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The Structural Parameters of Classical Lie Algebras

F. N. NDILI, G. C. CHUKWUMAH and P. N. OKEKE
Departmente of Physics, Uniuersity of Nigeria, Nsukka, Nigeria

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By considering as an example, the case of the conformal algebra and some of its subalgebras, we demonstrate how the problem of calculating the structural parameters of semi-simple Lie algebras can be reduced to a set of computational prescriptions, which can readily be programmed. The importance of such computational prescriptions in the analysis of the structure of larger Lie algebras like the quaternionic Lie algebras is then pointed out.

Demonstra-se como o problema de calcular os parâmetros estruturais das álgebras de Lie semi-simples pode ser reduzido a um conjunto de prescrições computacionais, fácilmente programáveis, considerando, como um exemplo, a álgebra conforme e algumas de suas sub-álgebras. A importância de tais prescrições na análise da estrutura de álgebras de Lie maiores, como as álgebras de Lie quaterniônicas, é também ressaltada.

1. Introduction

The important part which group theoretical arguments can play in the study of physical problems is well known, especially in such areas as high energy physics and elementary particles. By exploiting the structural properties and representations of Lie groups and Lie algebras, one can arrive at very important physical conclusions. Thus one is often required in the study of these physical problems, to analyse first the structure of the underlying Lie algebra.

Now although the general procedure to be followed in the analysis of the structure of Lie algebras is well known and is in fact available in several places (1-7), one still has that each algebra has to be analysed as a separate problem. For the larger algebras, the analysis can become involved and one may need to use a computer. This is possible if one re-states the conventional algebraic theorems (2, 3, 8), which form the basis for these structural analysis, in the form of readily programmable ad hoc prescriptions. The prescriptions can then be applied to the analysis of any Lie algebra. We illustrate this point by considering the case of the conformal algebra $so(4, 2)$.

The technique is to exploit various theorems (see the Appendix) on the Killing forms of a Lie algebra, and to characterize a given Lie algebra by means of a set of parameters whose values provide, at a glance, information about the algebraic structure and Iwasawa-type decompositions of the given algebra and its associated analytic group.

2. Killing Forms of so(4, 2)

Consider the commutation relations of the general pseudo-orthogonal group $SO(n-s, s)$ with generators Z_{ij} satisfying

$$[Z_{ij}, Z_{kl}] = g_{jk}Z_{il} - g_{ik}Z_{jl} + g_{il}Z_{jk} - g_{jl}Z_{ik} \tag{1}$$

and $Z_{ij} = -Z_{ji}$.

We specialize to the case of SO(4, 2) by putting

$$g_{ij} = g_i \delta_{ij}$$

with $g_1 = g_2 = g_3 = g_4 = -g_5 = -g_6 = 1$.

Choosing our generators as:

$$\begin{aligned} Z_{1i} &= X_{i-1} \ : i = 2, 3, 4, 5, 6 \\ Z_{2i} &= X_{3+i} \ : i = 3, 4, 5, 6 \\ Z_{3i} &= X_{6+i} \ : i = 4, 5, 6 \\ Z_{4i} &= X_{8+i} \ : 5, 6 \\ Z_{56} &= X_{15} \end{aligned} \tag{2}$$

We can write down the usual set of commutation relations and structure constants defined generally by

$$[X_i, X_j] = C_{ijk}X_k. \tag{3}$$

If we use the notation:

$$C_{ij} = \pm k, 0 \text{ for } [X_i, X_j] = \pm X_k,$$

we can display these commutation relations and structure constants as elements of a 15 x 15 skew-symmetric matrix C shown in Table I. The corresponding 15 x 15 adjoint matrices for the generators, can be read off from this table. The adjoint matrix Ad_{X_i} for a generator X_i is defined by

$$(Ad_{X_i})_{jk} = (C_i)_{jk} \equiv C_{ijk} \ ; \ j, k = 1, 2, \ldots, n,$$

where $n = 15$ for $SO(4, 2)$ and C_{ijk} are the structure constants.

From these adjoint matrices one deduces the values of the Killing forms on the generators of $SO(4, 2)$. The defining relation is:

