the creation of a ferromagnetic state in a neutron star, by whatever means, would take too long a time to have observable effects due to the extremely high conductivity (very long mean-free-paths of the relativistic electrons) inside the neutron star.[3]

Another early proposal for the source of the pulsar magnetic field was neutron ferromagnetism. According to Brownell and Callaway (Reference 4, hereafter called BC), the neutrons, numerically predominant in the star, may align their spins, thus undergo a ferromagnetic transition, at sufficiently high density: Feeling the strong short-range repulsion in the neutron-neutron interaction, neutrons can take advantage of the Pauli principle to keep further apart from each other by occupying triplet states of spin rather than singlet states; consequently the potential energy is reduced. Of course, the kinetic energy will increase due to the fact that some particles are forced to higher wave-vector states, as the occupation of lower wave-vector states is cut in half. Even so, at high densities, it may become energetically advantageous for the particles to align their spins.

In this paper, we investigate by microscopic theory the possibility of a ferromagnetic transition in pure neutron matter. First we use a t-matrix criterion due to BC, which is derived in Section 1, to assess the role of a nucleon-nucleon hard core in ferromagnetism. The hard core is seen to favor ferromagnetism of neutron matter, as already indicated by the qualitative argument just given.

In Section 2, we incorporate the effect of the attractive component of the nuclear force. This effect is found to vitiate the conclusion drawn with the hard core alone. There is no sign of a ferromagnetic instability of the pure neutron system in the density range up to twice the equilibrium density of symmetrical nuclear matter.

In Sections 3 and 4, we study the magnetic properties of neutron matter more thoroughly by performing a many-body calculation for the ground-state energy of a system of neutrons with arbitrary spin polarization. For densities up to five times that of ordinary nuclear matter, we find that the antiferromagnetic state is energetically preferred to states with nonzero spin polarization. A calculation of the magnetic susceptibility within the same scheme provides a further check of the now-popular contention that neutron matter does not undergo a ferromagnetic transition at neutron-star densities.

A microscopic description of the intense field of a neutron star must therefore be sought elsewhere than in the mechanism of neutron ferromagnetism.[3] Nevertheless, the general magnetic properties of neutron matter, explored here, remain of much interest, being vital to a comprehensive picture of the structure and dynamics of the interior of the star.

2 t-Matrix Criterion for **Ferromagnetism**

In this section, we derive the t-matrix criterion for the onset of a ferromagnetic transition, following BC. Consider a system of N identical fermions with small spin polarization,

$$s = \frac{n_+ - n_-}{n_+ + n_-}, \tag{1}$$

where n_+ and n_- are respectively the partial densities of particles with spin up and spin down relative to a tiny externa1 magnetic field. These partial densities satisfy the condition $n_+ + n_- = n$, where n is the number density of the system. Equivalently, we can write n_+ and n_- as functions of n and s:

$$n_\pm = \frac{n}{2}(1 \pm s). \tag{2}$$

We treat the system at $T = 0°K$ and the particles are assumed to interact through a two-body potential $v(1,2)$. Because of the singular nature of $v(1,2)$, which is supposed to contain a strong repulsive core, it is necessary to introduce a t-matrix or other effective potential approximation. In the t-matrix formalism, the operator t is defined as[5]

$$t = v - v\,\frac{Q}{W - H_0}\,t, \tag{3}$$

where the Pauli operator Q projects out of the Fermi sea, W is a starting energy, and H_0 is the Hamiltonian for two independent particles. The total energy per particle can be written in a two-body approximation as the sum of the free kinetic energy per particle ε_0 and a two-body interaction energy per particle,

$$\varepsilon = \varepsilon_0 + \frac{1}{N}\sum_{i>j}(ij \mid t \mid ij - ji), \tag{4}$$

where the sums go over space and spin quantum numbers of the occupied (Fermi sea) orbitals, and the starting energy is taken as the sum of self-consistent single-particle energies for Fermi sea orbitals i and j. We are of course interested only in the lowest state of given spin polarization. The interaction energy due to the small uniform field is neglected.

Define average t-matrix elements by

$$t_{\sigma_i\sigma_j} = \left(\frac{3}{4\pi k_F^3}\right)^2 \int d^3k_i d^3k_j \theta(\varepsilon_F - \varepsilon_i)\theta(\varepsilon_F - \varepsilon_j)\,(ij \mid t \mid ij - ji), \tag{5}$$

where θ is the unit step function, $k_F = (3\pi^2 n)^{1/3}$ is the radius of the Fermi sphere, ε_F is the Fermi energy, and ε_i, k_i, and σ_i are respectively the energy, wave number, and spin of orbital i. It is convenient to suppose that the N particles occupy unit volume, so that $N = n$. Now, (4) can be written as

$$\varepsilon = \varepsilon_0 + \frac{1}{N}(t_{++}n_+^2 + 2t_{+-}n_+n_- + t_{--}n_-^2), \tag{6}$$

where "+" means spin up and "–" means spin down.

