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On Spinor Representation of the Lorentz Group

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The spinor space representation of the Homogeneous Lorentz Group offered by Clifford numbers in Minkowski space is reviewed. Two-spinor calculus naturally follows when spinor matrix representation for these numbers is used. Representations of the improper four group are also discussed.

Examina-se aqui, a representação espinorial do grupo de Lorentz homogêneo, que decorre dos números de Clifford, no espaço de Minkowski. O cálculo de espinores, a duas componentes, resulta naturalmente quando se faz uso da representação matricial daqueles números. São também discutidas as representações do grupo impróprio.

1. Introduction

The purpose of the present paper is to discuss the four-component and two-component spinor analyses, starting from the representation of the Lorentz group in terms of Clifford numbers. The results are not new[1]; however, a good deal of clarification is achieved in the discussion. With an extensive use of spinors in Riemanni an space[2], this is perhaps desirable.

In Sections 2 and 3, we review the four-spinor representation of the restricted homogeneous Lorentz group, offered by Clifford algebra, in Minkowski space. Section 4 is devoted to 2-spinor calculus which naturally follows when we express the matrices γ^μ in the spinor representation and the $SL(2, C)$ group structure is made transparent. In Section 5, we discuss how a spin frame, in two dimensional spinor space, can be defined in terms of two legs like the four-legs or tetrads of vectors frequently used in Minkowski space. A set of null tetrad of vectors $\sigma^{\mu(A)(\dot{B})}$ is also constructed. Finally, in Sections 5 and 6, we discuss in detail the representations of the improper Four group, in spinor space, together with the transformations of bilinear invariants.

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2. Notation. Representation of the Lorentz Group by Clifford Numbers. Spinor Space

The homogeneous Lorentz group (H.L.G.) may be defined as the group of 4 x *4* real matrices *{A)* which satisfy

$$A^T G A = G, \tag{1}$$

where[3] $A = (\Lambda^\mu_\nu)$, $G = (g_{,,}) = (g_{,,})$, $(A^T)'' = A^\nu_{,,,}$ with $\mu, \nu = 0, 1, 2, 3$ and $g_{00} = 1, g_{kk} = -1, k = 1, 2, 3, g_{\mu\nu} = 0$ for $\mu \neq \nu$. We will be mostly concerned[4] here with restricted H.L.G., referred to simply as Lorentz group, for which

$$\Lambda^0{}_0 > 1, \text{ det } A = +1. \tag{2}$$

Equation (2-1), written explicitly, reads

$$(\Lambda^T)^\mu{}_\alpha \; g_{\alpha\beta} \; \Lambda^\beta{}_\nu = g_{\alpha\beta} \; \Lambda^\alpha{}_\mu \; \Lambda^\beta{}_\nu = g_{\mu\nu}. \tag{3}$$

Here the summation on repeated indices is understood.

The matrix group can be represented by the group of linear transformations on a four dimensional real linear vector space, called Minkowski space, with basis vectors $\underline{e}$, which transform as

$$\underline{e}'_\mu \ni L(\Lambda)\, \underline{e}, = \Lambda^\nu{}_\mu \, \underline{e}_\nu. \tag{4}$$

The contravariant components of a (real) vector $\underline{A}$ w.r.t. the basis $\{\underline{e}_\mu\}$, indicated by real components A^μ, e.g., $\underline{A} = A^\mu \underline{e}_,$ transform as

$$A'^\mu = \Lambda^\mu{}_\nu \, AV, \tag{5}$$

since

$$\underline{A}' \equiv L(\Lambda) A^\nu \, \underline{e}, = A^\nu \, \Lambda^\mu{}_\nu \, \underline{e}, \equiv A'^\mu \, \underline{e}_\mu. \tag{6}$$

The group of contragradient matrices $\{\Lambda^{-1T}\}$ is isomorphic to the matrix group *{A)*. Denoting the basis vectors in the corresponding representation space by $\underline{e}^\mu$, it can be realized as a group of linear transformations defined by

$$\underline{e}'^\mu = (A^{-1})^\mu{}_\nu \, \underline{e}^\nu. \tag{7}$$

The covariant components $A,$ of vector A w.r.t. this basis ($\underline{A} \equiv A, \, \underline{e}^\mu$) transform as

$$A'_\mu = (\Lambda^{-1})^\nu{}_\mu \; A, = (\Lambda^{-1T})^\mu{}_\nu \; A_\nu. \tag{8}$$

Equation *(2.1)* implies that

$$(\Lambda^T)^{-1} = G \, A \, G^{-1}, \tag{9}$$

so that the contragradient representation is equivalent to the representation {A}. We note also, from equations (2-5) and (2-8), that the Kronecker delta δ^{μ}_{ν} is an invariant tensor. From the fact that Λ^{-1} is also a Lorentz transformation, Eq. (2-3)implies g„ = (A–')", $(\Lambda^{-1})^{\beta}{}_{\nu}\, g_{\alpha\beta}$ which states that the indices μ and v are covariant tensor indices and that g„ is an invariant tensor.

It is clear that (g„ $\underline{e}^{\nu}$) transforms like e, for

$$(g_{\mu\nu}\ \underline{e}^{\nu})' = g_{\mu\nu}\ \underline{e}'^{\nu} = g_{\alpha\beta}\ \Lambda^{\alpha}{}_{\mu}\ \Lambda^{\beta}{}_{\nu}\ (\text{A}{-}^{1})^{\nu}{}_{\rho}\ \underline{e}^{\rho} = \Lambda^{\alpha}{}_{\mu}\ (g_{\alpha\rho}\ \underline{e}^{\rho}).$$

Thus, we may define

$$\underline{e}_{\mu} = g_{\mu\nu}\ \underline{e}^{\nu} \tag{10}$$

which is an alternative statement of the equivalence expressed by Eq. (2-9). This leads to

$$\text{A}, = \text{g}„\ \text{A}^{\text{P}}. \tag{11}$$

In other words: while the components (A^0, A', A^2, A^3) transform by the matrix A, the components (A^0, $-\text{A}^1$, $-\text{A}^2$, $-A^3$) transform according to the matrix $(\Lambda^{-1})^{\text{T}}$. The two representations are equivalent since the former can be obtained from the latter by a change of basis, in the representation space, according to Eq. (2-10).

We may then introduce a metric tensor $g^{\mu\nu}$. Using Eq. (2-11) to lower the indices we have:

$$g_{\mu\nu} = g_{\alpha\mu}\ g_{\beta\nu}\ g^{\alpha\beta} \tag{12}$$

which gives[5] (g„ = g„):

$$g^{\alpha\beta}\ g_{\beta\nu} = \delta^{\alpha}{}_{\nu}. \tag{13}$$

Thus, in matrix form $(g^{\alpha\beta}) = G^{-1}(=G)$, so that $g^{00} = g_{00} = +1$, $g^{kk} = g_{kk} = -1$, k $= 1, 2, 3$ and $g^{\mu\nu} = 0$ for $\mu \neq$ v. We may thus use $g^{\mu\nu}$ to raise the indices and g„ to lower them. We note that, because of the equivalence, the same representation space is involved for the two representations. The introduction of upper and lower indices is convenient in that (A, B^{μ}) is an invariant (while $A^{\mu} B^{\mu}$ is not so) under Lorentz transformations. The Lorentz group can thus, alternatively, be defined as the group of linear transformations which leaves the bilinear-form,

$$x^{\mu} x_{\mu} = g_{\mu\nu}\, x^{\mu} x^{\nu} = g^{\mu\nu}\, x_{\mu} x_{\nu}, \tag{14}$$

invariant, where $x^\mu \equiv (x^0, x^1, x^2, x^3)$ are the space time coordinates. Finally, from Eq. (2-3), we may derive[6]

$$g^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta}. \tag{15}$$

We may define an inner product, in Minkowski space, by

$$\underline{A} \cdot \underline{B} = A_\mu B^\mu = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu, \tag{16}$$

which implies

$$\underline{e}_\mu \cdot \underline{e}_\nu = g_{\mu\nu}, \qquad \underline{e}^\mu \cdot \underline{e}^\nu = g^{\mu\nu}. \tag{17}$$

The hypercomplex Clifford numbers may be used to construct a representation of the Lorentz group. The Clifford algebra, in Minkowski space, is defined by a set of four hypercomplex riumbers $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ which satisfy the anticommutation relations[7]

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} I. \tag{18}$$

Any product of $\gamma's$ can be reduced to, using Eq. (2-18), to one of the 16 elements I, γ^μ, $(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$, $\gamma_5\gamma^\mu$, γ_5, where $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$. Representing the $\gamma's$ by (r x r) matrices, we can show that the 16 elements are linearly independent so that r must be ≥ 4. It also follows that the representation of the algebra, by 4 x 4 matrices, is irreducible. In the following, the $\gamma's$ will be regarded as (4 x 4) (irreducible) matrices.

It may easily be shown that the 6 elements $\Sigma^{\mu\nu} = \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ satisfy the commutation relations of the Lie algebra of the generators $M^{\mu\nu}$ of the homogeneous Lorentz group, viz.,

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = i(g^{\mu\sigma}\Sigma^{\nu\rho} - g^{\mu\rho}\Sigma^{\nu\sigma} - g^{\nu\sigma}\Sigma^{\mu\rho} + g^{\nu\rho}\Sigma^{\mu\sigma}). \tag{19}$$

Thus, we can obtain a representation, by (4 x 4) complex matrices[9], $\{S(\Lambda)\}$, of the Lorentz group, in terms of Clifford numbers, with

$$S(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right], \tag{20}$$

where[10] $\omega_{\mu\nu} = \Lambda_{\mu\nu} - g_{\mu\nu}$. The corresponding representation space is 4-dimensional complex vector space, called *Spinor Space.* Equations (2-18) to (2-20) lead to

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu. \tag{21}$$

Denoting the basis vectors of spinor space by $\underline{Z}_a$ (a = 1, 2, 3, 4), and the components of a vector $\underline{\xi}$, w.r.t. this basis, by ξ^a, e.g., $\underline{\xi} = \xi^a \underline{Z}_a$, corresponding to a Lorentz transformation Λ, in Minkowski space, the transformation in spinor space is defined to be the linear transformation given by

$$\underline{Z}'_a \equiv U(\Lambda)\underline{Z}_a = S(\Lambda)^b{}_a\, \underline{Z}_b. \tag{22}$$

Here, $S^a{}_b$ are the matrix elements of the matrix S. The group property of the transformations (or the operators $U(\Lambda)$ defined on spinor space) may be easily verified. The components ξ^a are seen to transform as

$$\xi'^a \equiv (U(\Lambda)\underline{\xi})^a = S(\Lambda)^a{}_b\, \xi^b. \tag{23}$$

The contragradient representation constituted by the group of matrices[11] $[S^{-1}(\Lambda)]^{\mathrm{T}}$ is realized on a representation space, whose basis vectors will be indicated by $\underline{Z}^a$. The group of linear operators acts according to

$$\underline{Z}'^a = (S^{-1}(\Lambda))^a{}_b\, \underline{Z}^b, \tag{24}$$

and the components ξ_a of a vector $\underline{\xi} = \xi_a \underline{Z}^a$ transform as

$$\xi'_a = (S^{-1}(\Lambda))^b{}_a\, \xi_b = (S^{-1\,\mathrm{T}}(\Lambda))^a{}_b\, \xi_b. \tag{25}$$

We observe that $\xi^a \eta_a$ is an invariant under homogeneous Lorentz transformations.