$$B(X_i, X_j) = \text{trace}\ (Ad_{X_i}\ Ad_{X_j}).$$

$i \backslash j$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}
X_1	0	−6	−7	−8	−9	+2	+3	+4	+5	0	0	0	0	0	0
X_2	+6	0	−10	−11	−12	−1	0	0	0	+3	+4	+5	0	0	0
X_3	+7	+10	0	−13	−14	0	−1	0	0	−2	0	0	+4	+5	0
X_4	+8	+11	+13	0	−15	0	0	+1	0	0	+2	0	+3	0	−5
X_5	+9	+12	+14	+15	0	0	0	0	+1	0	0	+2	0	+3	+4
X_6	−2	+1	0	0	0	0	−10	−11	−12	+7	+8	+9	0	0	0
X_7	−3	0	+1	0	0	+10	0	−13	−14	−6	0	0	+8	+9	0
X_8	−4	0	0	−1	0	+11	+13	0	−15	0	+6	0	+7	0	−9
X_9	−5	0	0	0	−1	+12	+14	+15	0	0	0	+6	0	+7	+8
X_{10}	0	−3	+2	0	0	−7	+6	0	0	0	−13	−14	+11	+12	0
X_{11}	0	−4	0	−2	0	−8	0	−6	0	+13	0	−15	+10	0	−12
X_{12}	0	−5	0	0	−2	−9	0	0	−6	+14	+15	0	0	+10	+11
X_{13}	0	0	−4	−3	0	0	−8	−7	0	−11	−10	0	0	−15	−14
X_{14}	0	0	−5	0	−3	0	−9	0	−7	−12	0	−10	+15	0	+13
X_{15}	0	0	0	+5	−4	0	0	+9	−8	0	+12	−11	+14	−13	0

Table 1 – Matrix C

We obtain the following results for the case of $so(4, 2)$:

$$\begin{aligned} B(X_i, X_j) &= -\ 8\ \delta_{ij} \text{ for } i, j = 1, 2, 3, 6, 7, 10, 15 \\ &= +\ 8\ \delta_{ij} \text{ for } i, j = 4, 5, 8, 9, 11, 12, 13, 14 \end{aligned} \tag{4}$$

These results are in fact a special case of the general result for $so(n-s, s)$, namely:

$$B(Z_{ij}, Z_{kl}) = (4 - 2n)\ (g_{ik}\, g_{jl} - g_{il}\, g_{jk})$$

Now, according to theorem 2 of the Appendix which we now state as a prescription, we have that the generators of $so(4, 2)$ can be separated into the two sets:

$$\begin{aligned} L_K &= (X_1, X_2, X_3, X_6, X_7, X_{10}, X_{15}) \\ P &= (X_4, X_5, X_8, X_9, X_{11}, X_{12}, X_{13}, X_{14}), \end{aligned} \tag{5}$$

where the elements of P are non-compact while the elements of L, generate the maximal compact subalgebra of $so(4, 2)$.

Writing down separate commutation relations for the elements of L_K, one verifies that they close a Lie algebra which is isomorphic to the Lie algebra of $SO(4) \otimes SO(2)$.

We find:

$$\begin{aligned} L_K &= so(4) \oplus so(2) \\ &= so(3)_+ \oplus so(3)_- \oplus so(2), \end{aligned}$$

where $so(3)_+$ has the elements

$$\begin{aligned} X_1^+ &= \tfrac{1}{2}(X_6 + X_3) \\ X_2^+ &= \tfrac{1}{2}(X_7 - X_2) \\ X_3^+ &= \tfrac{1}{2}(X_1 + X_{10}) \end{aligned}$$

while $so(3)_-$ has the elements

$$\begin{aligned} X_1^- &= \tfrac{1}{2}(X_6 - X_3) \\ X_2^- &= -\tfrac{1}{2}(X_7 + X_2) \\ X_3^- &= \tfrac{1}{2}(X_1 - X_{10}) \end{aligned}$$

Usually one introduces hermitian infinitesimal generators J_l of $so(3)$ by defining $Z_{jk} = \mathrm{i}\, \varepsilon_{jkl} J_l$ giving familiar commutation relations of the form: $\mathbf{J} \times \mathbf{J} = i\,\mathbf{J}$.

This can be achieved here by putting $J_l^{\pm} = -\, i X_l^{\pm}$.

One also checks that between the elements of P and L_K we have

$$\begin{aligned} [L_K, P] &\subset P \\ [P, P] &\subset L_K \\ [L_K, L_K] &\subset L_K \end{aligned}$$

$$B(P, L_K) = B(L_K, P) = 0.$$

so that we can write $so(4, 2) = L_K \oplus P$

3. Eigenvalue Spectrum and Eigenfunctions of so(4, 2)

By considering further, the structure of the subset P we can cbtain the roots and eigenfunctions of $so(4, 2)$. The prescription is to pick out those elements of P which are mutually commuting. Denoting this subset of P by L, and writing out the commutation relations of all the elements of P, we find that we can choose:

$$L_A = (X_4, X_{12}). \tag{6}$$

L_A is an abelian subalgebra of P.

The centralizer of L_A in $so(4, 2)$, defined generally by

$$L^0 = \{X \in so(4,\ 2) : [L^*,\ X] = 0\},$$

is next deduced. Scanning through the commutation relations in Eq. (3) [or Table I], we obtain that L^0 may be chosen as:

$$L^0 = (X_4,\ X_7,\ X_{12}),$$

which is the maximal abelian or Cartan subalgebra of $so(4,\ 2)$.