Introducing average t operators for singlet and triplet pairs of particles, $t^{(s)}$ and $t^{(t)}$, by

$$t_{++} = t_{--} = t^{(t)},$$

and

$$t_{+-} = t_{-+} = \frac{1}{2} t^{(t)} + \frac{1}{2} t^{(s)}, \tag{7}$$

we have

$$\begin{aligned} \varepsilon &= \varepsilon_0 + \frac{1}{N} \left[t^{(t)} (n_+^2 + n_+ n_- + n_-^2) + t^{(s)} n_+ n_- \right] \\ &= \varepsilon_0 + N \left[\frac{3}{4} t^{(t)} + \frac{1}{4} t^{(s)} - \frac{1}{4} s^2 (t^{(s)} - t^{(t)}) \right] \end{aligned} \tag{8}$$

Denote the density of Fermi-gas single-particle states of a given spin direction by $G(E)$. The Fermi energy μ_0 of the non-interacting system is defined by

$$\int_0^{\mu_0} G(E)\, dE = \frac{N}{2}. \tag{9}$$

Letting $\mu_\pm$ be the Fermi energies of particles of up and down spins in the paramagnetic state of spin polarization s, we have

$$\int_0^{\mu_\pm} G(E)\, dE = \frac{1}{2} N(1 \pm s).$$

Subtracting (9) from (10), we get

$$\int_{\mu_0}^{\mu_\pm} G(E)\, dE = \pm \frac{1}{2} Ns. \tag{11}$$

Assuming s to be small, we can expand $G(E)$ in a Taylor series about μ_0. Keeping the first two terms, (11) becomes

$$\int_{\mu_0}^{\mu_\pm} [G(\mu_0) + G'(\mu_0)(E - \mu_0)]\, dE = \pm \frac{1}{2} Ns, \tag{12}$$

where $G'(E)$ denotes the derivative of the density of states. Carrying out the integration in (12), we find

$$\mu_\pm - \mu_0 = \pm \frac{Ns}{2G(\mu_0)} - \frac{G'(\mu_0)}{2G(\mu_0)} (\mu_\pm - \mu_0)^2. \tag{13}$$

To second order in s, (13) is equivalent to

$$\mu_{\pm} - \mu_0 = \pm \frac{Ns}{2G(\mu_0)} - \frac{N^2 s^2 G'(\mu_0)}{8G^3(\mu_0)} \tag{14}$$

The total energy of the non-interacting system can be calculated as

$$N\varepsilon_0 = \int_0^{\mu_+} EG(E)dE + \int_0^{\mu_-} EG(E)dE. \tag{15}$$

In the absence of spin polarization, $\mu_+ = \mu_- = \mu_0$, and the total energy of the antiferromagnetic state of the non-interacting system is

$$N\varepsilon_a = 2 \int_0^{\mu_0} EG(E)dE\,. \tag{16}$$

Again expanding $G(E)$ in a Taylor series about μ_0, Eq. (15) yields, to leading order in s,

$$\varepsilon_0 = \varepsilon_a + \frac{N^2 s^2}{4G(\mu_0)} \tag{17}$$

From Eq. (8) we then have, for the approximate energy of the interacting system with (small) spin polarization s,

$$\varepsilon = \varepsilon_a + n\left\{\frac{3}{4}t^{(t)} + \frac{1}{4}t^{(s)} + \frac{s^2}{4}\left[\frac{1}{G(\mu_0)} - t^{(s)} + t^{(t)}\right]\right\}. \tag{18}$$

Accordingly, the antiferromagnetic state is unstable against spin flips which increases s, when the following criterion is satisfied:

$$\mathrm{I} = G(\mu_0)\,[t''' - t^{(t)}] \geq 1. \tag{19}$$

Note that in the derivation of this criterion we have ignored three- and higher-body cluster effects in the sense of Brueckner theory[5], as well as any intrinsic s-dependence of the average t-matrix elements.[6,7]

Treating neutron matter as a system of hard spheres of diameter c and neglecting the attractive nuclear forces, BC have calculated the t-matrix elements of (5) by means of a reference spectrum method.[5] The criterion (19) for an instability toward ferromagnetism in neutron matter is found to be satisfied at $k_F c = 0.86$. Assuming a hard-core radius (hard sphere diameter) c = 0.4 – 0.5 fm, this implies a "critical" Fermi wave number $k_F = 2.15 - 1.72\ \mathrm{fm}^{-1}$, thus a "critical" density 2-1 times the equilibrium density of symmetrical nuclear matter. From this result, BC concluded that throughout the major portion of a typical neutron star, the neutron component is likely to be ferromagnetic.