The conjugate representation carried by the group of matrices $\{S^*(\Lambda)\}$ is realized on a space with basis vectors indicated by $\underline{Z}_{\dot{a}}$, with

$$\underline{Z}'_{\dot{a}} = S(\Lambda)^{b}{}_{\dot{a}} \underline{Z}_{\dot{b}}, \qquad S^{\dot{b}}{}_{\dot{a}} = (S^b{}_a)^*, \tag{26}$$

and

$$\xi'^{\dot{a}} = S(\Lambda)^{\dot{a}}{}_{\dot{b}}\, \xi^{\dot{b}}, \tag{27}$$

where the $\xi^{\dot{a}}$ are components of $\underline{\xi}$, w.r.t. the basis $\underline{Z}_{\dot{a}}$.

The representation contragradient to the conjugate one is realized on a vector space with basis vectors denoted by $\underline{Z}^{\dot{a}}$, with

$$\underline{Z}'^{\dot{a}} = (S^{-1}(\Lambda))^{\dot{a}}{}_{\dot{b}} \underline{Z}^{\dot{b}} \tag{28}$$

and

$$\xi'_{\dot{a}} = (S^{-1}(\Lambda))^{\dot{b}}{}_{\dot{a}} \xi_{\dot{b}}. \tag{29}$$

3. Invariant Tensors

It will be shown below that all these representations, in the present case, are equivalent to each other and that there is, essentially, only one irreducible representation. However, it is convenient to work with upper, lower, dotted and undotted indices (just as in the case of Minkowski space.

Eq. (2-20) can be written explicitly as $(\gamma^{\mu})^{a}{}_{b} \equiv \gamma^{\mu a}{}_{b})$:

$$\gamma^{\mu a}{}_{b} = \Lambda^{\mu}{}_{\nu} S(\Lambda)^{a}{}_{c} (S^{-1}(\Lambda))^{d}{}_{b} \gamma^{\nu c}{}_{d}. \tag{1}$$

The "mixed quantities" $\gamma^{\mu a}{}_{b}$, thus, are invariant or held fixed under the Lorentz transformation of the indices defined above and, under the tacit assumption, that the index 'μ', in γ^{μ}, is a space-time contravariant index, is consistently assigned. Since Λ^{-1} is a Lorentz transformation, we also have that

$$\gamma^{\mu a}{}_{b} = (\Lambda^{-1})^{\mu}{}_{\nu} (S^{-1}(\Lambda))^{a}{}_{c} (S(\Lambda))^{b}{}_{d} \gamma^{\nu c}{}_{d}. \tag{2}$$

Taking the complex conjugate of **Eq.** (3-1), one obtains

$$\gamma^{\mu \dot{a}}{}_{\dot{b}} = \Lambda^{\mu}{}_{\nu} S(\Lambda)^{\dot{a}}{}_{\dot{c}} (S^{-1}(\Lambda))^{\dot{d}}{}_{\dot{b}} \gamma^{\nu \dot{c}}{}_{\dot{d}}. \tag{3}$$

It may be remarked that the Kronecker deltas $\delta^{\dot{a}}{}_{\dot{b}}$, $\delta^{a}{}_{b}$ are also invariant tensors.

The equivalence of the representations indicated above follow from

$$\{\gamma^{\mu \mathrm{T}}, \gamma^{\nu \mathrm{T}}\}_{+} = \{\gamma^{\mu *}, \gamma^{\nu *}\}_{+} = \{\gamma^{\mu \dagger}, \gamma^{\nu \dagger}\}_{+} = 2 g^{\mu\nu} I, \tag{4}$$

which from the fundamental lemma[8] assures the existence of non-singular matrices A, B, C such that[12]

$$A \gamma^{\mu} A^{-1} = \gamma^{\mu \dagger}, \qquad B \gamma^{\mu} B^{-1} = \gamma^{\mu \mathrm{T}}, \qquad C \gamma^{\mu} C^{-1} = \gamma^{\mu *}. \tag{5}$$

One can show, then, that

$$B\, S(\Lambda) B^{-1} = S^{-1 \mathrm{T}}(\Lambda) \text{ or } B^{-1} = S(\Lambda) B^{-1} S^{\mathrm{T}}(\Lambda), \tag{6}$$

$$A\, S(\Lambda) A^{-1} = S^{\dagger -1}(\Lambda), \qquad C\, S(\Lambda) C^{-1} = S^{*}(\Lambda). \tag{7}$$

We can write **Eq.** (3-6) explicitly as

$$(S^{-1 \mathrm{T}})^{a}{}_{c} B_{cd} (S^{-1})^{d}{}_{b} = (S^{-1})^{c}{}_{a} B_{cd} (S^{-1})^{d}{}_{b} = B_{ab} \tag{8}$$

or

$$B_{ab} = (S^{-1}(\Lambda))^{c}{}_{a} (S^{-1}(\Lambda))^{d}{}_{b} B_{cd} = S(\Lambda)^{c}{}_{a} S(\Lambda)^{d}{}_{b} B_{cd}, \tag{9}$$

where the matrix $\mathrm{B} \equiv (B_{ab})$. This relation shows that B_{ab} is an invariant tensor, with a and b transforming as covariant indices. The B matrix plays the role of metric tensor in spinor space. Since $B_{ab}\underline{Z}^{b}$ transforms as $\underline{Z}_{a}$, we may define

$$\underline{Z}_a = B_{ab}Z^b, \tag{10}$$

which implies

$$\xi_a = B_{ba}\xi^b. \tag{11}$$

It may be shown that B may be chosen unitary and antisymmetric[13]. The metric tensor B^{ab} can be introduced by

$$B_{ab} = B_{ca}B_{db}B^{cd}, \tag{12}$$

so that $(B_{ab} = -B_{ba})$

$$B_{db}B^{cd} = -\delta^c{}_b, \; B_{ca}B^{cd} = \delta^d{}_a \tag{13}$$

and $B^{ab} = -B^{ba}$ as expected. Also, if $B \equiv (B_{ab})$ then $(B^{ab}) = -B^{-1}$. We may also choose a representation of the y matrices such that B is a real matrix; then, $B^* = B = -B^T = -B^{-1}$ and $(B^{ab}) = B$. Using Eq. (3-13), we have

$$\xi^a = B^{ab}\xi_b, \; \underline{Z}^a = -B^{ab}\underline{Z}_b \tag{14}$$

against $\xi_a = -B_{ab}\xi^b$ and $Z_a = B_{ab}\underline{Z}^b$.

We may define the inner product between two vectors $\underline{\xi}$ and $\underline{\eta}$ by

$$\underline{\xi}\cdot\underline{\eta} = \xi_a\eta^a = B_{ab}\xi^a\eta^b = B^{ab}\xi_a\eta_b. \tag{15}$$

From $\underline{Z}_a = \delta^b{}_a\underline{Z}_b$, etc., it follows[14] that

$$\begin{aligned} \underline{Z}_a\cdot\underline{Z}_b &= B_{ab}, \; \underline{Z}^a\cdot\underline{Z}^b = B^{ab}, \\ \underline{Z}^a\cdot\underline{Z}_b &= \delta^a{}_b = -\underline{Z}_a\cdot\underline{Z}^b. \end{aligned} \tag{16}$$

Other properties of inner product are

$$\begin{aligned} &\underline{\xi}\cdot\underline{\eta} = -\underline{\eta}\cdot\underline{\xi}, \; (\alpha\underline{\xi})\cdot\underline{\eta} = \underline{\xi}\cdot(\alpha\underline{\eta}) = \alpha\underline{\xi}\cdot\underline{\eta}, \; (\underline{\xi}_1 + \underline{\xi}_2)\cdot\underline{\eta} = \underline{\xi}_1\cdot\underline{\eta} + \underline{\xi}_2\cdot\underline{\eta}, \\ &\underline{\xi}\cdot(\underline{\eta}_1 + \underline{\eta}_2) = \underline{\xi}\cdot\underline{\eta}_1 + \underline{\xi}\cdot\underline{\eta}_2 \end{aligned} \tag{17}$$

and $\underline{\xi}\cdot\underline{\xi} \equiv \xi^a\xi_a = 0$ for all $\underline{\xi}$; also, $\underline{\xi}\cdot\underline{\eta} = 0$, for all $\underline{y}$, implies $\underline{5} \equiv 0$. The representation space is called Symplectic space and the transformations $S(\Lambda)$ leave invariant the nondegenerate skew symmetric bilinear form given by Eq. (3-15).

An exactly similar discussion can be carried out for conjugate and its contragradient representations. Since $(S^{-1T})^* = B^*S^*B^{*-1}$ the invariant metric tensors are $B_{\dot a\dot b}$ and $B^{\dot a\dot b}$ where $(B_{\dot a\dot b}) = B^*$ and $(B^{\dot a\dot b}) = -B^{*-1}$, which for a real matrix B, are the same as B_{ab} and B^{ab}. We observe that $\xi^{\dot a}\eta_{\dot a}$ is an invariant but $\xi^a\eta_a$ is not so and that $\xi^{\dot a}$ transforms like ξ^{a*} while $\xi_{\dot a} = B_{\dot a\dot b}\xi^b$ transforms like ξ_a^*.

We consider now the equivalence relation of Eq. (3-7). It can be written explicitly as

$$(S^{-1\,\mathrm{T}})^{\dot{a}}{}_{\dot{c}}\, A_{\dot{c}d}(S^{-1})^{d}{}_{b} = (S^{-1})^{\dot{c}}{}_{\dot{a}}\, A_{\dot{c}d}(S^{-1})^{d}{}_{b} = A_{\dot{a}b},$$

or

$$A_{\dot{a}b} = (S^{-1}(\Lambda))^{\dot{c}}{}_{\dot{a}}\,(S^{-1}(\Lambda))^{d}{}_{b}\, A_{\dot{c}d}, \tag{18}$$

where we write $A \equiv (A_{\dot{a}b})$. This relation shows that $A_{\dot{a}b}$ is an invariant tensor with one dotted and another undotted covariant index. Taking the complex conjugate we obtain invariant tensor $A_{a\dot{b}}$. Raising the indices by use of metric tensors, we obtain invariant tensors[15] A^{ab}, $A^{\dot{a}b}$, $A^{a}{}_{b}$, $A^{a}{}_{\dot{b}}$, etc. It is clear that they are useful in constructing invariants of type $\xi^{\dot{a}}\, A_{\dot{a}b}\,\eta^{b}$ and of type $\xi^{a*}A_{\dot{a}b}\,\xi^{b}$ which may not vanish, in contrast to $\xi^{a}\xi_{a} = 0$. We may choose[8] A to be hermitian, e.g., $A_{\dot{a}b} = A_{\dot{b}a} = (A_{b\dot{a}})^{*}$ and $A^{2} = I$.