Having thus extracted L^0, the next prescription is to construct eigenfunctions of L_A out of the remaining generators of $so(4,\ 2)$. Forming the following linear combinations:

$$\begin{array}{ll} Z^{\pm} = X_6 \mp X_9\ ; & W^{\pm} = X_{10} \pm X_{14} \\ P^{\pm} = X_1 \pm X_8\ ; & Q^{\pm} = X_3 \pm X_{13} \\ S_1 = R_1 + T_2\ ; & S_2 = R_1 - T_2 \\ S_3 = R_2 + T_1\ ; & S_4 = R_2 - T_1, \end{array} \tag{7}$$

where

$$\begin{array}{ll} R_1 = X_2 + X_5 & ;\ R_2 = X_2 - X_5, \\ T_1 = X_{11} + X_{15} & ;\ T_2 = X_{11} - X_{15}, \end{array}$$

we find from Eq. (3) that:

$$\begin{array}{llr}
[X_4,\ Z^+] = 0 & ;\quad [X_{12},\ Z^+] = Z^+ & \text{(8a)} \\
[X_4,\ Z^-] = 0 & ;\quad [X_{12},\ Z^-] = -Z^- & \text{(8b)} \\
[X_4,\ W^+] = 0 & ;\quad [X_{12},\ W^+] = W^+ & \text{(8c)} \\
[X_4,\ W^-] = 0 & ;\quad [X_{12},\ W^-] = -W^- & \text{(8d)} \\
[X_4,\ P^+] = P^+ & ;\quad [X_{12},\ P^+] = 0 & \text{(8e)} \\
[X_4,\ P^-] = -P^- & ;\quad [X_{12},\ P^-] = 0 & \text{(8f)} \\
[X_4,\ Q^+] = Q^+ & ;\quad [X_{12},\ Q^+] = 0 & \\
[X_4,\ Q^-] = -Q^- & ;\quad [X_{12},\ Q^-] = 0 & \text{(8h)} \\
[X_4,\ S_1] = S_1 & ;\quad [X_{12},\ S_1] = -S_1 & \text{(8i)} \\
[X_4,\ S_2] = -S_2 & ;\quad [X_{12},\ S_2] = -S_2 & \\
[X_4,\ S_3] = +S_3 & ;\quad [X_{12},\ S_3] = +S_3 & \text{(8k)} \\
[X_4,\ S_4] = -S_4 & ;\quad [X_{12},\ S_4] = +S_4. & \text{(8l)}
\end{array}$$

We conclude from Eqs. (8a)-(8d) that Z^+ and W^+ are degenerate eigenfunctions of L_A with the two-component eigenvalue

$$\vec{\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Similarly Z^- and W^- are degenerate eigenfunctions with the eigenvalue

$$-\vec{\alpha} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Equations (8e)-(8h) show that P^+ and Q^+ are degenerate eigenfunctions with the eigenvalue

$$\vec{\beta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

while P^- and Q^- have the eigenvalue

$$-\vec{\beta} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

From Eqs. (8i)-(8l) we get the other eigenfunctions as

S_3, with eigenvalue $\vec{\gamma} = \begin{pmatrix} +1 \\ +1 \end{pmatrix}$

S_2, with eigenvalue $-\vec{\gamma} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

S_4, with eigenvalue $+\vec{\eta} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

S_1, with eigenvalue $-\vec{\eta} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The roots of $so(4, 2)$ with respect to L_A are therefore given by

$$\begin{aligned} \vec{\alpha} &= \begin{pmatrix} \quad \end{pmatrix} \quad ; \quad \text{(muitipiicity 2)} \\ \vec{\beta} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \text{(multiplicity 2)} \\ \vec{\gamma} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ; \quad \text{(no degeneracy)} \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad ; \quad \text{(no degeneracy).} \end{aligned} \tag{9}$$

These roots may be plotted in a root diagram shown in Fig. 1.

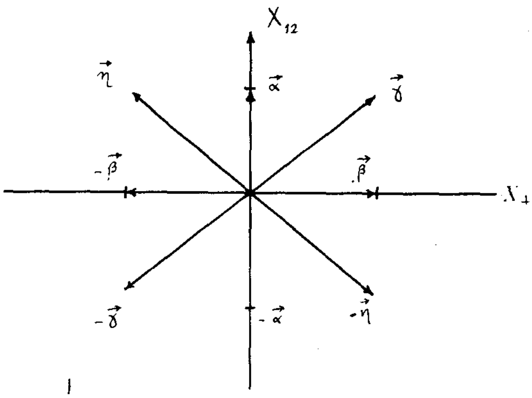

Fig. 1 – ***Roots of*** *so*(4, *2)* with ***respect to*** L_A.

We can collect together these degenerate eigenfunctions by writing:

$$
\begin{aligned}
L^{\alpha} &= (Z^+,\ \mathrm{W}+) \quad ; \quad L^{-\alpha} = (Z^-,\ \mathrm{w}-) \\
L^{\beta} &= (P^+,\ Q^+) \quad ; \quad L^{-\beta} = (P^-,\ Q^-) \\
L^{\gamma} &= (S_3) \quad ; \quad L^{-\gamma} = (S_2) \\
\mathrm{C} &= (S_4) \quad ; \quad L^{-\eta} = (S_1).
\end{aligned}
\tag{10}
$$

The hyperplanes orthogonal to the roots can be deduced from Fig. 1. We have

$$P_{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$P_{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$P_\gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$P_\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{11}$$

These hyperplanes are shown in Fig. 2.

