Revista Brasileira de Física, Vol. 5, N.° 1, 1975

Gradient Formula for the Four-Dimensional Hyperspherical Harmonics

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Recebido em 12 de Novembro de 1974

The gradient formula for the hyperspherical harmonics in 4 dimensions is derived, a result which is here obtained in two distinct ways: either by differentiation of a closed expression for the hyperspherical harmonics or by making use of the Wigner-Eckart theorem for the R_4 group. The result is useful for physical applications in view of the significance of the R_4 group in several physical problems.

Deriva-se a fórmula do gradiente para os hiperesféricos harmônicos em 4 dimensões. O resultado é obtido de duas maneiras distintas: por diferenciação de uma expressão fechada para os hiperesférims harmônicos ou pela aplicação do teorema de Wigner-Eckart para o grupo R_4. O resultado é útil para aplicações em vista da relevância do grupo R_4 em diversos problemas físicos.

Introduction

The gradient formula for the spherical harmonics $Y_{lm}(\theta, \varphi)$ in 3 dimensions is well known from textbooks on the quantum theory of angular momentum[1]. It provides an useful expression for $\nabla[F(r)\, Y_{lm}(\theta, q)]$, where $F(r)$ is an arbitrary differentiable function of the scalar distance r.

In the present work, the corresponding formula for the 4-dimensional hyperspherical harmonics $Y_{klm}(\Omega)$ is derived in two different ways. In the first, we start from a closed expression for the solid hyperspherical harmonics[2] $\mathcal{Y}_{klm}(x_1, x_2, x_3, x_4)$, making extensive use of their elementary properties. As an alternative, the sarne formula is obtained from the Wigner-Eckart theorem for the rotation group in 4 dimensions[3] (R_4).

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The results are of special relevance for applications, in view of the intrinsic interest of the R_4 group in several physical problems[4].

1. The Gradient Formula for $\nabla[F(R)Y_{klm}(\Omega)]$

The solid hyperspherical harmonics in 4 dimensions, $\mathcal{Y}_{klm}(x_1, x_2, x_3, x_4)$, are homogeneous and harmonic polynomials of degree k which carry a class of irreducible representations of the R_4 group in the $R_4 \supset R_3 \supset R_2$ chain, namely, those irreducible representations of the type $[k/2,\ k/2]$, in the notation of Ref. 3.

They can be expressed in the form[2]

$$\mathcal{Y}_{klm}(x_1,\ x_2,\ x_3,\ x_4) = C(k,\ l)\ G_{kl}(R,\ x_4)\ \mathcal{Y}_{lm}(x_1,\ x_2,\ x_3), \tag{1-1}$$

where R denotes the hyperdistance $R^2 = \sum_{\alpha=1}^{4} x_\alpha^2$, the $\mathcal{Y}_{lm}(x_1, x_2, x_3)$ are solid hannonics in 3 dimensions and G_{kl} (R, x_4) are functions given by

$$G_{kl}(R,\ x_4) = \sum_{\mu=0}^{\left[\frac{k-l}{2}\right]} \frac{(-)^\mu (k-\mu)!\, R^{2\mu} x_4^{k-l-2\mu}}{2^{2\mu} \mu!\ (k-l-2\mu)!}. \tag{1-2}$$

For a given k (a non-negative integer), l assumes the values 0,1, 2, . . . , k. Normalization to one, over the unit hypersphere, gives[5]

$$C(k,\ l) = 2^k(-i)^l \left[\frac{2(k+1)\,(k-l)!}{\pi\,(k+l+1)\,!}\right]^{\frac{1}{2}} \tag{1-3}$$