Other invariant hermitian tensors are[16] $\Gamma^{\mu} = (\Gamma^{\mu}_{\dot{a}b}) = (A_{\dot{a}c}\,\gamma^{\mu c}{}_{b})$, $A\,\Sigma^{\mu\nu} \equiv (\Gamma^{\mu\nu}_{\dot{a}b})$, $A\gamma_{5} \equiv (\Gamma_{5})_{\dot{a}b}$, $(i\,A\,\gamma_{5}\gamma^{\mu}) = (\Gamma^{\mu}_{5})_{\dot{a}b}$. For example, $\Gamma^{\mu}_{\dot{b}a} = (\Gamma^{\mu}_{\dot{b}a})^{*} = (A_{\dot{b}c}\,\gamma^{\mu c}{}_{a})^{*} = A_{b\dot{c}}\,\gamma^{\mu\dot{c}}{}_{\dot{a}} = \gamma^{\mu\dot{c}}{}_{\dot{a}}\,A_{\dot{c}b} = (\gamma^{\mu\dagger}A)_{\dot{a}b} = (A\,\gamma^{\mu})_{\dot{a}b} = \Gamma^{\mu}_{\dot{a}b}$ since, from Eq. (3-5), $A\gamma^{\mu} = \gamma^{\mu\dagger}\,A$. Tensor quantities may be constructed from a quantity like $\eta^{\dot{a}b}$, e.g., scalar $A_{\dot{a}b}\,\eta^{\dot{a}b}$, pseudoscalar $\Gamma_{5\dot{a}b}\,\eta^{\dot{a}b}$, four-vector $\Gamma^{\mu}_{\dot{a}b}\,\eta^{\dot{a}b}$, pseudo four-vector $\Gamma^{\mu}_{5\dot{a}b}\,\eta^{\dot{a}b}$, antisymmetric tensor $\Gamma^{\mu\nu}_{\dot{a}b}\,\eta^{\dot{a}b}$ (Ref. 17).

In particular, recalling that $\xi^{\dot{a}}$ transforms as ξ^{*a}, we have the well known bilinear covariants $\xi^{*a}\,A_{\dot{a}b}\,\xi^{b} \equiv \xi^{\dagger}\,A\,\xi$, $\xi^{\dagger}(A\,\gamma^{\mu})\xi = \xi^{*a}\Gamma^{\mu}_{\dot{a}b}\,\xi^{b}$, $\xi^{\dagger}\,A\,\gamma_{5}\gamma^{\mu}\,\xi$, $\xi^{\dagger}\,A\,\Sigma^{\mu\nu}\,\xi$ and $\xi^{\dagger}\,A\,\gamma_{5}\,\xi$ transforming as a scalar, vector, pseudo-vector, antisymmetric tensor and pseudo-scalar, in Minkowski space.

4. Spinor Representation of γ matrices. Two-Spinors

To bring out clearly the relationship of the 2-spinor calculus with the 4-spinor calculus, discussed above, we use a convenient matrix representation for traceless γ matrices.

We take

$$\gamma^{0\dagger} = \gamma^{0}, \qquad \gamma^{k\dagger} = -\,\gamma^{k}, \qquad k = 1, 2, 3, \tag{1}$$

so that $\gamma_{5} \equiv \gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = -\gamma_{5}^{\dagger}$; $\gamma_{5}^{2} = -\,I$. Clearly, $\gamma^{0}\gamma^{\mu}\gamma^{0} = \gamma^{\mu\dagger}$, so that we may identify $A \equiv (A_{\dot{a}b}) = \gamma^{0} = A' = A^{-1}$. Furthermore, we will take[18] the γ^{μ}'s to be odd matrices so that $\Sigma^{\mu\nu}$, hence, $S(\Lambda)$, will be even. A suitable representation is the spinor representation, defined by

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma_5 = i \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \tag{2}$$

where the σ^k's are the, 2 x 2, Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We also note that, for this representation, $\gamma^{0\mathrm{T}} = \gamma^0$, $\gamma^{2\mathrm{T}} = \gamma^2$, $\gamma^{1\mathrm{T}} = -\gamma^1$, $\gamma^{3\mathrm{T}} = -\gamma^3$ and $\gamma_5^{\mathrm{T}} = \gamma_5$. A real matrix B, satisfying **Eq.** (3-5), can be taken to be ($B = B^* = -B^{\mathrm{T}} = -B^{-1}$)

$$B = -\gamma_5 \gamma^0 \gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \tag{3}$$

so that

$$(B_{ab}) = (B^{ab}) = (B_{\dot{a}\dot{b}}) = (B^{\dot{a}\dot{b}}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4}$$

Also,

$$\omega_{\rho\sigma} \Sigma^{\rho\sigma} = \begin{pmatrix} \vec{\sigma}\cdot(\vec{a} - i\vec{b}) & 0 \\ 0 & \vec{\sigma}\cdot(\vec{a} + i\vec{b}) \end{pmatrix}, \tag{5}$$

where $b^k = \frac{1}{2}(\omega_{0k} - \omega_{k0})$; $a^k = \frac{1}{2}(\omega_{lm} - \omega_{ml})$, k, l, m cyclic and $\vec{\sigma}\cdot\vec{a} \equiv \sigma^1 a^1 + \sigma^2 a^2 + \sigma^3 a^3$. For $S(\Lambda)$, we find

$$S(\Lambda) = \begin{pmatrix} S_1(\Lambda) & 0 \\ 0 & S_1^{\dagger -1}(\Lambda) \end{pmatrix}, \tag{6}$$

where

$$S_1(\Lambda) = \exp\left[-\frac{i}{2}\, \vec{\sigma}\cdot(\vec{a} - i\vec{b})\right]. \tag{7}$$

The representation is unitary for space rotations but is, in general, non-unitary. In so far as the restricted Lorentz group is concerned, $S(\Lambda)$ appears as a direct sum of 2-dimensional representations.

The, 2 x 2, matrix groups $\{S_1(\Lambda)\}$ and $\{S_1^{\dagger -1}(\Lambda)\}$, themselves, constitute[19] two inequivalents representations of the Lorentz group. Under parity transformation, we will see below, the two get interchanged so that the representation is irreducible under the full Lorentz group. We note that det $(S_1(\Lambda)) = +1$, so that $\{S_1(\Lambda)\}$ and $\{S_1^{\dagger -1}(\Lambda)\}$ are two inequivalent representations of the $SL(2, C)$ group[20].

It is clear that the two upper components (ξ^1, ξ^2) of ξ^a transform, under a Lorentz transformation, among themselves according to the 2×2 matrix $S_1(\Lambda)$, while the lower components (ξ^3, ξ^4) according to $S_1^{\dagger -1}(\Lambda)$. A change of notation is thus suggested:

$$u^1 \equiv \xi^1, \qquad u^2 \equiv \xi^2, \qquad v_{\dot 1} \equiv \xi^3, \qquad v_{\dot 2} \equiv \xi^4, \tag{8}$$

$$u'^A = S_1(\Lambda)^A{}_B\, u^B, \quad v'_{\dot A} = (S_1^{\dagger -1}(\Lambda))^{\dot B}{}_{\dot A}\, v_{\dot B}, \tag{9}$$

where $A, B = 1, 2$ and $\dot A, \dot B = \dot 1, \dot 2$. Also, $\xi_1 = -\xi^2 = -u^2$, $\xi_2 = = \xi^1 = u^1$, $\xi_3 = -\xi^4 = -v_{\dot 2}$, $\xi_4 = \xi^3 = v_{\dot 1}$, and (ξ_1, ξ_2) transforms according to the matrix $S_1^{-1\mathrm{T}}(\Lambda)$, while (ξ_3, ξ_4) according to the matrix $S_1^*(\Lambda)$. We may, thus, introduce the notation[21]

$$u_1 \equiv \xi_1, \qquad u_2 \equiv \xi_2, \qquad -v^{\dot 1} \equiv \xi_3, \qquad -v^{\dot 2} \equiv \xi_4, \tag{10}$$

so that

$$u_A = -\varepsilon_{AB} u^B, \quad v_{\dot A} = -\varepsilon_{\dot A \dot B}\, v^{\dot B}, \tag{11}$$

where ε_{AB}, $\varepsilon_{\dot A\dot B}$ are Levi-Civitta symbols[22] and $u'_A = (S_1^{-1}(\Lambda))^B{}_A\, u_B$, $u'^{\dot A} = (S_1(\Lambda))^{\dot A}{}_{\dot B}\, u^{\dot B}$. We remark that the invariant tensor B is an even matrix in our representation:

$$B = (B_{ab}) = \left(\begin{array}{c|c} (\varepsilon_{AB}) & 0 \\ \hline 0 & (\varepsilon^{\dot A\dot B}) \end{array}\right), \quad (B^{ab}) = \left(\begin{array}{c|c} (\varepsilon^{AB}) & 0 \\ \hline 0 & (\varepsilon_{\dot A\dot B}) \end{array}\right), \tag{12}$$

where $(\varepsilon_{AB}) = (\varepsilon^{AB}) = (\varepsilon_{\dot A\dot B}) = (\varepsilon^{\dot A\dot B}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. From the identities like[23]

$$\varepsilon^{CD} S^A{}_{1C} S^B{}_{1D} = (\det S_1)\, \varepsilon^{AB} = \varepsilon^{AB}, \tag{13}$$

if det $S_1 = 1$, we see that ε^{AB}, $\varepsilon_{\dot A\dot B}$, $\varepsilon^{\dot A\dot B}$, ε_{AB} are invariant tensors. Hence, equation (4-11) expresses the equivalence of representation S_1 with $(S_1^{\mathrm{T}})^{-1}$ and S_1^* with $S_1^{-1\dagger}$. Since ξ^{a*} transforms like $\xi^{\dot a}$, we see that u^{A*} transforms like $u^{\dot A}$ while $v_{\dot A}^*$ transforms like v_A. We will adopt the customary practice of identifying $u^{\dot A} = u^{*A}$ and $u_{\dot A} = u_A^*$. There is no invariant quantity (like $A_{\dot a b}$) which relates the dotted and undotted components since the conjugate representation $(u^{\dot A})$ is not equivalent to the representation (u^A). Likewise, we define the basis vectors $\underline{h}$ by

$$\underline{h}_1 \equiv \underline{Z}_1, \qquad \underline{h}_2 \equiv \underline{Z}_2, \qquad \underline{h}^{\dot 1} \equiv \underline{Z}_3, \qquad \underline{h}^{\dot 2} \equiv \underline{Z}_4 \tag{14}$$

and

$$-\underline{h}^1 \equiv \underline{Z}^1 = -\underline{Z}_2 = -\underline{h}_2, \quad -\underline{h}^2 \equiv \underline{Z}^2 = \underline{Z}_1 = \underline{h}_1,$$
$$\underline{h}_{\dot{1}} = \underline{Z}^3 = -\underline{Z}_4 = -\underline{h}^2, \qquad \underline{h}_{\dot{2}} = \underline{Z}^4 = \underline{Z}_3 = \tag{15}$$

so that

$$\underline{\xi} = u^A \underline{h}_A + v_{\dot{A}} \underline{h}^{\dot{A}} = - u_A \underline{h}^A - v^{\dot{A}} \underline{h}_{\dot{A}} \tag{16}$$

and

$$\underline{h}_A = - \varepsilon_{AB} \underline{h}^B, \qquad \underline{h}_{\dot{A}} = - \varepsilon_{\dot{A}\dot{B}} \underline{h}^{\dot{B}}. \tag{17}$$