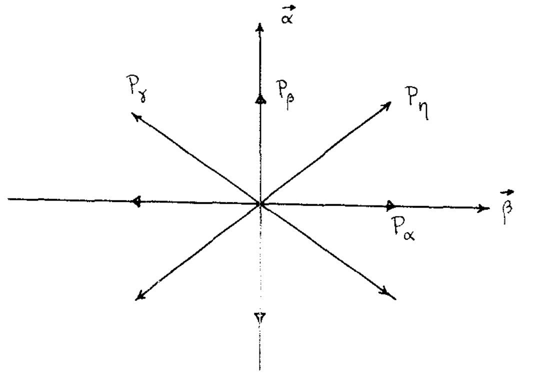

Fig. 2 – Hyperplanes in the root space of $so(4, 2)$

4. The Weyl Group of so(4, 2):

Resides the roots and hyperplanes, another structural parameter which one needs to compute for $so(4, 2)$ is the Weyl reflection group. The prescription is to calculate first the characteristic vectors $\vec{H}$ associated with the roots. If we denote the basis vectors of the vector space L_A by

$$\vec{H}_1 = X_4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \; \vec{H}_2 = X_{12} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then the vector $\vec{H}^1_\alpha$ associated with any root $\vec{\alpha}$ is defined generally by

$$B(H_i, \vec{H}^1_\alpha) = \alpha_i,$$

where α_i is the i^{th} component of the chosen root 2. Thus for $\vec{\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we get that

$$B(X_4, \vec{H}^1_\alpha) = 0,$$
$$B(X_{12}, \vec{H}^1_\alpha) = 1,$$

from which we conclude that

$$\vec{H}^1_\alpha = \frac{1}{8}\begin{pmatrix} 0 \\ X_{12} \end{pmatrix}.$$

Similarly, we find that

$$\vec{H}^1_\beta = \frac{1}{8}\begin{pmatrix} X_4 \\ 0 \end{pmatrix},$$

$$\vec{H}^1_\gamma = \frac{1}{8}\begin{pmatrix} X_4 \\ X_{12} \end{pmatrix}, \qquad (12)$$

$$\vec{H}^1_\eta = \frac{1}{8}\begin{pmatrix} -X_4 \\ X_{12} \end{pmatrix}$$

Then, the characteristic vectors in normalized form become:

$$\vec{H}_\alpha = 2X_{12}\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \vec{H}_\beta = 2X_4\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{H}_\gamma = \begin{pmatrix} X_4 \\ X_{12} \end{pmatrix}; \quad \vec{H}_\eta = \begin{pmatrix} -X_4 \\ X_{12} \end{pmatrix}. \qquad (13)$$

We the get the following integers:

$$\alpha(\vec{H}_\beta) = B(\vec{H}_\beta, \vec{H}^1_\alpha) = n_{\alpha\beta} = 0,$$
$$\beta(\vec{H}_\alpha) = B(\vec{H}_\alpha, \vec{H}^1_\beta) = n_{\beta\alpha} = 0,$$
$$\alpha(\vec{H}_\gamma) = B(\vec{H}_\gamma, \vec{H}^1_\alpha) = n_{\alpha\gamma} = 1,$$
$$\gamma(\vec{H}_\alpha) = B(\vec{H}_\alpha, \vec{H}^1_\gamma) = n_{\gamma\alpha} = 2,$$

from which we get for example that

$$\cos^2(\vec{\alpha}, \vec{y}) = \frac{n_{\alpha\gamma} \cdot n_{\gamma\alpha}}{4} = \frac{1}{2} \quad \text{or} \quad \cos(\vec{a}, \vec{y}) = \frac{1}{\sqrt{2}}.$$

This is consistent with the root diagram in Fig. 1.

We now compute the Weyl reflection operators associated with these roots. In general for each root $\vec{\alpha}$, we can define a transformation operator S_α which has the following properties:

(i) $$S_\alpha \vec{H} = \vec{H} - \alpha(\vec{H})\, \vec{H}_\alpha,$$

where $\vec{H}$ is an arbitrary vector in the vector space L_A. For the particular case where $\vec{H} = \vec{H}_\alpha$, we have:

$S_\alpha \vec{H}_\alpha = - \vec{H}_\alpha$ since $\alpha(\vec{H}_\alpha) = 2$ for any root $\vec{\alpha}$. (See reference (8)). In addition, we have

(ii) $$S_\alpha P_\alpha = P_\alpha,$$

(iii) $$(S_\alpha)^2 = I.$$

We make use of these properties by writing the operator S generally as

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then, using the properties as constraint equations for finding S, we obtain the matrix representations of the Weyl reflection operators for $so(4, 2)$. Thus for the root $\vec{\alpha}$, the constraints take the form:

$$S_\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_\alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_\alpha^2 = I .$$

This leads to the result

$$S_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Similarly we find:

$$S_\beta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S_\gamma = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad S_\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (14)$$

These matrices, together with the product matrices $(S_\gamma . S_\beta)$, $(S_\beta . S_\gamma)$, and $(S_\gamma . S_\beta)^2$, and the 2-dimensional identity matrix, form a 2-dimensional representation of a finite group of order 8 which is the Weyl reflection group for $so(4, 2)$.