The surface hyperspherical harmonics are defined by

$$Y_{klm}(\Omega) \equiv Y_{klm}(\theta,\ \varphi,\ \lambda) = R^{-k}\mathcal{Y}_{klm}(x_1,\ x_2,\ x_3,\ x_4), \tag{1-4}$$

where 0, φ and λ are the well-known angles of the polar parametrization of E_4:

$$\begin{aligned} x_1 &= \mathrm{R}\ \sin\ \lambda\ \sin\ \theta\ \cos\ \varphi, \\ x_2 &= R\ \sin\ \lambda\ \sin\ \theta\ \sin\ \varphi, \\ x_3 &= \mathrm{R}\ \sin\ \lambda\ \cos\ \theta, \\ x_4 &= \mathrm{R}\ \cos\ \lambda, \end{aligned}$$

with

$$0 \leq \lambda \leq \pi, \quad 0 \leq \theta \leq \pi \quad \text{and} \quad 0 \leq \varphi \leq 2\pi.$$

On utilizing the property[2] of the function $G_{kl}(R, x_4)$

$$G_{kl}(R, x_4) = \frac{l!}{2^{k-l}} R^{k-l} C_{k-l}^{l+1} \left(\frac{x_4}{R}\right),$$

where the C_{k-l}^{l+1} are Gegenbauer polynomials[6], one readily obtains the expression

$$Y_{klm}(\Omega) = (-2i)^l l! \left[\frac{2(k+1)(k-l)!}{\pi (k+l+1)!}\right]^{\frac{1}{2}} Y_{lm}(\theta, \varphi) C_{k-l}^{l+1}(\cos\lambda) \sin^l\lambda. \quad (1\text{-}5)$$

The functions (1-5) satisfy the rule

$$Y^*_{klm}(\Omega) = (-)^{l+m} Y_{k,l,-m}(\Omega).$$

As shown in Ref. 2, the function $G_{kl}(R, x_4)$ also obeys the following properties, which follow from (1-2):

$$x_4 G_{kl} = 2 G_{k+1,l} - (k+l+1) G_{k,l-1}, \quad (1\text{-}6a)$$
$$R^2 G_{kl} = 4 [G_{k+2,l} - (k+2) G_{k+1,l-1}], \quad (1\text{-}6b)$$
$$\partial_i G_{kl} = -\frac{1}{2} x_i G_{k-1,l+1}, \quad (1\text{-}6c)$$
$$\partial_4 G_{kl} = \frac{1}{2} (k+l+1) G_{k-1,l}. \quad (1\text{-}6d)$$

The components of the 4-dimensional gradient V are defined as

$$V = (-i V_, \ V_4), \quad (1\text{-}7)$$

with

$$V_4 = \partial_4,$$
$$V_q = \begin{cases} -\frac{1}{\sqrt{2}} (\partial_1 + i\partial_2), & (q = +1) \\ \partial_3, & (q=0) \\ +\frac{1}{\sqrt{2}} (\partial_1 - i \quad , & (q = -1). \end{cases}$$

The defmition (1-7) is a consequence of the vector character of the gradient **V**, as an irreducible vector operator T^1_{lm} of the R_4 group, associated to its irreducible representation of dimension 4, in the chain $R_4 \supset R_3 \supset R_2$. In fact, by a straightforward calculation, it can be shown from the Racah's defmition of T^1_{lm},

$$[\mathcal{G}_q, T^1_{lm}] = \sum_{l'm'} (1l'm'|\mathcal{G}_q|1lm) T^1_{l'm'},$$

in terms of the well known matrix elements of the generators $\mathcal{G}_q \equiv \begin{pmatrix} L_q \\ A_q \end{pmatrix}$ of the R_4 group[3], in the chain $R_4 \supset R_3 \supset R_2$, that if

$$T^1_{00} = \alpha \partial_4,$$
$$T^1_{1q} = \beta_q \nabla_q,$$

then necessarily $i\beta = a$, where $\beta = \beta_{+1} = \beta_0 = \beta_{-1}$. Setting a = 1, then $\beta = -i$, a fact which justifies the choice of the relative phases in (1-7). Similarly, the components of the vector operator x are defined as

$$\mathrm{x} \equiv \quad (-i\, x_q,\ x_4) \tag{1-8}$$

with

$$x_q = \begin{cases} -\dfrac{1}{\sqrt{2}}\,(x_1 + i\,x_2), & (q = +1) \\ \qquad x_3, & (q = 0) \\ +\dfrac{1}{\sqrt{2}}\,(x_1 - i\,x_2), & (q = -1). \end{cases}$$