3. Effect of Attractive Nuclear Forces on the Onset of Ferromagnetism in Neutron Matter

The conclusion drawn by Brownell and Callaway is based on the assumption that the interaction of two neutrons in the density range of interest is dominated by the repulsive core. However, the spin dependence of the extracore component of the two-neutron potential has an essential effect on the onset of a ferromagnetic transition in neutron matter. This was first pointed out by Clark and Chao.[8] As indicated by the t-matrix criterion (19) for an instability toward ferromagnetism, the ferromagnetic ground state is favored by a strong triplet attraction (weak triplet repulsion) and weak singlet attraction (strong singlet repulsion). Since the extra-core component of the two-neutron potential is known to have practically the opposite characteristics, being strongly attractive in singlet-even states and weak in triplet-odd states, it is to be expected that the inclusion of the extra-core potential may push the critica1 density for instability of the $s = 0$ state of neutron matter to a higher density than that predicted by BC. To put it more physically, note that in a realistic hard-core two-neutron interaction potential, there is a deep, narrow attractive well in singlet states just outside the (essentially) state-independent repulsive core. This well opposes the tendency toward spin alignment: to take fullest advantage of the attraction, a pair of neutrons should approach fairly closely, with zero total spin. Thus in the moderate density range in which a t-matrix calculation may be considered reliable, the ground state of neutron matter may in fact be antiferromagnetic rather than ferromagnetic. This should also be true for a realistic soft-core potential[8], as considered by Clark[10].

An empirical indication of the importance of the attractive forces (or more generally, extra-core interactions) for the occurence or absence of a ferromagnetic transition in a system of condensed fermions can be seen in liquid He^3. Liquid He^3 is the (bulk) system which most closely resembles neutron matter among terrestrial materials. Suppose we treat liquid He^3 in the same way as BC treated neutron matter, and to this end replace the interatomic interaction by a hard core of radius $c = 1.73$ Å. This radius is taken from the work of Cole[11], who has treated helium as a system of hard spheres. (Cole deduced an effective hard-sphere diameter from the experimental zero-point density and pressure relations of solid He^4 and He^3, upon the premise that it is the strong short-range repulsive component of the two-body interaction in these systems which governs the behavior of the wave function, the attractive component having little influence on the wave function of the system at high density.) The equilibrium density of liquid He^3 corresponds (in the absence of ferromagnetism) to a Fermi wave number $k_F = 0.78\,\text{Å}^{-1}$. Thus $x = k_F c = 1.35$. According to BC, the critical value of x for an instability toward ferromagnetism, for a system of hard spheres, is $x_c = 0.86$, so that liquid He^3, on this basis, would be expected to be ferromagnetic, in contradiction to experiment. It may further be noted that, in the presence of attraction, the situation in liquid He^3 should perhaps be more favorable for the occurence of ferromagnetism than that in neutron matter, since in the former system, as opposed to the latter, the triplet-state attraction (which aids alignment of the spins) is just as strong as the singlet-state attraction (which inhibits, alignment of the spins).

We now give quantitative evidence of the importance of the attraction, in terms of the BC criterion (19) for the occurence of ferromagnetism. However, we will interpret $t^{(s)}$ and $t^{(t)}$ in (19) more generally as the appropriate diagonal matrix elements of an effective two-body interaction, not necessarily the reaction operator. At $k_F = 2\,\text{fm}^{-1}$, which corresponds to a density near the critical density for the ferromagnetic transition as predicted by BC, one has $G(\mu_0) = mk_F/2\pi^2\hbar^2 = 2.44 \times 10^{-3}$ $\text{MeV}^{-1}\,\text{fm}^{-3}$, so that satisfaction of (19) requires $t^{(s)} - t^{(t)} > 410$ MeV fm^3. To see whether this condition is still satisfied when the proper attractive forces (extracore interactions) are incorporated, we extract an estimate (probably an over-estimate) of $t^{(s)} - t^{(t)}$ from a Jastrow evaluation of the energy per particle of the lowest nonmagnetic, normal state of neutron matter, for two reasonable neutron-neutron potentials. The calculational procedure is the same as that employed for nuclear matter by Bäckman, Chakkalakal, and Clark[12]. The result for the energy per particle is of the form

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \tag{20}$$

where ε_1 is the ground-state energy per particle of the Fermi gas with Fermi number k_F, and ε_2 and ε_3 are, respectively, the two-body and three-body cluster contributions to the Jastrow energy expectation value per particle. To the extent that higher cluster contributions are negligible, ε may be considered an upper bound on the true energy per particle. Numerical results for the Iwamoto-Yamada[13] (IY) and Hamada-Johnston[14] (HJ) potentials, as compiited by Chakkalakal[15], are presented in Table I. Note that $n = k_F^3/3\pi^2$. The three-body correction ε_3 is quite small in magnitude for all densities considered and will be discarded in the present analysis. The two-body contribution may be decomposed thus:

$$\varepsilon_2 = n\left[\frac{3}{4}t^{(t)} + \frac{1}{4}t^{(s)}\right] \tag{21}$$

Accordingly,

$$t^{(s)} - t^{(t)} = \frac{4\varepsilon_2}{n} - 4t^{(t)}. \tag{22}$$

In the case of the IY potential, the odd-state interaction consists of the core alone, so that $t^{(t)} > 0$; for more realistic potentials like the HJ potential, the net energetic contribution of the ^3p states and more generally the odd states is expected to be rather small in the density region under study[16]. Thus we take

$$t^{(s)} - t^{(t)} \approx \frac{4\varepsilon_2}{n} \tag{22a}$$

Over the density ranges considered, this is always negative, although it does increase as k_F increases. Evidently, the system is still far too dilute to experience a ferromagnetic transition, if prepared in the antiferromagnetic state. It is true that the core term of $t^{(s)} - t^{(t)}$ is large and positive, but this is more than compensated by the attractive contribution. At $k_F = 2\,\mathrm{fm}^{-1}$, we find $t^{(s)} - t^{(t)} < -300$ MeV fm^3 (HJ). Table I provides a summary of our estimates for $t^{(s)} - t^{(t)}$ and the left-hand side I of condition (19).