5. The Nilpotent Subalgebra

Given now these structural parameters: the roots, the hyperplanes and the Weyl reflection operators of $so(4, 2)$, we need to compute next the characteristic nilpotent sub-algebra of $so(4, 2)$. This is deduced from a consideration of the hyperplanes given in Eq. (11). The prescription is to select, arbitrarily, some hyperplane and, with respect to it, to classify as positive all roots on one side of the hyperplane and as negative, the remaining roots on the other side. Here we may select the hyperplane P_γ. Then $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ and $\vec{\eta}$ come out as positive roots while $-\vec{\alpha}$, $-\vec{\beta}$, $-\vec{\gamma}$ and $-\vec{\eta}$ become negative. [Strictly speaking, the two roots $\vec{\beta}$ and $-\vec{\beta}$ lie on the selected hyperplane P_γ and cannot therefore be described as lying on one side or the other of P_γ. To get over this problem one should select a hyperplane which does not contain any of the roots].

From Eq. (8), the eigenfunctions with the positive roots are:

$$L_N^+ = (Z^+,\ W^+;\ P^+,\ Q^+;\ S_3;\ S_4). \tag{15}$$

Then the standard theorems[8] lead us to the prescription that: these positive root eigenfunctions form a nilpotent subalgebra of the parent algebra $so(4, 2)$. This can be verified. We find $L_N^3 = 0$.

6. Iwasawa Decomposition of so(4, 2)

Having thus identified the generators of $so(4, 2)$ which form the subalgebras L_K, L_A and L_N^+, one applies the Iwasawa theorem[3,8], according to which one can write simply:

$$so(4,2) = so(3)_+ \ (+)\ so(3)_-\ (+)\ so(2)_{X_{15}}\ (+)\ L_A\ (+)\ L_N^+$$

where L_A and L_N^+ are given by equations (6) and (15), respectively.

At the group level, this Iwasawa decomposition of $SO(4, 2)$ becomes:

$$G = K.A.N, \tag{16b}$$

where K, A and N are the connected analytic subgroups of $SO(4, 2)$ which correspond to the Lie algebras L_K, L_A and L_N^+ respectively. These subgroups can be parametrized in the usual way as follows. First one constructs suitable matrix representations of the generators of $SO(4, 2)$. In general, we always choose for the generators of $so(n\text{-}s, s)$, the representation:

$$\begin{aligned} Z_{ij} &= e_{ij} - e_{ji} \text{ for } \mathbf{i}, \mathbf{j} \leq (\text{n-s}), \\ Z_{ij} &= -e_{ij} + e_{ji} \text{ for } i, j > (\text{n-s}), \\ Z_{ij} &= +e_{ij} + e_{ji} \text{ for } \mathbf{i} \leq (\text{n-s});\ j > (\text{n-s}), \\ Z_{ij} &= -e_{ij} - e_{ji} \text{ for } i > (\text{n-s}), j \leq (\text{n-s}), \end{aligned} \tag{17}$$

where e_{ij} is the matrix with 1 at the i^{th} row and j^{th} column, but otherwise zero everywhere.

We get for example:

$$Z_{15} = X_4 = \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix},$$

$$Z_{36} = X_{12} = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix},$$

etc.

One checks that these matrices reproduce the commutation relations in Eq. (3).

Then, the abelian subgroup A can be parametrized as

$$A = e^{\theta X_4}\, e^{\varphi X_{12}} = \begin{pmatrix} \cosh\theta & 0 & 0 & 0 & \sinh 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\varphi & 0 & 0 & \sinh\varphi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sinh\theta & 0 & 0 & 0 & \cosh\theta & 0 \\ 0 & 0 & \sinh\varphi & 0 & 0 & \cosh\varphi \end{pmatrix}$$

with $-\infty \leq \theta,\ \varphi \leq +\infty$.

Similarly, writing

$$N = e^{C_1 Z^+} e^{C_2 W^+} e^{C_3 P^+} e^{C_4 Q^+} e^{C_5 S_3} e^{C_6 S_4}$$

and using Eqs. (7) and (17), we get

$$e^{C_1 Z^+} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & C_1 & 0 & 0 & -C_1 \\ 0 & -C_1 & 1-C_1^2/2 & 0 & 0 & C_1^2/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -C_1 & -C_1^2/2 & 0 & 0 & 1+C_1^2/2 \end{pmatrix}$$

$$e^{C_2 W^+} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-C_2^2/2 & C_2 & 0 & C_2^2/2 \\ 0 & 0 & -C_2 & 1 & 0 & C_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -C_2^2/2 & C_2 & 0 & 1+C_2^2/2 \end{pmatrix}$$