Our airn now is to compute $\nabla\, [F(R)\, Y_{klm}]$. From (1-4), (1-7) and (1-8), it follows that

$$\nabla\,[F(R)\,Y_{klm}] = R^{-(k+1)} \left(\frac{dF}{dR} - \frac{k}{R}\,F\right) x\,\mathcal{Y}_{klm} + FR^{-k}\nabla\,\mathcal{Y}_{klm} \tag{1-9}$$

and we see that our task is then to compute $x\,\mathcal{Y}_{klm}$ and $\nabla\,\mathcal{Y}_{klm}$.

In view of the distinguished role played by the variable x_4 in (1-1), the calculation of those quantitities proceeds separately for their fourth and q-components.

From (1.6a) and (1.6b) it follows that

$$x_4\, G_{kl} = \frac{k-l+1}{k+1}\, G_{k+1,l} + \frac{1}{4}\,\frac{k+l+1}{k+1}\, R^2\, G_{k-1,l}\,. \tag{1-10}$$

Therefore, from (1-1) and (1-3), one can write

$$x_4\,\mathcal{Y}_{klm} = \frac{1}{2}\left[\frac{(k-l+1)\,(k+l+2)}{(k+1)\,(k+2)}\right]^{\frac{1}{2}} \mathcal{Y}_{k+1,lm} + $$
$$+\ \frac{1}{2}\left[\frac{(k+l+1)\,(k-l)}{k(k+1)}\right]^{\frac{1}{2}} R^2\,\mathcal{Y}_{k-1,lm}. \tag{1-11}$$

Similarly, from (1-6d) one gets

$$\nabla_4 \mathcal{Y}_{klm} = \left[\frac{(k+1)(k-l)(k+l+1)}{k}\right]^{\frac{1}{2}} \mathcal{Y}_{k-1,lm}. \tag{1-12}$$

Hence, for the fourth component of ∇ in (1-9), one gets

$$\nabla_4 [F(R)\, Y_{klm}] = \frac{1}{2}\left(\frac{dF}{dR} - \frac{k}{R} F\right)\left[\frac{(k-l+1)(k+l+2)}{(k+1)(k+2)}\right]^{\frac{1}{2}} Y_{k+1,lm} +$$

$$+ \frac{1}{2}\left(\frac{dF}{dR} + \frac{k+2}{R} F\right)\left[\frac{(k+l+1)(k-l)}{k(k+1)}\right]^{\frac{1}{2}} Y_{k-1,lm}. \tag{1-13}$$

To calculate the q-component of the gradient, we make use of the result[1]

$$x_q \mathcal{Y}_{lm} = \left(\frac{l+1}{2l+3}\right)^{\frac{1}{2}} (lm1q|l+1,\ m+q)\, \mathcal{Y}_{l+1,m+q} \tag{1-14}$$

$$- \ r^2 \left(\frac{l}{2l-1}\right)^{\frac{1}{2}} (lm1q|l-1,\ m+q)\, \mathcal{Y}_{l-1,m+q},$$

valid for the 3-dimensional solid spherical harmonics $\mathcal{Y}_{lm}(x_1, x_2, x_3)$, where $r^2 = \sum_4 (-)^q x_q x_{-q}$.