Potential	k_F (fm^{-1})	ε_2 (MeV)	ε_3 (MeV)	E (MeV)	$(t^{(s)} - t^{(t)})_{\mathrm{est}}$ (MeV fm^3)	I_{est},
IY	0.9	−4.21	0.09	5.95	−680	−0.68
	1.3	−7.75	−0.15	13.11	−420	−0.86
	1.7	−8.55	0.04	27.42	−210	−0.73
HJ	0.9	−4.29	0.10	5.88	−700	−0.69
	1.3	−9.66	−0.06	11.28	−520	−1.07
	1.7	−16.51	−0.22	19.19	−400	−1.40
	2.1	−22.69	0.20	32.33	−290	−1.56

Table I - Ground-state properties of antiferromagnetic neutron matter, as calculated by Chakkalakal[15], and associated estimates for ingredients of the ferromagnetic criterion (19).

From the foregoing analysis we conclude that, in the framework of a two-body cluster treatment, the effect of the extra-core component of the two-neutron potential is to push the critica1 density for the onset of ferromagnetism in neutron matter to a value substantially higher than that predicted in the presence of the repulsive core alone. To further uncover the magnetic properties of neutron matter, we carry out, in the following sections, calculations of its paramagnetic susceptibility and of its ground-state energy for arbitrary spin polarization.

4. Preliminary Discussion of Magnetic Susceptibility

The energy of a system of identical spin$-\frac{1}{2}$ fermions is a function of the spin polarization parameter s defined in (1). As in the earlier considerations, we shall be concerned only with the lowest state of given s, whose energy we denote by $\varepsilon(s)$. We would in fact like to know what value of s the system takes in its ground state. The ground state is said to be antiferromagnetic if $s=0$ minimizes $\varepsilon(s)$ and ferromagnetic if $s=+1$ or $s=-1$ minimizes $\varepsilon(s)$. (Note that in Section 3 all energy quantities referred to $s=0$.)

It is generally useful to calculate the magnetic susceptibility χ. This quantity measures the energy required to prodiice a small spin polarization, starting from s = 0, i.e., starting from the antiferromagnetic state. We consider a system with a small spin polarization s due to a small perturbing magnetic field of strength H. The unperturbed energy (meaning the energy with the field *off*) is approximated by a Taylor series expansion about $s=0$, truncated at the s^2 term:

$$\varepsilon(s) \cong \varepsilon(0) + \left.\frac{\partial\varepsilon(s)}{\partial s}\right|_{s=0} s + \frac{1}{2}\left.\frac{\partial^2\varepsilon(s)}{\partial s^2}\right|_{s=0} s^2. \tag{23}$$

Because of spatial isotropy, the energy of the unperturbed system cannot depend upon the sign of the spin polarization. Thus $\varepsilon(s) = \varepsilon(-s)$ and the energy of the unperturbed system at $s=0$ must be an extremum, in that $\partial\varepsilon(s)/\partial s\,|_{s=0} = 0$. For the rest of the discussion, we will consider only the case $s \geq 0$.

The total energy $\tilde{\varepsilon}(s)$ of the system is the sum of the unperturbed energy and the energy of interaction with the (weak) applied magnetic field H. Thus,

$$\tilde{\varepsilon}(s) \cong \varepsilon(0) + \frac{1}{2} \left. \frac{\partial^2 \varepsilon(s)}{\partial s^2} \right|_{s=0} s^2 - \gamma H s, \tag{24}$$

where γ is the neutron magnetic moment. This approximation to $\tilde{\varepsilon}(s)$ assumes a relative extremum with respect to s at

$$s = \frac{\gamma H}{\left. \dfrac{\partial^2 \varepsilon(s)}{\partial s^2} \right|_{s=0}} . \tag{25}$$

This extremum is a relative minimum if and only if $\partial^2 \varepsilon(s)/\partial s^2 |_{s=0}$ is positive (which implies that energy must be added to polarize the unperturbed system). Otherwise it is a (physically irrelevant) relative maximum at $s < 0$. The magnetic susceptibility is defined as the ratio of the induced magnetization per unit volume in the direction of the field, $n\gamma s$, to the field magnitude H. Thus we have, using (25),

$$\chi = \frac{n\gamma^2}{\left. \dfrac{\partial^2 \varepsilon(s)}{\partial s^2} \right|_{s=0}} . \tag{26}$$

Strictly speaking, this formula is meant to apply only when (25) minimizes the perturbed energy, that is, when the antiferromagnetic state of the unperturbed system is locally stable, $\partial^2 \varepsilon(s)/\partial s^2 |_{s=0} > 0$. But it is useful to evaluate χ of (26) in any case. A result $\chi > 0$ indicates the antiferromagnetic state is at least locally stable. On the other hand, a singularity of $\chi\,(\chi \to \infty)$ with increase of some parameter (say density) followed by negative values of χ with further increase of this parameter, signals the onset and prevelance of an instability of the antiferromagnetic state against spin flips leading toward ferromagnetism.