$$e^{C_3 P^+} = \begin{pmatrix} 1-C_3^2/2 & C_3 & 0 & 0 & C_3^2/2 & 0 \\ -C_3 & 1 & 0 & 0 & C_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -C_3^2/2 & C_3 & 0 & 0 & 1+C_3^2/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{C_4 Q^+} = \begin{pmatrix} 1-C_4^2/2 & 0 & 0 & C_4 & C_4^2/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -C_4 & 0 & 0 & 1 & C_4 & 0 \\ -C_4^2/2 & 0 & 0 & C_4 & 1+C_4^2/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

$$e^{C_5 S_3} = \begin{pmatrix} 1 & 0 & C_5 & 0 & 0 & -C_5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -C_5 & 0 & 1 & 0 & C_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & C_5 & 0 & 1 & -C_5 \\ -C_5 & 0 & 0 & 0 & C_5 & 1 \end{pmatrix}$$

$$e^{C_6 S_4} = \begin{pmatrix} 1 & 0 & C_6 & 0 & 0 & -C_6 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -C_6 & 0 & 1 & 0 & -C_6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -C_6 & 0 & 1 & C_6 \\ -C_6 & 0 & 0 & 0 & -C_6 & 1 \end{pmatrix}$$

where the C_i are arbitrary real parameters with:

$$-\infty \leq C_i \leq +\infty; \quad i = 1, 2, \ldots, 6$$

For the maximal compact subgroup, we can use Euler-type paramete-rization and write:

$$K = e^{\theta_1 X_1} e^{\theta_2 X_2} e^{\theta_3 X_3} e^{\theta_4 X_6} e^{\theta_5 X_7} e^{\theta_6 X_{10}} e^{\theta_7 X_{15}},$$

where

$$e^{\theta_1 X_1} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{\theta_2 X_2} = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{\theta_3 X_3} = \begin{pmatrix} \cos\theta_3 & 0 & 0 & \sin\theta_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\sin\theta_3 & 0 & 0 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{\theta_4 X_6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta_4 & \sin\theta_4 & 0 & 0 & 0 \\ 0 & -\sin\theta_4 & \cos\theta_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{\theta_5 X_7} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta_5 & 0 & \sin\theta_5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\sin\theta_5 & 0 & \cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

$$e^{\theta_6 X_{10}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta_6 & \sin\theta_6 & 0 & 0 \\ 0 & 0 & -\sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{\theta_7 X_{15}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_7 & -\sin\theta_7 \\ 0 & 0 & 0 & 0 & \sin\theta_7 & \cos\theta_7 \end{pmatrix}$$

Putting these results (18), (19) and (20) into Eq. (16b), we arrive at the Iwasawa decomposition and planar parameterization of $SO(4, 2)$. The results may be compared for example with those of Kihlberg, Muller and Halbwachs[9] for the covering group $SU(2, 2)$.

7. Structural Parameters of Subalgebras of so(4, 2).

Now it is well known that the conformal algebra $so(4, 2)$ has many interesting subalgebras. These include the De Sitter algebras $so(4, 1)$ and $so(3, 2)$; the Lorentz algebra $so(3, 1)$ and the Poincaré algebra $p = so(3, 1) \circledS T_4$; also the 2-dimensional conformal algebra $so(2, 2)$ which is isomorphic to $so(2, 1) \otimes so(2, 1)$. Using the structural parameters already computed for $so(4, 2)$ we can deduce the structural parameters of these subalgebras. All that is necessary is to identify those elements of $so(4, 2)$ which form a given subalgebra. The identification is not unique in some cases, in the sense that one can find different sets of generators of $so(4, 2)$ which close, for example, the algebra of $so(3, 1)$ or $so(4, 1)$.

Now, since we are working with the metric

$$g = (+,\ +,\ +,\ +,\ -,\ -),$$

we can select by inspection the generators of these subalgebras as follows:

$so(3, 1)$: $Z_{23}, Z_{24}, Z_{25}, Z_{34}, Z_{35}, Z_{45}$
$so(4, 1)$: $Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{23}, Z_{24}, Z_{25}, Z_{34}, Z_{35}, Z_{45}$
$so(3, 2)$: $Z_{23}, Z_{24}, Z_{25}, Z_{26}, Z_{34}, Z_{35}, Z_{36}, Z_{45}, Z_{46}, Z_{56}$.
$so(2, 2)$: $Z_{34}, Z_{35}, Z_{36}, Z_{45}, Z_{46}, Z_{56}$.

In terms of the generators X_i of $so(4, 2)$, we can thus write

$$so(3, 1) = (X_6, X_7, X_8, X_{10}, X_{11}, X_{13}) \tag{21a}$$
$$so(4, 1) = (X_1, X_2, X_3, X_4, X_6, X_7, X_8, X_{10}, X_{11}, X_{13}) \tag{21b}$$
$$so(3, 2) = (X_6, X_7, X_8, X_9, X_{10}, X_{11}, X_{12}, X_{13}, X_{14}, X_{15}) \tag{21c}$$
$$so(2, 2) = (X_{10}, X_{11}, X_{12}, X_{13}, X_{14}, X_{15}). \tag{21d}$$

Other subalgebras can similarly be considered.