In order to obtain

$$-ix_q \mathcal{Y}_{klm} = C(k,\ l)\, G_{kl}(R,\ x_4)\, (-ix_q) \mathcal{Y}_{lm}, \tag{1-15}$$

one sees, from (1-14), that one has to calculate $G_{kl}\, \mathcal{Y}_{l+1,m+q}$ and $r^2\, G_{kl}\, \mathcal{Y}_{l-1,m+q}$. This can be easily done by noting that, from (1-6b),

$$G_{kl} = \frac{1}{k+1}\, G_{k+1,l+1} - \frac{R^2}{4(k+1)}\, G_{k-1,l+1}. \tag{1-16}$$

Further, we note that $r^2 = R^2 - x_4^2$ and, from (1-6a) and (1-16), one gets

$$x_4^2\, G_{kl} = 4\, G_{k+2,l} - 2(2k+2l+3)\, G_{k+1,l-1} +$$

$$+ \frac{(k+l+1)(k+l)}{k+1}\left[G_{k+1,l-1} - \frac{R^2}{4} G_{k-1,l-1}\right]. \tag{1-17}$$

Hence,

$$r^2 G_{kl} = \frac{(k+l+1)\,(k+l)}{4(k+1)} R^2 G_{k-1,l-1} - \frac{(k-l+2)\,(k-l+1)}{k+1} G_{k+1,l-1} \quad ($$

and we finally get from (1-3), (1-15), (1-16) and (1-18),

$$- i x_q \mathscr{Y}_{klm} =$$

$$= \frac{1}{2} \left[\frac{l+1}{2l+3}\right]^{\frac{1}{2}} \left[\frac{(k+l+3)\,(k+l+2)}{(k+1)\,(k+2)}\right]^{\frac{1}{2}} (lm1q \,|\, l+1,\ m+q)\, \mathscr{Y}_{k+1,l+1,m+q}$$

$$- \frac{1}{2} R^2 \left[\frac{l+1}{2l+3}\right]^{\frac{1}{2}} \left[\frac{(k-l)\,(k-l-1)}{k(k+1)}\right]^{\frac{1}{2}} (lm1q \,|\, l+1,\ m+q)\, \mathscr{Y}_{k-1,l+1,m+q}$$

$$- \frac{1}{2} \left[\frac{l}{2l-1}\right]^{\frac{1}{2}} \left[\frac{(k-l+2)\,(k-l+1)}{(k+1)\,(k+2)}\right]^{\frac{1}{2}} (lm1q \,|\, l-1,\ m+q)\, \mathscr{Y}_{k+1,l-1,m+q}$$

$$+ \frac{1}{2} R^2 \left[\frac{l}{2l-1}\right]^{\frac{1}{2}} \left[\frac{(k+l+1)\,(k+l)}{k(k+1)}\right]^{\frac{1}{2}} (lm1q \,|\, l-1,\ m+q)\, \mathscr{Y}_{k-1,l-1,m+q} \tag{1-19}$$

Now, we briefly indicate the calculation of

$$-i \nabla_q \mathscr{Y}_{klm} = C(k,\ l)\, G_{kl}\,(-i \nabla_q \mathscr{Y}_{lm}) + C(k,\ l)\, \mathscr{Y}_{lm}\,(-i \nabla_q G_{kl}). \tag{1-20}$$

First, we recall the result[1]

$$\nabla_q \mathscr{Y}_{lm} = -(2l+1) \left[\frac{l}{2l-1}\right]^{\frac{1}{2}} (lm1q|l-1,\ m+q)\, \mathscr{Y}_{l-1,m+q}. \tag{1-21}$$

Further, from (1-6c), one has

$$\nabla_q G_{kl} = -\frac{1}{2} x_q G_{k-1,l+1}. \tag{1-22}$$

One also has, from (1-6b) and (1-18),

$$r^2 G_{k-1,l+1} = 2(2l+1) G_{kl} - (k+l+1)\,(k+l)\, G_{k-1,l-1}.$$

Therefore, it follows from (1-20), by using (1-14), (1-21) and (1-22), that

$$- i \nabla_q \mathscr{Y}_{klm} =$$

$$- \left(\frac{l+1}{2l+3}\right)^{\frac{1}{2}} \times$$

$$\times \left[\frac{(k+1)(k-l)(k-l-1)}{k} \right]^{\frac{1}{2}} (lm1q \,|\, l+1,\ m+q)\, \mathcal{Y}_{k-1,l+1,m+q}$$

$$+ \left(\frac{l}{2l-1} \right)^{\frac{1}{2}} \times$$

$$\times \left[\frac{(k+1)(k+l)(k+l+1)}{k} \right]^{\frac{1}{2}} (lm1q \,|\, l-1,\ m+q)\, \mathcal{Y}_{k-1,l-1,m+q}. \qquad (1\text{-}23)$$