Under a Lorentz transformation, the basis vectors $\underline{h}$ transform as

$$\underline{h}'_A = S_1(\Lambda)^B{}_A \underline{h}_B, \qquad \underline{h}'^A = (S_1^{-1}(\Lambda))^A{}_B \underline{h}^B,$$
$$\underline{h}'_{\dot{A}} = (S_1(\Lambda))^{\dot{B}}{}_{\dot{A}} \underline{h}_{\dot{B}}, \qquad \underline{h}'^{\dot{A}} = (S_1^{-1}(\Lambda))^{\dot{A}}{}_{\dot{B}} \underline{h}^{\dot{B}}. \tag{18}$$

From

$$\varepsilon_{AB} \varepsilon^{BC} = - \delta^C{}_A, \qquad \varepsilon_{\dot{A}\dot{B}} \varepsilon^{\dot{B}\dot{C}} = - \delta^{\dot{C}}, \tag{19}$$

it follows

$$u^A = \varepsilon^{AB} u_B, \qquad \underline{h}^A = \varepsilon^{AB} \underline{h}_B, \tag{20}$$

and similar expression for dotted indices.
From Eq. (3-16), we find the following inner products:

$$\underline{h}_A \cdot \underline{h}^B = - \underline{h}^A \cdot \underline{h}_B = \delta^A_B = \underline{h}_{\dot{A}} \cdot \underline{h}^{\dot{B}} = - \underline{h}^{\dot{A}} \cdot \underline{h}_{\dot{B}},$$
$$\underline{h}_A \cdot \underline{h}^{\dot{B}} = \underline{h}_{\dot{A}} \cdot \underline{h}^B = \underline{h}^A \cdot \underline{h}_{\dot{B}} = \underline{h}^{\dot{A}} \cdot \underline{h}_B = 0, \tag{21}$$

and

$$\underline{h}_A \cdot \underline{h}_B = \varepsilon_{AB} = \varepsilon^{AB} = \underline{h}^A \cdot \underline{h}^B,$$
$$\underline{h}_{\dot{A}} \cdot \underline{h}_{\dot{B}} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \underline{h}^{\dot{A}} \cdot \underline{h}^{\dot{B}}. \tag{22}$$

The vector spaces generated by undotted and dotted basis vectors are orthogonal. The inner product of two vectors $\underline{u}$ and $\underline{\phi}$ in undotted space is

$$\underline{u} \cdot \underline{\phi} = u^A \underline{h}_A \cdot \phi^B \underline{h}_B = \varepsilon_{AB} u^A \phi^B = \varepsilon^{AB} u_A \phi_B = - u^A \phi_A = u_A \phi^A \tag{23}$$

and satisfies the properties given in **Eq.** (3-17). The representation space is a symplectic space $Sp(2)$ in two dimensions. The same goes for the dotted vector space and the linear independence of basis vectors follows from equation (4-22). We remark that $u^A \phi_A = - u_A \phi^A$ and $u^A u_A = 0$.

In the spinor representation, the Eq. *(2-21)* leads to[24]

$$\sigma^\mu = \Lambda^\mu{}_\nu S_1(A)\sigma^\nu S_1^\dagger(\Lambda), \tag{24}$$

from which follows that

$$\sigma^\mu \equiv (\sigma^{\mu A\dot{B}}) \tag{25}$$

and

$$\sigma^{\mu A\dot{B}} = \Lambda^\mu{}_\nu S_1(\Lambda)^A{}_C S_1(\Lambda)^{\dot{B}}{}_{\dot{D}} \sigma^{\nu C\dot{D}}, \tag{26}$$

showing that $\sigma^{\mu A\dot{B}}$ defined by equation *(4-25)* is an invariant mixed quantity like $(\gamma^{\mu a}{}_b)$. On lowering the indices with the invariant metric tensor ε_{AB} and using

$$\varepsilon_{AC} S_1(\Lambda)^C{}_D \varepsilon^{DB} = -(S_1^{-1}(\Lambda))^B{}_A = -S_1(\Lambda^{-1})^B{}_A, \tag{27}$$

we can show that

$$\sigma^\mu_{A\dot{B}} = \Lambda^\mu{}_\nu (S_1^{-1}(\Lambda))^C{}_A (S_1^{-1}(\Lambda))^D{}_{\dot{B}} \sigma^\nu{}_{C\dot{D}}. \tag{28}$$

Both $(\sigma^{\mu A\dot{B}})$ and $(\sigma^\mu_{A\dot{B}})$ are hermitian matrices, that is $\sigma^{\mu A\dot{B}} = o^{pBA}$, $\sigma^\mu_{A\dot{B}} = \sigma^\mu_{\dot{B}A} \equiv (\sigma^\mu{}_{B\dot{A}})^*$. From $(\sigma^\mu{}_{A\dot{B}}) = (\varepsilon_{AC}\ \varepsilon_{\dot{B}\dot{D}}\, \sigma^{\mu C\dot{D}}) = -(E\ \sigma^\mu\ E)$ we see $(\sigma^0_{A\dot{B}}) = \sigma^0$, $(\sigma^1{}_{A\dot{B}}) = -\sigma^1$, $(\sigma^2{}_{A\dot{B}}) = \sigma^2$, $(\sigma^3{}_{A\dot{B}}) = -\sigma^3$, so that

$$(\sigma^0{}_{\dot{A}B}) = \sigma^0; \quad (\sigma^k{}_{\dot{A}B}) = (\sigma^k{}_{A\dot{B}})^* = -\sigma^k, \quad k = 1, 2, 3. \tag{29}$$

The matrices $(\sigma^{\mu A}{}_{\dot{B}})$ are not all hermitian contrai-y to (o^{pA^i}), $(\sigma^\mu_{\dot{A}B})$ and $(\sigma^\mu_{A\dot{B}})$. It can be easily shown

$$\sigma^{\mu A\dot{B}} \sigma^\nu{}_{\dot{B}C} + \sigma^{\nu A\dot{B}} \sigma^\mu{}_{\dot{B}C} = 2 g^{\mu\nu} \delta^A_C, \tag{30}$$

from which follows that

$$\sigma^{\mu AB} \sigma^\nu{}_{A\dot{B}} = 2 g^{\mu\nu}. \tag{31}$$

Then,

$$\sigma_{\mu C\dot{D}} \sigma^{\mu AB} \sigma^\nu{}_{A\dot{B}} = 2 g^{p\nu} \sigma_{\mu C\dot{D}} = 2 \sigma^\nu{}_{C\dot{D}},$$

which implies[25]

$$\sigma_{\mu C\dot{D}} \sigma^{\mu AB} = 2 \delta^A{}_C \delta^B{}_{\dot{D}}. \tag{32}$$

Other similar relations follow by raising or lowering the indices and taking complex conjugation. From Eqs. *(4-26)* and (4-31), we have

$$2g^{\mu\nu} = \Lambda^\mu{}_\rho (S_1(\Lambda))^A{}_C (S_1(\Lambda))^{\dot{B}}{}_{\dot{D}} \sigma^{\rho C\dot{D}} \sigma^\nu{}_{A\dot{B}} \tag{33}$$

or

$$\Lambda^{\nu\alpha} \equiv g^{\mu\nu}(\Lambda^{-1})^\alpha{}_\mu = \frac{1}{2} S_1(\Lambda)^A{}_C S_1(\Lambda)^{\dot{B}}{}_{\dot{D}} \sigma^{\alpha C\dot{D}} \sigma^\nu{}_{A\dot{B}},$$

so that

$$\Lambda^{\mu}{}_{\nu} = \frac{1}{2} S_1(\Lambda)^A{}_C\, S_1(\Lambda)^{\dot{B}}{}_{\dot{D}}\, \sigma_\nu{}^{C\dot{D}}\, \sigma^\mu{}_{A\dot{B}} = \frac{1}{2} g_{\alpha\nu}\, \mathrm{Tr}\, \{(\sigma^\mu{}_{\dot{B}A})\, S_1(\Lambda)\, \sigma^\alpha S_1^\dagger(\Lambda)\}. \tag{34}$$

The following explicit form of γ^μ will be useful latter:

$$(\gamma^{\mu a}{}_b) = \left(\begin{array}{c|c} 0 & (\sigma^{\mu A\dot{B}}) \\ \hline (\sigma^\mu{}_{\dot{A}B}) & 0 \end{array}\right), \tag{35}$$

$$(A_{\dot{a}b}\, \gamma^{\mu b}{}_c)\ \left(\begin{array}{c|c} (\sigma^\mu{}_{\dot{A}B}) & 0 \\ \hline 0 & (\sigma^{\mu A\dot{B}}) \end{array}\right). \tag{36}$$

The results obtained here are the main tools, of 2-spinor calculus, discussed in Refs. 1 and 2.

Eq. (4-26) shows that $\sigma^{\mu A\dot{B}} u_{A\dot{B}}$ transforms as a 4-vector in Minkowski space while, for a vector U^μ, $(U_\mu\, \sigma^\mu{}_{A\dot{B}})$ transforms as $u_{A\dot{B}}$. Thus, we may establish a correspondence between $u_{A\dot{B}}$ and a 4-vector U^μ by the relation[26]

$$U^\mu = \frac{1}{2}\, \sigma^{\mu A\dot{B}}\, u_{A\dot{B}}, \quad u_{A\dot{B}} = \sigma_{\mu A\dot{B}}\, U^\mu. \tag{37}$$

U^μ is real if $u_{A\dot{B}}$ is hermitian, it is a null vector if $u_{A\dot{B}} = \xi_A\, \eta_{\dot{B}}$ and a real null vector if $u_{A\dot{B}} = \pm\, \xi_A\, \xi_{\dot{B}}$ (e.g., a 2-spinor, ξ_A, determines a real null vector). Since u^{AB} are components of a vector, in the direct product spinor space spanned by $\{\underline{h}_A \otimes \underline{h}_B\}$,

$$\underline{u} \equiv u^{A\dot{B}}(\underline{h}_A \otimes \underline{h}_{\dot{B}}) = U_\mu\, \sigma^{\mu A\dot{B}}(\underline{h}_A \otimes \underline{h}_{\dot{B}}) \equiv U_\mu \underline{E}^\mu,$$

where

$$\underline{\mathrm{E}} = \sigma^{\mu AB}\, (\underline{h}_A\ \mathrm{O}\ \underline{h}_{\dot{B}}) \tag{39}$$

constitute a basis for a representation of the Lorentz group. In fact,

$$\begin{aligned} \underline{E}'^\mu &= (\Lambda^{-1})^\mu{}_\nu\, \sigma^{\nu C\dot{D}}\, S_1^{-1}(\Lambda)^A{}_C\, S_1^{-1}(\Lambda)^{\dot{B}}{}_{\dot{D}})\, S_1(\Lambda)^E{}_A\, S_1(\Lambda)^{\dot{F}}{}_{\dot{B}}\, (\underline{h}_E \otimes \underline{h}_{\dot{F}}) \\ &= (\Lambda^{-1})^\mu{}_\nu\, \sigma^{\nu C\dot{D}}\, \underline{h}_C \otimes \underline{h}_{\dot{D}} = (\Lambda^{-1})^\mu{}_\nu\, \underline{E}^\nu, \end{aligned} \tag{40}$$

thus establishing the correspondence e" ↔ E". Explicitly,

$$\begin{aligned} \underline{E}^0 &= (\underline{h}_1 \otimes \underline{h}_{\dot{1}} + \underline{h}_2 \otimes \underline{h}_{\dot{2}}), & \underline{E}^1 &= (\underline{h}_2 \otimes \underline{h}_{\dot{1}} + \underline{h}_1 \otimes \underline{h}_{\dot{2}}), \\ \underline{E}^2 &= -\, i\, (\underline{h}_1 \otimes \underline{h}_{\dot{2}} - \underline{h}_2 \otimes \underline{h}_{\dot{1}}), & \underline{E}^3 &= (\underline{h}_1 \otimes \underline{h}_{\dot{1}} - \underline{h}_2 \otimes \underline{h}_{\dot{2}}). \end{aligned} \tag{41}$$