In general, having identified the generators of any subalgebra, the Killing forms restricted to this subalgebra can be deduced from Eq. (4). The same prescriptions as before, can then be used to compute the structural parameters of the subalgebras. The same holds for other semi-simple Lie algebras. We then observe generally that the entire analysis as set out above can readily be programmed.

Summary

To decompose a given semi-simple Lie algebra L, in the Iwasawa fashion, the following prescriptions are to be followed.
(1) The Killing form $B(X_i, X_i)$ of each generator X_i is calculated, using the known structure constants of the algebra.
(2) The maximal compact subalgebra L_K and its orthogonal complement P are deduced.
(3) Next we find the Cartaqsubalgebra L_A of Land calculate the corresponding root vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$, etc. The hyperplanes P_α, P_β, P_γ then follow.
(4) The sets of positive and negative root eigenfunctions L_N^+ and L_N^- are next calculated, leading to the Iwasawa decomposition of the algebra L.

(5) Extension to the group level is made on the basis of the known topological properties of connected analytic subgroups.

Appendix

We collect together in this appendix the most important theorems which are needed for the structural analysis of classical Lie algebras.

Theorem 1

A Lie algebra L is compact if and only if its Killing form is negative definite, that is,

$$B(X_i, X_j) \leq 0$$

for all X_i, X_j and i, j = 1, 2 . . . . n.

Proof:

Let L be a compact Lie algebra. For such an algebra, one has that the operators Ad_{X_i} which are the inner derivations of L, form a Lie algebra known as the algebra of inner derivations. This algebra is isomorphic to a subalgebra of the special orthogonal group $SO(n)$, implying that

$$\left(Ad_X\right)_{ik} = -\left(Ad_X\right)_{ki}$$

since the generators of $SO(n)$ correspond to the algebra of antisymmetric matrices.

We can then write

$$B(X, X) = \sum_{i,k} \left(Ad_X\right)_{ik} \left(Ad_X\right)_{ki}$$

$$= - \sum_{i,k} \left[\left(Ad_X\right)_{ik}\right]^2 \leq 0,$$

as required. For further details we refer the reader to p. 123 of Helgason[3].

As a corollary to this theorem, we add that for the non-compact part of a given non-compact Lie algebra, the Killing form is positive definite.

Theorem 2:

Any semi-simple Lie algebra L which is of the non-compact type, has a direct sum decomposition of the form:

$$L = L_K \oplus P$$

where the subsets L_K and $\boldsymbol{P}$ satisfy the following conditions:

(i) $[L_K, L_K] \subset L_K$
(ii) $[P, P] \subset L_K$

(iii) $[L_K, P] \subset P$
(iv) $B(L_K, L_K) < 0$
$B(P, P) > 0$
(v) $B(L_K, P) = 0$

These assertions are easily demonstrated by means of examples. Rigorous proofs are harder to construct. Useful discussions are given in chapter 5 of Hermann[2]. See also p. 60 of Strom[4].

Proposition 1

With respect to a non-compact Lie algebra L, we can introduce a linear map θ such that for

$$L = L_K \oplus P$$

we have:

$$\theta(X) = X \text{ for all } X \in L_K,$$
$$\theta(X) = -X \text{ for all } X \in P,$$
and
$$B(\theta X, \theta Y) = B(X, Y) \text{ for all } X, Y \in L.$$

This automorphism 0 is known as the Cartan involution of Lwith respect to L_K.

Theorem 3

By means of the Cartan involution θ, one can define a new bilinear form:

$$B_1(X, Y) = -B(X, OY),$$

for any two elements X and Y of L, where $B_1(X, Y)$ is symmetric, positive definite, and non-degenerate.

Proof:

The proof of this assertion follows directly from the definition of 0. Thus we have

$$B_1(X, Y) = -B(X, OY).$$

Suppose that both X and Y belong to $L_{,,}$ then $\theta Y = Y$ so that

$$B_1(X, Y) = -B(X, Y),$$

which is positive and non-degenerate from theorems 1 and 2.

Next, suppose that both X and Y belong to P, then

$$\theta Y = -Y$$
and
$$B_1(X, Y) = -B(X, -Y) = B(X, Y).$$

This is again positive definite from theorem 2.
Suppose that $X \in L$, and $Y \in P$, then

$$B_1(X, Y) = -B(X, -Y) = B(X, Y) = 0 \text{ from theorem 2.}$$

Similarly if $X \in P$ and $Y \in L_K$ we get that

$$B_1(X,\ Y) = 0 \text{ as required.}$$

That $B_1(X,\ Y)$ is also always symmetric and non-degenerate follows from the basic properties of the Killing form $B(X,\ Y)$.