By substituting (1-23) and (1-19) in (1-9), one finally gets

$$-i\nabla_q [F Y_{klm}] = \frac{1}{2} \left(\frac{dF}{dR} - \frac{k}{R} F \right) \left\{ \left(\frac{l+1}{2l+3} \right)^{\frac{1}{2}} \times \right.$$

$$\times \left[\frac{(k+l+3)(k+l+2)}{(k+1)(k+2)} \right]^{\frac{1}{2}} (lm1q \,|\, l+1,\ m+q)\ Y_{k+1,l+1,m+q}$$

$$- \left(\frac{l}{2l-1} \right)^{\frac{1}{2}} \times$$

$$\times \left[\frac{(k-l+2)(k-l+1)}{(k+1)(k+2)} \right]^{\frac{1}{2}} (lm1q \,|\, l-1,\ m+q)\ Y_{k+1,l-1,m+q}$$

$$- \frac{1}{2} \left(\frac{dF}{dR} + \frac{k+2}{R} F \right) \left\{ \left(\frac{l+1}{2l+3} \right)^{\frac{1}{2}} \times \right.$$

$$\times \left[\frac{(k-l)(k-l-1)}{k(k+1)} \right]^{\frac{1}{2}} (lm1q \,|\, l+1,\ m+q)\ Y_{k-1,l+1,m+q}$$

$$- \left(\frac{l}{2l-1} \right)^{\frac{1}{2}} \times$$

$$\left. \times \left[\frac{(k+l)(k+l+1)}{k(k+1)} \right]^{\frac{1}{2}} (lm1q \,|\, l-1,\ m+q)\ Y_{k-1,l-1,m+q} \right\}. \qquad (1\text{-}24)$$

The expressions (1-13) and (1-24), together, give the gradient formula for the hyperspherical harmonics in 4 dimensions.

2. The Gradient Formula from the Wigner-Eckart Theorem for the R_4 Group.

In this section, we wish to obtain the previously deduced gradient formula, (1-13) and (1-24), by means of the Wigner-Eckart theorem for the R_4 group.

We recall that the gradient operator ∇ is an irreducible tensor operator of rank one of the R_4 group, i.e., the tensor operator associated to its irreducible representation $k=1$: T^1_{lm}. In this case, l assumes the values 0 and 1 and therefore the components of ∇ are given (except for an overall phase), according to (1-7), by

$$T^1_{00} = \partial_4, \qquad (2\text{-}1)$$
$$T^1_{1q} = -i\nabla_q \quad (q = +1,\ 0,\ -1).$$

Similar considerations are valid for the vector $\mathbf{x}$. In its general form, the Wigner-Eckart theorem for a tensor operator $T^{jj'}_{\lambda\mu}$ associated to the irreducible representation $[jj']$ of R_4, in the chain $R_4 \supset R_3 \supset R_2$, reads[3]

$$(j_1 j_1' l_1 m_1 | T^{jj'}_{\lambda\mu} | j_2 j_2' l_2 m_2) = \left[\begin{matrix} j_2 & j_2' & j & j' \\ l_2 & m_2 & \lambda & \mu \end{matrix}\middle| \begin{matrix} j_1 & j_1' \\ l_1 & m_1 \end{matrix}\right] (j_1 j_1' \| T^{jj'} \| j_2 j_2'), \qquad (2\text{-}2)$$

where $(j_1 j_1' \| T^{jj'} \| j_2 j_2')$ is the reduced matrix element and the bracket is the Wigner coefficient of the R_4 group:

$$\left[\begin{matrix} j_2 & j_2' & j & j' \\ l_2 & m_2 & \lambda & \mu \end{matrix}\middle| \begin{matrix} j_1 & j_1' \\ l_1 & m_1 \end{matrix}\right] = [(2l_2+1)(2\lambda+1)(2j_1+1)(2j_1'+1)]^{1/2} (l_2 m_2 \lambda \mu | l_1 m_1) \,.$$
$$\left\{\begin{matrix} j_2 & j_2' & l_2 \\ j & j' & \lambda \\ j_1 & j_1' & l_1 \end{matrix}\right\}, \qquad (2\text{-}3)$$

the well-known expression in terms of a $9j$-symbol. In this notation, (2-1) reads

$$T^{1/2\ 1/2}_{0\ \ \ 0} \equiv T^1_{00}$$
$$T^{1/2\ 1/2}_{1\ \ \ q} \equiv T^1_{1q}\,. \qquad (2\text{-}4)$$

If the basic states in (2-2) are hyperspherical harmonics, then $j_2 = j_2' = k_2/2$ and we write

$$|j_2,\, j_2,\, l_2,\, m_2) = Y_{2j_2,\, l_2 m_2} \equiv Y_{k_2 l_2 m_2}\,. \qquad (2\text{-}5)$$

Fixing this notation, from (2-2) one can write (dropping out the subscript 2)

$$T^1_{00}\, Y_{klm} = (2l+1)^{1/2}\, k \begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & l \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{k-1}{2} & \frac{k-1}{2} & l \end{Bmatrix} (k-1 \parallel T^1 \parallel k)\, Y_{k-1,lm}$$

$$+ (2l+1)^{1/2} (k+2) \begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & l \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{k+1}{2} & \frac{k+1}{2} & l \end{Bmatrix} (k+1 \parallel T^1 \parallel k)\, Y_{k+1,lm} \cdot \qquad (2\text{-}6)$$

The last formula allows one to calculate the reduced matrix elements for x and V if, for instance, one takes into account the following elementary properties of the basic states:

$$\frac{x_4}{R} Y_{k00} = \frac{1}{2} (Y_{k-1,00} + Y_{k+1,00}) \qquad (2\text{-}7)$$

$$R\, \nabla_4\, Y_{k00} = \frac{k+2}{2} Y_{k-1,00} - \frac{k}{2} Y_{k+1,00}, \qquad (2\text{-}8)$$

which are particular cases of (1-11) and (1-13), respectively. In fact, comparing *(2-7)* with (2-6) and using the result

$$\begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{k\mp1}{2} & \frac{k\mp1}{2} & 0 \end{Bmatrix} = [2(k+1)\ (k+1\pm1)]^{-1/2}, \qquad (2\text{-}9)$$

one readily gets the reduced matrix elements of x:

$$\left(k-1 \parallel \frac{x}{R} \parallel k\right) = \left[\frac{k+1}{2k}\right]^{\frac{1}{2}}$$

$$\left(k+1 \parallel \frac{x}{R} \parallel k\right) = \left[\frac{k+1}{2(k+2)}\right]^{\frac{1}{2}}. \qquad (2\text{-}10)$$

Similarly, through (2-8), one has the corresponding reduced matrix elements for V:

$$
\begin{aligned}
(k-1 \,\|\, R\nabla \,\|\, k) &= (k+2)\left[\frac{k+1}{2k}\right]^{\frac{1}{2}} \\
(k+1 \,\|\, R\nabla \,\|\, k) &= -\,k\left[\frac{k+1}{2(k+2)}\right]^{\frac{1}{2}}.
\end{aligned} \tag{2-11}
$$

Clearly, (2-10) and (2-11) allow the deterrnination of the full expressions for x Y_{klm} and V Y_{klm} by means of (2-6), provided the 9-j symbols occuring in it are known.