The inner product is found to be

$$\underline{E}^{\mu} \cdot \underline{E}^{\nu} = \sigma^{\mu A\dot{B}} \sigma^{\nu C\dot{D}} (\underline{h}_A \cdot \underline{h}_C)(\underline{h}_{\dot{B}} \cdot \underline{h}_{\dot{D}}) = \sigma^{\mu A\dot{B}} \sigma^{\nu C\dot{D}} \varepsilon_{AC}\, \varepsilon_{\dot{B}\dot{D}} = \sigma^{\mu A\dot{B}} \sigma^{\nu}{}_{A\dot{B}} = 2g^{\mu\nu}. \tag{42}$$

5. Spin Frame

The expression in Eq. (4-37) reminds us of the tetrad formalism frequently used in general relativity. The formalism is useful for our discussion, in Riemannian space, where the metric tensor $g^{\mu\nu}$ becomes a function of space-time coordinates while, at the same time, we introduce a local cartesian frame of reference at each point in space-time. The tetrads or four legs then connect the world component A^{μ} with local components $A^{(\mu)}$. We will limit ourselves to the discussion in which the metric tensors remain constant, i.e., independent of the coordinates. The discussion, in 2-dimensional spinor-space, goes in close analogy to the case of the 4-dimensional Minkowski space which we first briefly review.

Consider four vectors $n_{(\alpha)}$, (a) = (O), (1), (2), (3), such that

$$\underline{n}_{(\alpha)} \cdot \underline{n}_{(\beta)} = g_{(\alpha)(\beta)}, \tag{1}$$

where $g_{(\alpha)(\alpha)} = (1, -1, -1, -1)$, $g_{(\alpha)(\beta)} = 0$ for (a) $\neq$ (β), e.g., $\underline{n}_0$ is timelike and $\underline{n}_{(1)}$, $\underline{n}_{(2)}$, $\underline{n}_{(3)}$ are spacelike. They are clearly linearly independent and we may write

$$\underline{A} = A^{(\alpha)} \underline{n}_{(\alpha)} = A_{(\alpha)} \underline{n}^{(\alpha)}, \tag{2}$$

where we define $g^{(\alpha)(\beta)} \equiv g_{(\alpha)(\beta)}$ and $A^{(\alpha)} = g^{(\alpha)(\beta)} A_{(\beta)}$. We expand $\underline{n}_{(\alpha)}$ w.r.t. the basis {$\underline{e}$,}:

$$\begin{aligned} \underline{n}_{(\alpha)} &= h^{\mu}_{(\alpha)}\, \underline{e}_{\mu} = h_{(\alpha)\mu}\, \underline{e}^{\mu}, \\ \underline{n}^{(\alpha)} &= g^{(\alpha)(\beta)}\, \underline{n}_{(\beta)} = h^{(\alpha)}{}_{\mu}\, \underline{e}^{\mu} = h^{(\alpha)\mu}\, \underline{e}_{\mu}, \end{aligned} \tag{3}$$

hence follows that

$$\underline{A} = A^{\mu}\, \underline{e}_{\mu} = A^{(\alpha)}\, h^{\mu}_{(\alpha)}\, \underline{e}_{\mu}, \tag{4}$$

or

$$A^{\mu} = A^{(\alpha)}\, h^{\mu}_{(\alpha)}, \tag{5}$$

and similar relations obtained by raising and lowering the indices. The normalization conditions of $\underline{e}$, and $\underline{n}_{(\alpha)}$ give[27]:

$$\begin{aligned} g_{(\alpha)(\beta)}\, h^{(\alpha)}_{\mu}\, h^{(\beta)}_{\nu} &= g_{\mu\nu}, \\ g_{\mu\nu}\, h^{\mu}_{(\alpha)}\, h^{\nu}_{(\beta)} &= g_{(\alpha)(\beta)}. \end{aligned} \tag{6}$$

From the discussion in Sec. 2, we find that, under a Lorentz transformation,

$$\underline{n}'_{(\alpha)} \equiv L(\Lambda)\,\underline{n}_{(\alpha)} = h^{\mu}_{(\alpha)}\,\Lambda^{\nu}{}_{\mu}\,\underline{e}_{\nu} \equiv h'^{\mu}_{(\alpha)}\underline{e}_{\mu}, \tag{7}$$

e.g., $h'^{\mu}_{(\alpha)} = $ A", $h^{\nu}_{(\alpha)}$, so that the index (a) is unaffected. From $A^{(\alpha)} = h^{(\alpha)}{}_{\mu}\,A^{\mu}$, easily shown,we see that the $A^{(\alpha)}$ components are unchanged too. Thus, tetrads of vectors $h^{\mu}_{(\alpha)}$ (or $n_{(\alpha)}$) define a (fixed) frame of reference w.r.t. which any vector A^{μ} can be decomposed. The linear independence of $h^{\mu}_{(\alpha)}$ is easily demonstrated. We also note that $\underline{A}\cdot \underline{B} = A^{\mu}B_{\mu} = A^{(\alpha)}B_{(\alpha)}$ and that the inner product remains invariant under a Lorentz transformation as well as under a rotation of the frame of reference, that is, when

$$n_{(\alpha)} \to \underline{N}_{(\alpha)} \text{ with } \underline{N}_{(\alpha)}\ \ \underline{N}_{(\beta)} = g_{(\alpha)(\beta)} \text{ or, equivalently}^{28},$$

$$h^{\mu}_{(\alpha)} \to R^{(\beta)}{}_{(\alpha)}\,h^{\mu}{}_{(\beta)} \text{ with } g_{(\sigma)(\rho)}\,R^{(\sigma)}{}_{(\alpha)}\,R^{(\rho)}{}_{(\beta)} = g_{(\alpha)(\beta)}.$$

For the 2-spinor space, a spin frame may be defined in terms of two vectors $n_{(1)}$ and $n_{(2)}$, in a complex two dimensional space, with basis vectors $\underline{h}_1$ and $\underline{h}_2$ which satisfy, like $\underline{h}_A$, the normalization condition

$$\underline{n}_{(A)}\cdot\,\underline{n}_{(B)} = \varepsilon_{(A)(B)}, \tag{8}$$

where $\varepsilon_{(1)(2)} = -\,\varepsilon_{(2)(1)} = 1$, $\varepsilon_{(A)(B)} = 0$ for (A) $\neq$ (B), (Ref. 29). The spin frame is completely specified in terms of the components $h^{B}{}_{(A)}$ of the vectors $n_{(A)} = h^{B}{}_{(A)}\,\underline{h}_B = -\,h_{(A)B}\,\underline{h}^B$ just as $h^{\mu}{}_{(\alpha)}$ did so in the earlier case. From Eq. (4-23), it follows that

$$\varepsilon_{(A)(B)} = \varepsilon_{CD}\,h^{C}{}_{(A)}\,h^{D}{}_{(B)} = h_{D(A)}\,h^{D}{}_{(B)} = -\,h^{D}{}_{(A)}\,h_{(B)D}. \tag{9}$$

This leads to[30]

$$\varepsilon^{AB} = \varepsilon^{(C)(D)}\,h^{A}{}_{(C)}\,h^{B}{}_{(D)} = h^{A}{}_{(C)}\,h^{(C)B} = -\,h^{(D)A}\,h^{B}{}_{(D)}, \tag{10}$$

where $\varepsilon^{(A)(B)} = \varepsilon_{(A)(B)}$ and they are used to raise or lower the indices inside brackets in a fashion identical to that of ε_{AB} and ε^{AB}: for example, $\underline{n}^{(A)} = \varepsilon^{(A)(B)}\,\underline{n}_{(B)}$ and $u_{(A)} = -\,\varepsilon_{(A)(B)}\,u^{(B)}$, etc. From

$$\underline{u} = u^{(A)}\,\underline{n}_{(A)} = -\,u_{(A)}\,\underline{n}^{(A)} = u^{B}\,\underline{h}_B = -\,u_B\,\underline{h}^B, \tag{11}$$

we have the expansion

$$u^{A} = h^{A}{}_{(B)}\,u^{(B)} = -\,h^{(B)A}\,u_{(B)}. \tag{12}$$

The inverse relations[31] are

$$\begin{aligned} u^{(A)} &= -\,h^{(A)}_{B}\,u^{B} = h^{(A)B}\,u_{B}, \\ \underline{h}_A &= -\,h^{(C)}_{A}\,\underline{n}_{(C)} = h_{(C)A}\,\underline{n}^{(C)} \end{aligned} \tag{13}$$

and others obtained by raising and lowering the indices. **A** Lorentz transformation A induces, according to the discussion in Sec. 4, the transformation

$$\underline{n}'_{(A)} = h^B_{(A)}\, \underline{h}'_B \qquad S_1(\Lambda)^C{}_B\, h^B_{(A)}\, \underline{h}_C \equiv h'^B_{(A)}\, \underline{h}_B \tag{14}$$

or

$$h'^B_{(A)} = S_1(\Lambda)^B{}_C\, h^C_{(A)}. \tag{15}$$

Similarly, we have $h'_{(A)B} = (S_1^{-1}(\Lambda))^C{}_B\, h_{(A)C}$. Thus, the components $u^{(A)}$ are unaltered. For the inner product we note $u \cdot \phi = u_A \phi^A = -u^A \phi_A = u_{(A)} \phi^{(A)} = -u^{(A)} \phi_{(A)}$. It thus remains invariant under Lorentz transformations as well as under spin-frame rotations. The latter constitute the transformations defined by

$$\underline{N}_{(A)} = S^{(B)}{}_{(A)}\, \underline{n}_{(B)} \equiv H^B{}_{(A)}\, \underline{h}_B \tag{16}$$

such that $\underline{N}_{(A)} \cdot \underline{N}_{(B)} = \varepsilon_{(A)(B)}$. It follows that $\varepsilon_{(A)(B)}\, S^{(A)}{}_{(C)}\, S^{(B)}{}_{(D)} = \varepsilon_{(C)(D)}$ so that the complex matrix $(S^{(A)}_{(B)})$ is unimodular und belongs to the $SL(2, C)$ group. Also, $H^B_{(A)} = S^{(C)}_{(A)}\, h^B_{(C)}$ and $\underline{u} = u^{(A)}\, \underline{n}_{(A)} = U^{(A)}\, \underline{N}_{(A)}$ implies $U^{(A)} = (S^{-1})^{(A)}{}_{(B)}\, u^{(B)}$. We observe that, while $\overline{u}^{(A)}$ is unaltered under Lorentz transformations, u^A is unaltered under spin-frame rotations. An exactly analogous discussion goes for the complex 2-spinor space with dotted indices spanned by {h;, h;).