Theorem 4

With respect to the positive-definite, symmetric and non-degenerate bilinear form $B_1(X,\ Y)$, the adjoint operators Ad_X for all $X \in P$, are hermitian and have real eigenvalues.

Proof:

To prove the theorem, we need to show that for all $X \in P$ and for any Y, Z belonging to L, we have

$$B_1(Ad_X(Y),\ Z) = B_1(Y,\ Ad_X(Z)).$$

Now we have:

(i) $B_1(Ad_X(Y),\ Z) = B_1([X,\ Y],\ Z)$

(ii) $B_1(X,\ Y) = -\ B(X,\ \theta(Y))$

(iii) $B([X,\ Y],\ Z) = -\ B(Y, [X,\ Z])$

(iv) $Ad_X(\theta Z) = [X,\ \theta Z] = \theta[(\theta X),\ Z]$
$= \theta Ad_{(\theta X)}(Z) = -\ \theta Ad_X(Z)$
since $\theta X = -\ X$ for $X \in P$.

Using these results we get that

$$\begin{aligned} B_1(Ad_X(Y),\ Z) &= B_1([X,\ Y],\ Z) \\ &= -B\ ([X,\ Y],\ \theta Z) \\ &= +B\ (Y,\ [X,\ \theta Z]) \\ &= -B\ (I{:}\,\theta[X,\ Z]) \\ &= +B_1(Y,\ [X,\ Z]). \end{aligned}$$

That is,

$$B_1(Ad_X(Y),\ Z) = B_1(Y,\ Ad_X(Z)) \text{ as required.}$$

Proposition 2

Since the elements of P are hermitian, and have real eigenvalues, one can introduce a subset of mutually commuting elements of P for which one can construct simultaneous eigenvectors. These eigenvectors can be labelled, as in Quantum mechanics, by the real eigenvalues of the mutually commuting hermitian operators.

The required subset of mutually commuting hermitian operators can be chosen as the set of adjoint operators

$$\{Ad_X\ :\ X \in L_A\}$$

where (i) L_A is abelian; (ii) $L_A \subset P$; and (iii) L_A is maximal among the subalgebras of L, satisfying conditions (i) and (ii).

Denoting the elements of L_A by H and the remaining elements of L by X, Y, ... , we can write:

$$Ad_H(X) = [H,\ X] = \alpha(H)X,$$

where $\alpha(H) = \alpha_H$ is a real number, the eigenvalue of the hermitian operator Ad_H; X is the eigenvector.

If L_A has dimension r, with elements H_i, where i = 1, 2 . . . . r, then the quantities $\alpha(H_i)$ given by

$$Ad_{H_i}(X) = \alpha(H_i)X = \alpha_i X$$

constitute the components of a vector 2 in an r-dimensional vector space. This vector $\vec{\alpha}$ is called the root vector. For a given Lie algebra L, root vectors occur in pairs, the total number of root vectors being equal to the number of linearly independent eigenvectors.

For detailed discussions regarding this proposition and the properties of the roots, we refer the reader to references (2, 3, 4).

Proposition 3

Given a root vector $\vec{\alpha}$ we can consider the set of all vectors which lie on a hyperplane orthogonal to 2. Such a hyperplane we denote by P_α defined by

$$P_\alpha = \{H \in L_A : \alpha(H) = 0\}$$

With respect to a given hyperplane P_α, we can divide the root space into two half-spaces P_α^+ and P_α^- defined by

$$P_\alpha^+ = \{H \in L_A : \alpha(H) > 0\},$$
$$P_\alpha^- = \{H \in L_A : \alpha(H) < 0\}.$$

We can, on this basis, classify all the roots of a given algebra L into positive roots ($\alpha(H) > 0$) and negative roots ($\alpha(H) < 0$). Also we denote by L_N^+ the set of all eigenvectors with positive roots with respect to a selected hyperplane, and by L_N^- the set of all eigenvectors with negative roots with respect to the same hyperplane. We have the following theorem:

Theorem 5:

The subsets L; and L_N^- are nilpotent subalgebras of L.

Proof:

The proof that the algebras L_N^+ and L_N^- are nilpotent proceeds as follows. The partitioning into positive and negative roots is such that all eigenfunctions in L_N^+ have only positive roots, while the elements of L_N^- have only negative roots. Thus, if L^α is a subset of L_N^+ with positive root 2, while L is another subset with positive root $\vec{\beta}$, we should have that

$$[E, L^\beta] \subset L^{\alpha+\beta}$$

where $L^{\alpha+\beta}$ exists if and only if $(a+\beta)$ # 0. This would however lead to arbitrarily large positive roots. The iteration must therefore eventually terminate, implying that L_N^+ and L_N^- are nilpotent algebras.

Finally, we have:

Theorem 6

A semi-simple Lie algebra L can always be decomposed uniquely as follows

$$L = L_K \oplus L_A \oplus L_N^+.$$

This is known as the Iwasawa decomposition of the semi-simple Lie algebra L. Theorem 6 can be proved on the basis of the earlier theorems. Details of the proof can be found in Helgason[3] and Strom[4].

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