These considerations on χ and its sign bear on the behavior of the unperturbed system at and near $s = 0$; the behavior at finite s remains to be examined. It is possible that the unperturbed energy of the system is not a monotonic function of s. Therefore, a plot of $\varepsilon(s)$ against s may have relative minima in the interval $0 < s < 1$. In particular, the ferromagnetic state may conceivably have lower energy than the antiferromagnetic state, even if $\chi > 0$. It is thus necessary to calculate

$\varepsilon(s)$ as a function of s for $0 \leq s \leq 1$ to conclude deiinitely that the ground state is antiferromagnetic or else to find the finite polarization for which the system has its lowest energy,

5. Magnetic Susceptibility and Spin **Polarization** Energy of **Neutron** Matter

We now perform a two-body Jastrow calculation of the energy of a system of N interacting neutrons with arbitrary spin polarization s.

In the framework of the method of correlated basis functions[17,18], the trial ground-state wave function for a many-Fermion system is written as

$$\psi(12\ldots N) = F(12\ldots N)\Phi(12\ldots N). \tag{27}$$

Here the model wave function Φ is taken to be a Slater determinant describing the ground state of N non-interacting neutrons. For a uniform system, the appropriate single-particle wave functions are plane waves, unit-normalized and satisfying periodic boundary conditions in a cubic box. As in Section 2, it is convenient to assume a unity normalization volume, for the N-particle system. Thus the spatial portion of a given single-particle wave function is simply $\exp(i\mathbf{kr})$, where k is the corresponding wave vector. A number or number density n_+ of particles is assigned to up-spin orbitals, and n_-, to down-spin orbitals, in accordance with Eq. (2). The correlation factor F is taken to be a product

$$F = \prod_{1 \leq i < j \leq N} f(r_{ij}) \tag{28}$$

of two-body correlation factors f, one for each pair of particles.

The expectation value of the energy per particle with respect to the delineated trial wave function, called $\varepsilon(s)$, is cliister expanded[12,18] in the effects of the correlations introduced by F. Stopping at the leading correlation effect, i.e., assuming rapid convergence of the cluster expansion (cf. Table I), we set

$$\varepsilon(s) = \varepsilon_1(s) + \varepsilon_2(s)\,. \tag{29}$$

The one-body term $\varepsilon_1(s)$ is just the kinetic enei-gy per particle of the noninteracting system, in which n_+ particles of spin up fill a Fermi sphere of radius k_{F+} and n_- particles of spin down fill a Fermi sphere of radius k_{F-}. We make use of the relations

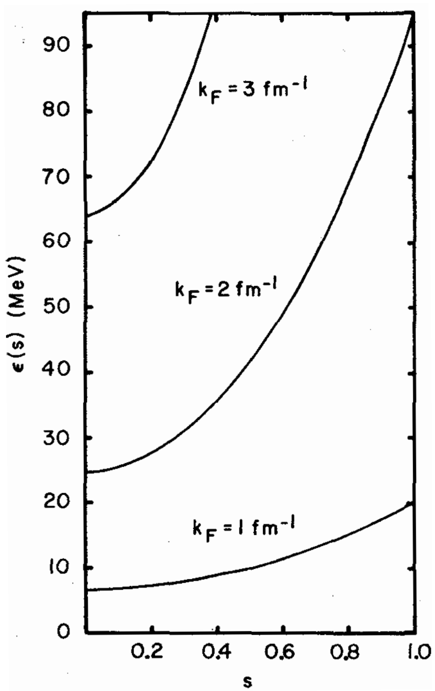

Figure 1 – Energy per particle $\varepsilon(s)$ of neutron rnatter as a function of the spin-polarization parameter s, for $k_F = 1,2,3\ \mathrm{fm}^{-1}$ and odd-state choice (a) (rnixed Serber potential).

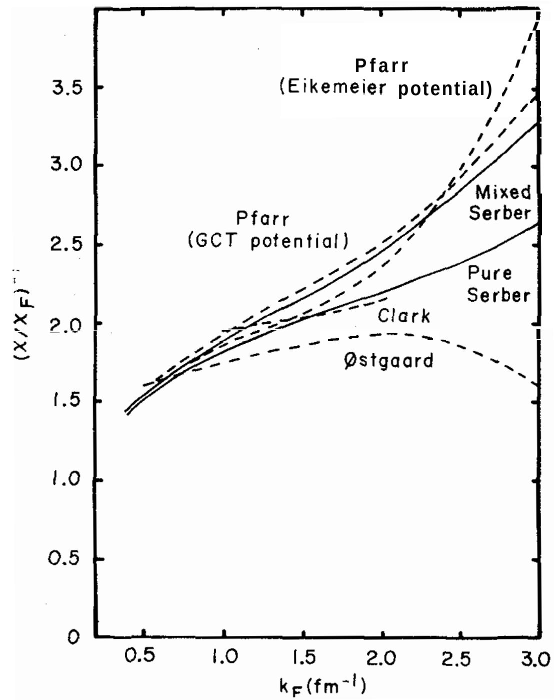

Figure 2 – Magnetic susceptibility χ of neutron matter in units of the magnetic susceptibility of the non-interacting neutron systern, plotted against Fermi wave nurnber k_F. The solid curves labeled "Pure Serber" and "Mixed Serber" are the results of the present work.