An arbitrary spinor $u^{A\dot{B}}$, likewise, may be expanded w.r.t. the spin frame, $u^{A\dot{B}} = h^A_{(C)}\, h^{\dot{B}}_{(\dot{D})}\, u^{(C)(\dot{D})}$. For the case of $\sigma^{\mu A\dot{B}}$, we have

$$\sigma^{\mu A\dot{B}} = h^A_{(C)}\, h^{\dot{B}}_{(\dot{D})}\, \sigma^{\mu(C)(\dot{D})} \tag{17}$$

or

$$\sigma^{\mu(A)(\dot{B})} = h^{(A)}_C\, h^{(\dot{B})}_{\dot{D}}\, \sigma^{\mu C\dot{D}}. \tag{18}$$

Under a Lorentz transformation, the quantities $\sigma^{\mu(A)(\dot{B})}$ transform like a four vector, viz.,

$$\sigma'^{\mu(A)(\dot{B})} = \Lambda^\mu{}_\nu\, \sigma^{\nu(A)(\dot{B})}. \tag{19}$$

Moreover, we may easily show that $\sigma^{\mu(A)(\dot{B})}\, \sigma_{\mu(C)(\dot{D})} = 2\, \delta^{(A)}_{(C)}\, \delta^{(\dot{B})}_{(\dot{D})}$ and $\sigma^{\mu(A)(\dot{B})}\, \sigma^\nu_{(A)(\dot{B})} = 2\, g^{\mu\nu}$ so that $\sigma^{\mu(1)(\dot{1})}$, $\sigma^{\mu(1)(\dot{2})}$, $\sigma^{\mu(2)(\dot{1})}$ and $\sigma^{\mu(2)(\dot{2})}$ are a linearly independent set. We may thus expand[3] any four-vector U^μ in terms of them

$$U^\mu = u_{(A)(\dot{B})}\, \sigma^{\mu(A)(\dot{B})} = u_{A\dot{B}}\, \sigma^{\mu A\dot{B}}, \tag{20}$$

where

$$u_{(A)(\dot{B})} = \frac{1}{2}\, \sigma_{\mu(A)(\dot{B})}\, U^\mu. \tag{21}$$

The explicit expressions for $\sigma^{\mu(A)(\dot{B})}$ are

$$\begin{aligned}
\sigma^{0(A)(\dot{B})} &= h^{(A)}_1 h^{(\dot{B})}_{\dot{1}} + h^{(A)}_2 h^{(\dot{B})}_{\dot{2}}, & \sigma^{1(A)(\dot{B})} &= h^{(A)}_1 h^{(\dot{B})}_{\dot{2}} + h^{(A)}_2 h^{(\dot{B})}_{\dot{1}}, \\
\sigma^{2(A)(\dot{B})} &= i(-h^{(A)}_1 h^{(\dot{B})}_{\dot{2}} + h^{(A)}_2 h^{(\dot{B})}_{\dot{1}}), & \sigma^{3(A)(\dot{B})} &= h^{(A)}_1 h^{(\dot{B})}_{\dot{1}} - h^{(A)}_2 h^{(\dot{B})}_{\dot{2}}.
\end{aligned} \tag{22}$$

We also note that

$$\sigma^{\mu(A)(\dot{B})}\,\sigma_{\mu}{}^{(A)(\dot{B})} = 0, \qquad (A),\ (\dot{B})\ \text{fixed.} \tag{23}$$

Thus, $\sigma^{\mu(A)(\dot{B})}$ constitute a basis in the Minkowski space of four null tetrad of vectors, two of which are real, e.g., $\sigma^{\mu(1)(\dot{1})}$ and $\sigma^{\mu(2)(\dot{2})}$ and $\sigma^{\mu(1)(2)}$ and $\sigma^{\mu(2)(1)}$ are complex conjugate of each other[33].

6. Representation of the Four Group in Spinor Space

The following 4 x 4 matrices A„ A„ A„„ together with the identity matrix, constitute an Abelian group called thc Four-group:

$$\Lambda_s = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix}, \qquad \Lambda_t = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$$

$$\Lambda_{st} \equiv \Lambda_s \Lambda_t = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{1}$$

They correspond to space reflection, time inversion and space-time inversion, in Minkowski space. Combined with the restricted Lorentz group, we obtain the Full or Extended Lorentz group. We can show that, if we stick to linear transformations in representation space, it is impossible to represent the four group by 2 x 2 matrices while maintaining the mixed quantities $\sigma^{\mu A\dot{B}}$ fixed according to Eq. (4-26). For A„ we have

$$\begin{aligned} (\sigma^{kA\dot{B}}) &= -\,S_1(\Lambda_s)\,(\sigma^{kA\dot{B}})\,S_1^{\dagger}(\Lambda_s), \qquad k = 1,\ 2,\ 3, \\ (\text{o}^{0\text{A}\dot{\text{B}}}) &= S_1(\text{A,})\,(\text{o}^{0\text{A}\dot{\text{B}}})\,S_1^{\dagger}(\Lambda_s), \end{aligned} \tag{2}$$

while, for A„

$$\begin{aligned} (\sigma^{kA\dot{B}}) &= +\,S_1(\Lambda_t)\,(\sigma^{kA\dot{B}})\,S_1^{\dagger}(\Lambda_t); \qquad k = 1,\ 2,\ 3, \\ (\sigma^{0A\dot{B}}) &= -\,S_1(\Lambda_t)\,(\text{o}^{0\text{A}\dot{\text{B}}})\,S_1^{\dagger}(\Lambda_t). \end{aligned} \tag{3}$$

In either case, we require $S_1\, \text{o}^{\text{k}}\, S_1^{-1} = -\,\sigma^k$ (or $S_1\,\sigma^k = -\,\sigma^k S_1$) for k = 1, 2, 3. It is easily verified that it is not possible to attain this in terms of 2 x 2 matrices for which (o^0, σ^1, σ^2, σ^3) is a complete set. The situation is different in 4-dimensional spinor space and the improper transformation can be represented by linear transformations by means of 4 x 4 matrices.

The restricted Lorentz group is invariant sub-group of the full group and one verifies the following relations:

$$\begin{aligned} \Lambda_s^{-1}\Lambda_R\Lambda_s &= \Lambda_R, & \Lambda_t^{-1}\Lambda_R\Lambda_t &= \Lambda_R, \\ \Lambda_s^{-1}\Lambda_L\Lambda_s &= \Lambda_L^{-1}, & \Lambda_t^{-1}\Lambda_L\Lambda_t &= \Lambda_L^{-1}, \\ \Lambda_{st}^{-1}\Lambda_R\Lambda_{st} &= \Lambda_R, & \Lambda_{st}^{-1}\Lambda_L\Lambda_{st} &= \Lambda_L, \end{aligned} \tag{4}$$

where Λ_R is a space rotation and Λ_L pure Lorentz transformation, say, in ***(01)*** plane. Hence, we require the corresponding representation matrices in spinor space to satisfy:

$$\begin{aligned} &S^{-1}(\Lambda_s)\,\gamma^k\gamma^l\,S(\Lambda_s) = \gamma^k\gamma^l = S^{-1}(\Lambda_t)\,\gamma^k\gamma^l\,S(\Lambda_t), \\ &S^{-1}(\Lambda_s)\,\gamma^0\gamma^k\,S(\Lambda_s) = -\gamma^0\gamma^k = S^{-1}(\Lambda_t)\,\gamma^0\gamma^k\,S(\Lambda_t). \end{aligned} \tag{5}$$

At the same time, we require that the $\gamma^{\mu a}{}_b$ behave as invariant "mixed quantities" under the full group, according to Eq. ***(3-1)***. This leads to

$$S(\Lambda_s)\,\gamma^k\,S^{-1}(\Lambda_s) = -\gamma^k, \qquad S(\Lambda_s)\,\gamma^0\,S^{-1}(\Lambda_s) = \gamma^0 \tag{6}$$

and

$$\begin{aligned} S(\Lambda_t)\,\gamma^k\,S^{-1}(\Lambda_t) &= \gamma^k, \\ S(\Lambda_t)\,\gamma^0\,S^{-1}(\Lambda_t) &= -\gamma^0. \end{aligned} \tag{7}$$

It is easily shown that these imply[34]

$$\gamma_5' \equiv S(\Lambda)\,\gamma_5\,S^{-1}(\Lambda) = (\det\,\Lambda)\,\gamma_5, \tag{8}$$

or, written explicitly,

$$\gamma'^a_{5\,b} \equiv S(\Lambda)^a{}_c S^{-1}(\Lambda)^d{}_b\,\gamma^c_{5d} = (\det\,\Lambda)\,\gamma^a_{5b} \tag{9}$$

From Eq. (2-18), it follows that we may choose

$$S(\Lambda_s) = a\,\gamma^0, \qquad S(\Lambda_t) = b\,\gamma_5\,\gamma^0, \tag{10}$$

and, then,

$$\begin{aligned} S(\Lambda_{st}) &\equiv S(\Lambda_s)\,S(\Lambda_t) = -ab\,\gamma_5 \\ &= -S(\Lambda_t)\,S(\Lambda_s) \equiv -S(\Lambda_{ts}). \end{aligned} \tag{11}$$

We note that, though ***A,, = A,,,*** one has $S(\Lambda_{st}) = -S(\Lambda_{ts})$. Hence, we have double valued representations of the four group in spinor space[35]. The constants ***'a'*** and ***'b'*** may be fixed by requiring that the parity and time inversion operations, applied twice, lead to the identity transformation up to a ($\pm$) sign due to double-valuedness of the representation. Thus,

$$\begin{aligned} [S(\Lambda_s)]^2 &= a^2\,I = \pm I \qquad \text{or } a^2 = \pm 1, \\ [S(\Lambda_t)]^2 &= b^2\,I = \pm I \qquad \text{or } b^2 = \pm 1, \end{aligned} \tag{12}$$

so that[36] $a = \pm 1, \pm i$; $b = \pm 1, \pm i$; $|a|^2 = 1$, $|b|^2 = 1$, $a^4 = 1$ and $b^4 = 1$. We find, then, the following relations[37]

$$\begin{aligned} A\, S(\Lambda_s) A^{-1} &= a\, \gamma^{0\dagger} = aa^*(a\, \gamma^0)^{\dagger -1} = S(\Lambda_s)^{-1\dagger}, \\ A\, S(\Lambda_t) A^{-1} &= b\, \gamma_5^\dagger\, \gamma^{0\dagger} = bb^*(b\, \gamma_5^{-1}\, \gamma^{0-1})^{\dagger -1} = -(b\, \gamma_5\, \gamma^0)^{\dagger -1} = \\ &= -S(\Lambda_t)^{\dagger -1}, \end{aligned} \tag{13}$$

which may be combined with Eq. (3-7) as

$$\begin{aligned} A\, S(\Lambda) A^{-1} &= S(\Lambda)^{\dagger -1}, \qquad \Lambda^0{}_0 \geq 1, \\ &= -S(\Lambda)^{\dagger -1}, \quad \Lambda^0{}_0 \leq -1, \end{aligned} \tag{14}$$

for the full group. This may be interpreted as the transformation of a matrix A, according to

$$A' \equiv S(\Lambda)^{\dagger -1}\, A\, S^{-1}(\Lambda) = \mathrm{Sgn}(\Lambda^0{}_0)\, A \tag{15}$$

or

$$A'_{\dot{a}b} \equiv (S^{-1}(\Lambda))^{\dot{c}}{}_{\dot{a}}\, (S^{-1}(\Lambda))^d{}_b\, A_{\dot{c}d} = \mathrm{Sgn}\,(\Lambda^0{}_0)\, A_{\dot{a}b}, \tag{16}$$

where $\mathrm{Sgn}(\Lambda^0{}_0) = \pm 1$ according to whether $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$.