With the help of the properties of 9-j symbols[1], one gets

$$
\begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & l \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{k\pm1}{2} & \frac{k\pm1}{2} & l \end{Bmatrix} = \frac{1}{(k+1)\,(k+1\pm1)}\left[\frac{\left(k-l+\frac{1\pm1}{2}\right)\left(k+l+\frac{3\pm1}{2}\right)}{2(2l+1)}\right]^{\frac{1}{2}}. \tag{2-12}
$$

Hence,

$$
\begin{aligned}
\frac{x_4}{R}\,Y_{klm} &= \frac{1}{2}\left[\frac{(k+l+1)\,(k-l)}{k(k+1)}\right]^{\frac{1}{2}} Y_{k-1,\,lm} + \\
&+\frac{1}{2}\left[\frac{(k+l+2)\,(k-l+1)}{(k+1)\,(k+2)}\right]^{\frac{1}{2}} Y_{k+1,\,lm}
\end{aligned} \tag{2-13}
$$

and

$$
\begin{aligned}
R\nabla_4\,Y_{klm} &= -\,\frac{1}{2}\,k\left[\frac{(k-l+1)\,(k+l+2)}{(k+1)\,(k+2)}\right]^{\frac{1}{2}} Y_{k+1,\,lm} \\
&+\frac{1}{2}\,(k+2)\left[\frac{(k+l+1)\,(k-l)}{k(k+1)}\right]^{\frac{1}{2}} Y_{k-1,\,lm}
\end{aligned} \tag{2-14}
$$

which are in agreement with (1-11) and (1-13) for $F(R) = 1$.

In order to obtain $x_q Y_{klm}$ and $\nabla_q Y_{klm}$, we note that

$$
\begin{aligned}
T^1_{1q} Y_{klm} = {} & [3(2l+1)]^{1/2}\, k(lm1q|l+1,\ m+q) \times (k-1 \,\|\, T^1 \,\|\, k) \begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & l \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{k-1}{2} & \frac{k-1}{2} & l+1 \end{Bmatrix} Y_{k-1,l+1,m+q} \\
& + [3(2l+1)]^{1/2}\, k(lm1q|l-1,\ m+q) \times (k-1 \,\|\, T^1 \,\|\, k) \begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & l \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{k-1}{2} & \frac{k-1}{2} & l-1 \end{Bmatrix} Y_{k-1,l-1,m+q} \\
& + [3(2l+1)]^{1/2}\, (k+2)\,(lm1q|l+1,\ m+q) \times (k+1 \,\|\, T^1 \,\|\, k) \begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & l \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{k+1}{2} & \frac{k+1}{2} & l+1 \end{Bmatrix} Y_{k+1,l+1,m+q} \\
& + [3(2l+1)]^{1/2}\, (k+2)\,(lm1q|l-1,\ m+q) \times (k+1 \,\|\, T^1 \,\|\, k) \begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & l \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{k+1}{2} & \frac{k+1}{2} & l-1 \end{Bmatrix} Y_{k+1,l-1,m+q}.
\end{aligned}
\qquad (2\text{-}15)
$$

For the 9-j symbols in (2-15), one gets

$$
\begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & l \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{k-1}{2} & \frac{k-1}{2} & l\pm 1 \end{Bmatrix} = \mp \frac{1}{k(k+1)} \left[\frac{\left(l+\frac{1}{2}\pm\frac{1}{2}\right)(k\mp l)\,(k\mp l\mp 1)}{6(2l\pm 1)\,(2l+2\pm 1)} \right]^{\frac{1}{2}},
$$

$$\begin{Bmatrix} \frac{k}{2} & \frac{k}{2} & l \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{k+1}{2} & \frac{k+1}{2} & l\pm 1 \end{Bmatrix} = \pm \frac{1}{(k+1)(k+2)} \left[\frac{\left(l+\frac{1}{2}\pm\frac{1}{2}\right)(k\pm l+2)(k\pm l+2\pm 1)}{6(2l\pm 1)(2l+2\pm 1)} \right]^{\frac{1}{2}}. \tag{2-16}$$