$$n_\pm = \sum_{k_\pm = 0}^{k_{F\pm}} 1 \to \frac{4\pi}{(2\pi)^3} \int_0^{k_{F\pm}} k_\pm^2 \, dk_\pm = \frac{k_{F\pm}^3}{6\pi^2}. \tag{30}$$

From (2) we see that

$$k_{F\pm} = k_F(1 \pm s)^{1/3}. \tag{31}$$

The two-body term ε_2 is the sum over all pairs of occupied plane-wave orbitals, of matrix elements of an effective two-body potential $w(1, 2)$, that is,

$$N\varepsilon_2(s) = \sum_{i>j} (ij \,|\, w_2(1, 2) \,|\, ij - ji)$$

with

$$w_2(1, 2) = \frac{\hbar^2}{M} (\nabla f(r_{1,2}))^2 + f^2(r_{1,2}) v(1, 2), \tag{33}$$

$v(1, 2)$ being the bare two-body potential. Separating the contributions to $\varepsilon_2(s)$ according to the spin projections of the particles, we have

$$\begin{aligned} N\varepsilon_2(s) = \frac{1}{2} \sum_{i_+, j_+} (i_+ j_+ \,|\, w_2 \,|\, i_+ j_+ - j_+ i_+) + \sum_{i_+, j_-} (i_+ j_- \,|\, w_2 \,|\, i_+ j_- - j_- i_+) \\ + \frac{1}{2} \sum_{i_-, j_-} (i_- j_- \,|\, w_2 \,|\, i_- j_- - j_- i_-), \end{aligned} \tag{34}$$

where of course the "$+$" sign tags orbitals with spin up and the "$-$" sign tags orbitals with spin down.

To perform the spin sums, we rewrite the above in terms of orthonormal two-particle spin functions $X_S^{M_S}$ which are simultaneous eigenfunctions of the operators S^2 and S_z, where $S = s(1) + s(2)$. As usual, eigenvalues of S^2 are denoted by $S(S + 1)$, and of S_z by M_S. For neutrons, we have triplet spin states with $S = 1$ and singlet spin states with $S = 0$, thus the familiar expressions

$$\begin{aligned} X_1^1 &= \alpha(1)\alpha(2), \\ X_1^0 &= \frac{1}{\sqrt{2}} [\alpha(1)\beta(2) + \beta(1)\alpha(2)], \quad X_0^0 = \frac{1}{\sqrt{2}} [\alpha(1)\beta(2) - \beta(1)\alpha(2)], \\ X_1^{-1} &= \beta(1)\beta(2), \end{aligned} \tag{35}$$

where α and β denote respectively the usual normalized one-body spin-up and spin-down functions. The mixed-spin states in the middle term of (34) are expressed as linear combinations of X_1^0 and X_0^0

$$\alpha(1)\beta(2) = \frac{1}{\sqrt{2}}\,[X_1^0 + X_0^0],$$
$$\beta(1)\alpha(2) = \frac{1}{\sqrt{2}}\,[X_1^0 - X_0^0]. \tag{36}$$

We assume the two-body neutron-neutron interaction potential to be spin dependent,

$$v(12) = v_0(r_{1,2})\Lambda_0 + v_1(r_{1,2})\Lambda_1, \tag{37}$$

where v_0 is the singlet and v_1 the triplet potential, while Λ_0 and Λ_1 are the singlet and triplet spin projection operators

$$\Lambda_0 = \frac{1 - \sigma_1\,.\,\sigma_2}{4}, \qquad \Lambda_1 = \frac{3 + \sigma_1\,.\,\sigma_2}{4} \tag{38}$$

Summing over spins, Eq. (34) becomes

$$N\varepsilon_2(s) = \frac{1}{2}\sum_{\mathbf{k}_i=0}^{\mathbf{k}_{F+}}\sum_{\mathbf{k}_j=0}^{\mathbf{k}_{F+}}(\mathbf{k}_i\mathbf{k}_j\,|\,u(r_{1,2}) + f^2v_1(r_{1,2})\,|\,\mathbf{k}_i\mathbf{k}_j - \mathbf{k}_j\mathbf{k}_i)$$
$$+\sum_{\mathbf{k}_i=0}^{\mathbf{k}_{F+}}\sum_{\mathbf{k}_j=0}^{\mathbf{k}_{F-}}\{(\mathbf{k}_i\mathbf{k}_j\,|\,u(r_{1,2})\,|\,\mathbf{k}_i\mathbf{k}_j) + \frac{1}{2}(\mathbf{k}_i\mathbf{k}_j|f^2v_0(r_{1,2})|\mathbf{k}_i\mathbf{k}_j + \mathbf{k}_j\mathbf{k}_i)$$
$$+\frac{1}{2}(\mathbf{k}_i\mathbf{k}_j\,|f^2v_1(r_{1,2})\,|\,\mathbf{k}_i\mathbf{k}_j - \mathbf{k}_j\mathbf{k}_i)\}$$
$$+\frac{1}{2}\sum_{\mathbf{k}_i=0}^{\mathbf{k}_{F-}}\sum_{\mathbf{k}_j=0}^{\mathbf{k}_{F-}}(\mathbf{k}_i\mathbf{k}_j\,|\,u(r_{1,2}) + f^2v_1(r_{1,2})\,|\,\mathbf{k}_i\mathbf{k}_j - \mathbf{k}_j\mathbf{k}_i),$$