The metric matrix, B, in the spinor representation of the γ matrices of Sec. 4, is given by $B = -\gamma_5\, \gamma^0\, \gamma^2$. We find

$$\begin{aligned} B' &\equiv S^{-1\mathrm{T}}(\Lambda_s)\, B\, S^{-1}(\Lambda_s) = a^2 B, \\ B' &\equiv S^{-1\mathrm{T}}(\Lambda_t)\, B\, S^{-1}(\Lambda_t) = -b^2 B \end{aligned} \tag{17}$$

or $B'_{ab} = S^{-1}(\Lambda_s)^c{}_a\, S^{-1}(\Lambda_s)^d{}_b\, B_{cd} = a^2\, B_{ab}$, etc. Taking inverse of Eq. (6-17), we find similar relations for B^{ab}. We note that $B'^{ac}\, B'_{cb} = B^{ac}\, B_{cb} = -\delta^a{}_b$, since the Kronecker delta is an invariant tensor under the full group. The tensors $A_{\dot{a}b}$, B_{ab} and B^{ab} are invariant only up to a sign under the four group.

7. Transformation of Spinor and Bilinear Invariants

The transformation of spinors, given by Eqs. (2-23), (2-25), (2-27) and (2-29), reads, in terms of 2-spinors of Sec. 4, as follows:

$$\textit{Parity:}\quad u'^A = a\, v_{\dot{A}},\ v'^A = -a^{*-1}\, u_{\dot{A}},\ u'_A = -a^{-1}\, v^{\dot{A}},\ v'_A = a^*\, u^{\dot{A}}, \tag{1}$$

$$\textit{Time-inversion:}\ u'^A = i\, b\, v_{\dot{A}}, v'^A = i\, b^{-1*}\, u_{\dot{A}}, u'_A = i\, b^{-1}_{\ } v^{\dot{A}},\ v'_A = i b^*\, u^{\dot{A}}, \tag{2}$$

and the relations obtained by taking their complex conjugate. From Eqs. (4-12) and (6-17), it follows that

$$(\varepsilon_{AB})' = a^2(\varepsilon^{\dot{A}\dot{B}}), \qquad\qquad (\varepsilon^{AB})' = a^2(\varepsilon_{\dot{A}\dot{B}}), \tag{3}$$

for parity and

$$(\varepsilon_{AB})' = -b^2(\varepsilon^{\dot{A}\dot{B}}), \qquad (\varepsilon^{AB})' = -b^2(\varepsilon_{\dot{A}\dot{B}}), \tag{4}$$

for time inversion[39]

The bilinear invariants, of Sec. 3, take the following form in spinor representation

$$S(\text{Scalar}) = \xi^{\dot{a}} A_{\dot{a}b}\eta^b = u^{\dot{A}}\chi_{\dot{A}} + v_A\phi^A, \tag{5}$$

where
$$\xi^a = \begin{pmatrix} u^A \\ v_{\dot{A}} \end{pmatrix}, \qquad \eta^a = \begin{pmatrix} \phi^A \\ \chi_{\dot{A}} \end{pmatrix}; \tag{6}$$

$$P(\text{Pseudoscalar}) = \xi^{\dot{a}} A_{\dot{a}b}\gamma^b{}_{5c}\eta^c = -i(u^{\dot{A}}\chi_{\dot{A}} - v_A\phi^{\dot{A}}); \tag{7}$$

V(4-Vector):

$$\begin{aligned} V^\mu &= \frac{1}{2}\xi^{\dot{a}} A_{\dot{a}c}\gamma^{\mu c}{}_b\eta^b \\ &= \frac{1}{2}\sigma^\mu_{B\dot{A}}(\phi^B u^{\dot{A}} + v^B\chi^{\dot{A}}); \end{aligned} \tag{8}$$

A(Pseudo vector):

$$A^\mu = \frac{1}{2}\xi^{\dot{a}} A_{\dot{a}c}(i\gamma_5\gamma^\mu)^c{}_b\eta^b = \frac{1}{2}\sigma^\mu_{B\dot{A}}(\phi^B u^{\dot{A}} - v^B\chi^{\dot{A}}); \tag{9}$$

T(Antisymmetric tensor): $F^{\mu\nu} = \xi^{\dot{a}} A_{\dot{a}c}(\Sigma^{\mu\nu})^c{}_b\eta^b$ is, apart from a factor, $\simeq [u^{\dot{A}}\sigma^\mu{}_{AB}\sigma^{\nu B\dot{C}}\chi_{\dot{C}} + v_A\sigma^{\mu A\dot{B}}\sigma^\nu_{\dot{B}C}\phi^C] - (\mu\leftrightarrow\nu)$.

For $\xi^a = \eta^a$, one has

$$\begin{aligned} &S = u^{\dot{A}}v_{\dot{A}} + u^A v_A = S^*, \; P = -i(u^{\dot{A}}v_{\dot{A}} - u^A v_A) = P^*, \\ &V^\mu = \frac{1}{2}\sigma^\mu{}_{B\dot{A}}(u^B u^{\dot{A}} + v^B v^{\dot{A}}) = V^{\mu *}, \\ &A^\mu = \frac{1}{2}\sigma^\mu{}_{B\dot{A}}(u^B u^{\dot{A}} - v^B v^{\dot{A}}) = A^{\mu *}. \end{aligned} \tag{10}$$

We observe that the invariants S and P vanish for 4-spinors of the type $\begin{pmatrix} u^A \\ \lambda u_{\dot{A}} \end{pmatrix} = \xi = \eta$. It is easilg shown that $V^\mu V^\mu := A^\mu A_\mu = 4(u^{\dot{A}}\chi_{\dot{A}}) \times (\phi^B v_B)$ which, for $\xi = \eta$, reduces to $4(u^A v_{\dot{A}})(u^B v_B)$ and is real. Hence, $\xi = \eta = \begin{pmatrix} u^A \\ \lambda u_{\dot{A}} \end{pmatrix}$ defines a real null vector, or that a 2-spinor defines a real null vector (Ref. 2). The two scalars S and P and the two vectors V and A behave differently under improper transformations. For example,

$$(\xi^{\dagger} A\, \gamma^{\mu} \xi)' \equiv \xi'^{\dagger} A'\, \gamma'^{\mu}\, \xi' = \xi^{\dagger}\, S^{\dagger}\, S^{-1\dagger} A S^{-1}\, \Lambda^{\mu}{}_{\nu}\, S\, \gamma^{\nu} S^{-1}\, S\, \xi$$
$$= \Lambda^{\mu}{}_{\nu}\, (\xi^{\dagger}\, A\, \gamma^{\nu}\, \xi) = \xi'^{\dagger}\, A'\, \gamma^{\mu}\, \xi', \tag{11}$$

the last equality following from the invariance of $\gamma^{\mu a}{}_{b}$ under the full group. From Eq. (6-15), it follows that

$$\begin{aligned} \xi'^{\dagger}(A\,\gamma^{\mu})\xi' &= \Lambda^{\mu}{}_{\nu}\,(\xi^{\dagger}\, A\,\gamma^{\nu}\,\xi), \quad \Lambda \in \mathscr{L}^{\uparrow}_{+}, \\ &= -\,\xi^{\dagger}\, A\,\gamma^{k}\,\xi, \qquad k = 1, 2, 3 \\ &= +\,\xi^{\dagger}\, A\,\gamma^{0}\,\xi, \qquad \mu = 0 \end{aligned} \Bigg\} \; \Lambda = \Lambda_s \text{ or } \Lambda_t \tag{12}$$

$$= + \;\xi^{\dagger}\, \mathrm{A}\, \gamma^{\mu}\, \xi, \qquad \mathbf{A = A,,.}$$

For pseudo-vectors[40], $\xi^{\dagger}$ A $\gamma_5\, \gamma^{\mu}\xi$, we have opposite sign for A $= \Lambda_s$ or A,. For scalars and pseudoscalars, we obtain

$$\begin{aligned} \xi'^{\dagger}\, A\, \xi' &= \xi^{\dagger}\, A\, \xi, && \text{for} \quad \Lambda = \Lambda_s, \\ &= -\,\xi^{\dagger}\, A\, \xi, && \text{for} \quad \Lambda = \Lambda_t,\ \Lambda_{st}, \\ \xi'^{\dagger}\, A\,\gamma_5\, \xi' &= -\,\xi^{\dagger}\, A\,\gamma_5\,\xi, && \text{for} \quad \Lambda = \Lambda_s,\ \Lambda_{st}, \\ &= +\,\xi^{\dagger}\, A\,\gamma_5\,\xi, && \text{for} \quad \Lambda = \Lambda_t, \end{aligned} \tag{13}$$

and, for tensors,

$$\begin{aligned} \xi'^{\dagger}\, A\,\gamma^{k}\,\gamma^{l}\,\xi' &= \pm\,\xi^{\dagger}\, A\,\gamma^{k}\,\gamma^{l}\,\xi, \\ \xi'\, A\,\gamma^{0}\,\gamma^{k}\,\xi' &= \pm\,\xi^{\dagger}\, A\,\gamma^{0}\,\gamma^{k}\,\xi, \end{aligned}$$

the upper sign holding for A, and lower for A,.