From (2-15), (2-16), (2-10) and (2-11), one gets the following expressions:

$$\begin{aligned} -i\frac{x_q}{R} Y_{klm} = \\ = &-\frac{1}{2}\left(\frac{l+1}{2l+3}\right)^{\frac{1}{2}} \left[\frac{(k-l)(k-l-1)}{k(k+1)}\right]^{\frac{1}{2}} (lm1q|l+1,\ m+q)\ Y_{k-1,l+1,m+q} \\ &+\frac{1}{2}\left(\frac{l}{2l-1}\right)^{\frac{1}{2}} \left[\frac{(k+l)(k+l+1)}{k(k+1)}\right]^{\frac{1}{2}} (lm1q|l-1,\ m+q)\ Y_{k-1,l-1,m+q} \\ &+\frac{1}{2}\left(\frac{l+1}{2l+3}\right)^{\frac{1}{2}} \left[\frac{(k+l+3)(k+l+2)}{(k+1)(k+2)}\right]^{\frac{1}{2}} (lm1q|l+1,\ m+q)\ Y_{k+1,l+1,m+q} \\ &-\frac{1}{2}\left(\frac{l}{2l-1}\right)^{\frac{1}{2}} \left[\frac{(k-l+1)(k-l+2)}{(k+1)(k+2)}\right]^{\frac{1}{2}} (lm1q|l-1,\ m+q)\ Y_{k+1,l-1,m+q} \end{aligned} \tag{2-17}$$

and

$$\begin{aligned} -iR\nabla_q Y_{klm} = \\ = &-\frac{k+2}{2}\left(\frac{l+1}{2l+3}\right)^{\frac{1}{2}} \left[\frac{(k-l)(k-l-1)}{k(k+1)}\right]^{\frac{1}{2}} (lm1q|l+1,\ m+q)\ Y_{k-1,l+1,m+q} \\ &+\frac{k+2}{2}\left(\frac{l}{2l-1}\right)^{\frac{1}{2}} \left[\frac{(k+l)(k+l+1)}{k(k+1)}\right]^{\frac{1}{2}} (lm1q|l-1,\ m+q)\ Y_{k-1,l-1,m+q} \\ &-\frac{k}{2}\left(\frac{l+1}{2l+3}\right)^{\frac{1}{2}} \left[\frac{(k+l+3)(k+l+2)}{(k+1)(k+2)}\right]^{\frac{1}{2}} (lm1q|l+1,\ m+q)\ Y_{k+1,l+1,m+q} \\ &+\frac{k}{2}\left(\frac{l}{2l-1}\right)^{\frac{1}{2}} \left[\frac{(k-l+1)(k-l+2)}{(k+1)(k+2)}\right]^{\frac{1}{2}} (lm1q|l-1,m+q)\ Y_{k+1,l-1,m+q}. \end{aligned} \tag{2-18}$$

The above relations (2-17) and (2-18) are in complete agreement with (1-19) and (1-24) for $F(R) = 1$, and of course, give rise to the same gradient formula for a general $F(R)$. It is enough to substitute them in (1-9).

References and Notes

1. M. E. Rose, *Elementary Theory of Angular Momentum*, New York (1957). A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton (1957).
2. J. A. Castilho Alcarás and P. Leal Ferreira, J. Math Phys. 6, 578 (1965).
3. L. C. Biedenharn, J. Math. Phys. 2, 433 (1961).
4. Our actual motivation for studying the gradient formula was due to a research work by the authors on the solutions of the Bethe-Salpeter equation for the Fermi-quark-antiquark system with strong binding, where the formula was found to be particularly useful.
5. The $C(k, l)$ given by (1-3) differs by a phase $(-i)^l$ from the corresponding expression in Ref. 2. This choice of the phase is in agreement with the relative phase of the components of the gradient operator given by (1-7).
6. E. D. Rainville, *Special Functions*, New York (1960).