with $u(r_{1,2}) \equiv (\hbar^2/M)(\nabla f(r_{1,2}))^2$. We next carry out the spatial integrations over the center-of-mass coordinate and the finally the wave vector summations. Using the well-known formula[19] for summation (integration) over a Fermi sphere,

$$\sum_{\mathbf{k}_\pm=0}^{k_{F\pm}}\exp[i\mathbf{k}_\pm\,.\,\mathrm{r}] = n_\pm\, l(k_{F\pm}\mathrm{r}), \tag{40}$$

where $l(t) = (3/t^3)\,(\sin t - t\cos t)$, we arrive at the following result for the energy per particle of a system of neutrons with spin polarization s:

$$\varepsilon(s) = \frac{3}{10}\,\frac{\hbar^2k_F^2}{2M}[(1+s)^{5/3} + (1-s)^{5/3}] + \frac{\pi}{2}n(1+s)^2\int_0^\infty[u(r) +$$
$$+f^2v_1(r)]\;[1 - l^2(k_{F+}r)]r^2\,dr +$$

$$+\pi n(1-s^2)\int_0^\infty \Big\{u(r)+\frac{1}{2}f^2 v_0(r)\,[1+l(k_{F+}r)\,l(k_{F-}r)]\,+$$

$$+\frac{1}{2}f^2 v_1(r)\,[1-l(k_{F+}r)\,l(k_{F-}r)]\Big\}\,r^2\,dr$$

$$+\frac{\pi}{2}n(1-s)^2\int_0^\infty [u(r)+f^2 v_1(r)]\;[1-l^2(k_{F-}r)]\,r^2\,dr. \qquad (41)$$

It is convenient to express the magnetic susceptibility χ in units of χ_F, the magnetic susceptibility of the non-interacting system. The latter is easily found to be

$$\chi_F=\frac{n\gamma^2}{\left.\dfrac{\partial^2\varepsilon_1(s)}{\partial s^2}\right|_{s=0}}=\frac{M\gamma^2}{\pi^2\hbar^2}\,k_F. \qquad (42)$$

(Note this "unit" depends on the density.) A straightforward calculation then yields the desired quantity,

$$(\chi/\chi_F)^{-1}=1+\left[\frac{\partial^2\varepsilon_2}{\partial s^2}\Big/\frac{\partial^2\varepsilon_1}{\partial s^2}\right]_{s=0}=$$

$$=1+\frac{k_F}{\pi}\left(\frac{\hbar^2}{M}\right)^{-1}\int_0^\infty\{-2u+f^2v_0\,[1+l^2(t)]+[2u+f^2v_1]\,[1-l^2(t)]-$$

$$-\frac{8}{3}[u+f^2v_1]\,tl(t)\frac{\partial l(t)}{\partial t}+[f^2v_0-3f^2v_1]\left[\frac{t}{3}l(t)\left(\frac{t}{3}\,\frac{\partial^2 l(t)}{\partial t^2}-\frac{2}{3}\,\frac{\partial l(t)}{\partial t}\right)-\right.$$

$$\left.-\frac{2}{9}t^2\left(\frac{\partial l(t)}{\partial t}\right)^2\right]\Big\}\,r^2\,dr, \qquad (43)$$

where $t=k_F r$.

For the purpose of numerical evaluation, we choose a potential of the Ohmura type[20] for the neutron-neutron interaction. This potential has, in even states, the following expression

$$\begin{aligned} v_0(r_{1,2}) &= \infty, && r\le c,\\ &= -A_0\,\exp[-\alpha_0(r-c)], && r>c, \end{aligned} \qquad (44)$$

where $c=0.4$ fm, $A_0=235.414$ MeV, $\alpha_0=2.03435\ \mathrm{fm}^{-1}$. It fits the singlet scattering length and effective range and reproduces, approximately, the energy dependence of the 'S-wave phase shift. We consider two possibilities for the odd-state character: (a) a mixed Serber potential, in which the hard core is assumed to be state-independent but the extra-core portion of the odd-state interaction is set to zero, and (b) a pure Serber potential, in which the odd-state interaction is set identically zero.

In case (a), the two-body correlation factor is taken as

$$\begin{aligned} f(r) &= 0, & r \leq c, \\ &= 1-\exp\left[-v(r-c)\right], & r > c, \end{aligned} \tag{45}$$

independent of parity state. The density-dependent variational parameter v is determined by minimization of ε_2 of (41), with s put zero. In case (b), the two-body correlation factor should obviously be unity in odd states. With $f = 1$ in odd states and retaining the form (45) in even states, the two-body correlation factor becomes state- (parity-) dependent. However, Eqs. (41), (43) were derived assuming a state--independent correlation factor. Some straightforward modifications of them are required, to deal adequately with choice (b). For the parity--dependent correlation factor just prescribed, one needs only write u in the form (37), with the even-state componetit u_0 computed from the r derivative of (45), and $u_1 \equiv 0$ Correspondingly, all u terms appearing in (41), (43) are to be dropped, while $f^2 v_0$ is to be replaced by $f^2 v_0 + u_0$. The f's that appear finally refer to form (45), all odd-state potential contributions being zero, as in case (a). Again, the parameter v is determined by minimizing the two-body energy contribution (still called ε_2) at $s = 0$.

Having chosen the potential and fixed the two-body correlations as described, we find that for k_F values up to $k_F = 3$ fm^{-1}, i.e., densities up to 0.91 neutrons per fm^3, the minimum of $\varepsilon(s)$ of (41), as a function of s, always occurs at $s = 0$. This result provides clear support for the proposition that the ground state of neutron-star matter is antiferromagnetic. Further, $\varepsilon(s)$ is found to increase monotonically with s. The results obtained would not be significantly affected if we redetermined v at each $s > 0$ by minimization of ε_2 at that s. The optimal value of v remains near 5 fm^{-1} in all cases. In Figure 1, curves of $\varepsilon(s)$ vs. s for $k_F = 1$, 2, and 3 fm^{-1} are displayed, for odd-state choice (a). The corresponding curves for potential choice (b) are similar, but their upward curvature is somewhat less. The magnetic susceptibility is found to be positive (as expected), with $(\chi/\chi_F)^{-1}$ increasing monotonically as the density increases. There is no hint of a lerromagnetic instability of the material.

Plots of $(\chi/\chi_F)^{-1}$ vs. k_F for odd-state potential choices (a) and (b) are shown in Figure 2. Included for comparison are the results of magnetic susceptibility calculations of Clark,[10] Østgaard,[6] and Pfarr.[21] Their results seem to be generally in line with those of the present investigation. Clark, using Brueckner theory to evaluate $\partial^2\varepsilon(s)/\partial s^2|_{s=0}$ for the realistic Reid soft-core potential, finds no sign of a ferromagnetically favorable instability at densities below 5 x 10^{14} g/cm^3. Østgaard, also using Brueckner theory, but for the semi-realistic potential of Moszkowski and Scott, obtains essentially the same results as those of Clark at low densities. However, at high densities the quantity $(\chi/\chi_F)^{-1}$ rapidly decreases and passes through zero at $k_F = 4.1$ fm^{-1}, indicating the onset of a ferromagnetic instability at thç corresponding density. Pfarr, employing a unitary transformation method[22] to define the effective-potential matrix elements, has calculated the magnetic susceptibility of neutron matter for thc Gammel-Christian-Thaler potential and the Eikemeier potential. Contrary to Østgaard's result, Pfarr detects no evidence for a ferromagnetic instability, up to $k_F = 4$ fm^{-1}. Østgaard's deviant finding is due to his incorrect use of unity for the effective mass and to the complete neglect of odd-state interactions.

Pearson and Saunier[23] have calculated the spin-polarization energy for neutron matter using their effective interaction potential and a first--order perturbation method. Their results show that the energy per particle increases monotonically with the polarization for densities up to $k_F = 1.8$ fm^{-1}, the highest Fermi wave number considcred. Thus, they also found ferromagnetism to be energetically unfavorable in neutron inatter.

The most refined calculation of neutron-matler susceptibility is probably that of Bäckman, Källman, and Sjöberg[24], who carried out a microscopic evaluation of the Landau Fermi-liquid parameters, using Brueckner theory. Their results are in agreement with ours and also with those of Clark and Pfarr.

6. Conclusion

Our arguments and supporting calculations have provided a strong basis for the currently held view that the ground state of ncutron matter is antiferromagnetic in the density range of interest foi- neutron stars. The absence of a ferromagnetic transition in neutron matter may be traced to tlie extra-core component of the neutron-neutron potential. which opposes thc onset of ferromagnetism. Although the repulsive core of the potential acts to promote ferromagnetism, this effect is dominant only at very high densities. For k_F up to 3 fm^{-1}, correspon-ding to densities up to about five times the equilibrium density n_0 of ordinary, symmetric nuclear matter, it appears that the system is still much too dilute to undergo a fcrromagnetic transition. At twice n_0, the antiferromagnetic state is energetically preferred over the ferromagnetic state by of the order of a hundred MeV. There is really not much point in calculating k_F values beyond about 3 fm^{-1} because of (i) inherent inadequacies of the two-body potential representation of the neutron-neutron force, which become increasingly apparent, (ii) importance of three-body and other corrections to the cluster method, and (iii) contamination of pure neutron matter by protons and hypei-ons.

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