The choice of the phase factors 'a' and 'b' may be narrowed down by appealing to the antilinear operation of charge conjugation associated with Dirac equation:

$$(i\,\gamma^{\mu a}{}_{b}\,\partial_{\mu} - m\,\delta^{a}{}_{b})\,\xi^{b} = e\,A_{\mu}\,\gamma^{\mu a}{}_{b}\,\xi^{b}. \tag{14}$$

On taking the complex conjugate, multiplying by $A_{\dot{a}c}$ and using A;, $\gamma^{\mu c}{}_{b} =$ $= A_{\dot{c}b}\,\gamma^{\mu\dot{c}}{}_{\dot{a}}$, we obtain

$$i\,A_{\dot{b}d}\,\gamma^{\mu d}{}_{c}\,\partial_{\mu}\,\xi^{*b} + m\,A_{c\dot{b}}\,\xi^{*b} = -\,e\,A_{\mu}\,A_{\dot{b}d}\,\gamma^{\mu d}{}_{c}\,\xi^{*b}.$$

From $\{-(\gamma^{\mu})^{\mathrm{T}},\ -(\gamma^{\nu})^{\mathrm{T}}\} = 2g^{\mu\nu}$, it follows that there exists a nonsingular matrix C, such that[41], $\mathbb{C}\gamma^{\mu}\,\mathbb{C}^{-1} = -\,(\gamma^{\mu})^{\mathrm{T}}$ or $\gamma^{\mu a}{}_{c}\,\mathrm{C}^{cb} = -\,\mathrm{C}^{ac}\,\gamma^{\mu b}{}_{c}$, where $\mathbb{C}^{-1} \equiv (\mathbb{C}^{ab})$ and whose invariance under the restricted group, may easily be verified. Hence,

$$(i\,\gamma^{\mu a}{}_{b}\,\partial_{\mu} - m\,\delta^{a}{}_{b})\,\eta^{b} = -\,e\,A_{\mu}\,\gamma^{\mu a}{}_{b}\,\eta^{b}, \tag{15}$$

where the charge conjugate spinor r] is given by

$$\eta^{a} \text{ E } \mathbb{C}^{ca} A_{\dot{b}c} \xi^{*b}. \tag{16}$$

It corresponds to a Dirac particle with charge (--e). **A** candidate for $\mathbb{C}$ is

$$\mathbb{C} = \lambda \gamma_5 B^{-1}. \tag{17}$$

Requiring that charge conjugation applied twice leads back to the original spinor, gives $|\lambda|^2 = 1$.

We verify[42] that under a restricted transformation

$$\eta'^{a} = \mathbb{C}^{ca} A_{\dot{b}c} \xi^{*\prime b} = \mathbb{C}^{ca} A_{\dot{b}c} S(\Lambda)^{\dot{b}}{}_{\dot{d}} \xi^{\dot{d}} = S(\Lambda)^{a}{}_{b} \eta^{b}; \tag{18}$$

now impose[8] that r]"' $\equiv \mathbb{C}^{ca} A_{\dot{b}c} \xi^{*\prime b}$ satisfies the same relation under the improper transformations as well: this leads to

$$\begin{aligned} \mathbb{C}'^{-1} \equiv S(\Lambda)\, \mathbb{C}^{-1} S^{T}(A) &= \quad \text{C-'}, \quad \text{for } \Lambda^{0}{}_{0} \geq 1, \\ &= -\mathbb{C}^{-1}, \quad \text{for } \Lambda^{0}{}_{0} \leq -1. \end{aligned} \tag{19}$$

This, in turn requires, $a^2 = -1$, $b^2 = -1$, e.g., $a = \pm i$, $b = \pm i$

References and Notes

1. An exhaustive list of references on spinor analysis may be found in W. L. Bade and H. Jehle, Rev. Mod. Phys. 25, 714 (1953) and W. C. Parke and H. Jehle, Lectures in Theoretical Physics, Vol. VII a, Univ. of Boulder Press (1964) p. 297. See also, E. M. Corson, Tensors, Spinors and Relatisvitic Wave Equations (Blackie and Sons Ltd., London, 1953) and J. Aharoni, The Special Theory of Relativity, (Oxford University Press, 1965).
2. See for example, F. A. E. Pirani, *Lectures in General Relativity,* (Prentice Hall, N. J., 1965); C. J. Isham, A. Salam'and J. Strathdee, IC/72/123 (1972).
3. The first or upper index labels row and the second or lower index labels columns when $\Lambda^{\mu}{}_{\nu}$ or g„ are written as matrices. We will avoid using matrices corresponding to $\Lambda_{\mu}{}^{\nu}$.
4. See Sec. 7, where representations of the Four Group are considered.
5. It follows that $g^{\mu\nu}$ is a contravariant tensor in indices μ and v.
6. $g^{\mu\nu}$ g„, $= g_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} g^{\mu\nu} \equiv g_{\alpha\beta} g^{\alpha\beta}$ or $g^{aB} = g^{\mu\nu} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu}$.
7. $(\gamma^0)^2 = \mathrm{I}$ (identity), $(\gamma^k)^2 = -I$, $k = $ I, 2, 3. The tacit assurnption, that the 'μ' appearing in γ^{μ} is a Minkowski space index, will be shown below.
8. See for example, S. S. Schweber: *Relativistic Quantum Field Theory* (Row, Peterson and Co., 1961), Chapters 1 and 4.
9. Due to the appearence of half-angles, the representation is double-valued for space rotations, e.g., both matrices $\pm S(\Lambda)$ represent the same rotation. There is no ambiguity in sign due to half-angles for pure Lorentz transformations. We will adopt the normalization det $S(\Lambda) = 1$.

10. *Note that, excepting the identity matrix,* all *other elements of the Clifford algebra are traceless. Note also* $\omega_{\mu\nu}$ + **o, ,** $+ g_{\alpha\beta}\, \omega^{\alpha}{}_{\mu}\, w^{p}$, = 0. *The relations,* $S(\Lambda)\, S(\Lambda') = S(\Lambda\Lambda')$ *and* $S(\Lambda^{-1}) = S^{-1}(\Lambda)$, *may easily be verified.*
11. *T indicates transposition:* $(A^{T})^{a}{}_{b} = A^{b}{}_{a}$. *The greek letters label space-time indices while roman letters, the spinor indices. It* is *clear that* $\{S^{-1T}(\Lambda)\}$, $\{S^{*}(\Lambda)\}$ *and* $\{S^{-1}{}'(A)\}$ *constitute representations isomorphic to the group* $\{S(\Lambda)\}$.
12. *We also have matrices corresponding to a* negative *sign on the right hand* side. *Note that* $A\gamma^{\mu} A$-' $= \gamma^{\mu T *} = (B\gamma^{\mu} B^{-1})^{*} = B^{*} C\, \gamma^{\mu}\, C$-$^{1} B^{-1 *}$. *Hence, (A-' B* C)* is *a* multiple *of identity. If we impose* $\det(A) = \det(B) = \det(C) = 1$, *these matrices are defined* only *up to a factor* ± 1, $\pm i$
13. *It cannot be chosen to be symmetric as it will lead tu ten antisymmetric, linearly independent, 4 x 4 matrices* $(B\gamma_5\, \gamma^{\mu})$, $(B\Sigma^{\mu\nu})$.
14. *Thus,* $Z_a \cdot Z_b = 0$ *when* $a = b$ *and* $Z_a \cdot Z_b = -Z_b \cdot Z_a$. Also, Eqs. *(3-16) ensure that the* Z_a *are* linearly *independent* vectors (when $B_{ab} = -B_{ba}$ is *a non-singular matrix).*
15. *These tensors are* like *the metric tensor and may be used to relate the dotted basis with the undotted one.*
16. *We have* $\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$, $\gamma_5' = A\, y$, A-', $\Sigma^{\mu\nu} = A\, \Sigma^{\mu\nu} A^{-1}$, $\{\gamma_5, \gamma^{\mu}\}_{+} = 0$.
17. *In fact,* $\eta_{\dot{a}b}$ *regarded as a (4 x 4) matrix can be decomposed in terms of the 16 linearly independent matrices,* A, $A\gamma^{\mu}$, $A\, \Sigma^{\mu\nu}$, $i\, A\gamma_5\, \gamma^{\mu}$, $A\gamma_5$, *which form a basis for* all *4-dim. matrices of the form* $(\eta_{\dot{a}b})$. *A similar situation* holds *for* (η_{ab}), $(\eta_{\dot{a}\dot{b}})$, *etc.*
18. *Since the y's constitute an irreducible representation, they cannot be* all *chosen to be even matrices.*
19. *They* become *identical for the space rotation sub-group. The representation* $\{S_1(\Lambda)\}$ *is the socalled* $D(1/2, 0)$ *representation while* $\{Si^{-1}(\Lambda)\}$ *is* $D(0, 1/2)$ *of the* $SL(2, C)$ *group.* $\{S(\Lambda)\}$ *corresponds to* $D(1/2, 0) \oplus D(0, 1/2)$. *See, for example, Corson' or Schweber*[5].
20. *M. A. Naimark, Linear Representations of the Lorentz Group (Pergamon Press, N. Y., 1964); I. M. Gel' fand, M. I. Graev and N. Ya. Vilenkin, Integral Geometry and Representation Theory (Academic Press, N. Y., 1966); M. Carmeli, J. Math. Phys.* **11,** *1917 (1970).*
21. *The minus sign for* ξ_3 *and* ξ_4 *is for* convenience.
22. $\varepsilon_{12} = \varepsilon^{12} = 1$, $\varepsilon_{AB} = -\varepsilon_{BA}$, $\varepsilon_{AB} = \varepsilon^{AB}$ *and same definition for dotted indices.*
23. *In matrix notation,* $\varepsilon^{-1} S_1\, E = S_1^{T}$, *where* $E = (\varepsilon^{AB}) = (\varepsilon_{AB})$.
24. *Use* $\gamma^{\mu} S(\Lambda) = \Lambda^{\mu}{}_{\nu}\, S(\Lambda)\, y^{\nu}$ *and* $(\Lambda^{k}{}_{l}\sigma^{l} + \Lambda^{k}{}_{0}\, \sigma^{0})$. $(A^{k}, \sigma^{m} - A^{k}{}_{0}\, \sigma^{0}) = (\Lambda^{k}{}_{1})^2 + (\Lambda^{k}{}_{2})^2 + (\Lambda^{k}{}_{3})^2 - (\Lambda^{k}{}_{0})^2 = 1$, etc.
25. In *fact* $\sigma_{\mu C\dot{D}}\, \sigma^{\mu A\dot{B}} = 2\, \delta^{A}_{C}\, \delta^{\dot{B}}_{\dot{D}} + F^{A\dot{B}}_{C\dot{D}}$ *such that* $F^{A\dot{B}}_{C\dot{D}}\, \sigma^{\nu}_{A\dot{B}} = 0$. *It is easily shown that* $F^{A\dot{B}}_{C\dot{D}} \equiv 0$.
26. $U_{\mu} U^{\mu} = \det\, (u^{A\dot{B}})$. *Note that* $g^{\mu\nu} = \frac{1}{4}\, \sigma^{\mu}_{A\dot{B}}\, \sigma^{\nu}_{C\dot{D}}\, (2\varepsilon^{AC}\, \varepsilon^{\dot{B}\dot{D}})$ *implies* $g^{A\dot{B}, C\dot{D}} = 2\, \varepsilon^{AC}\, \varepsilon^{\dot{B}\dot{D}}$.
27. *The* first *relation leads to the second since* $g^{\mu\nu}\, g_{(\alpha)(\beta)}\, h^{(\alpha)}_{\mu}\, h^{(\beta)}_{\nu} = g^{\rho\nu}\, g_{\mu\nu} = g_{(\alpha)(\beta)}\, g^{(\alpha)(\beta)}$.
28. $\underline{N}_{(\alpha)} = R^{(\beta)}{}_{(\alpha)}\, \underline{n}_{(\beta)}$ *then* $h^{\mu}{}_{(\alpha)} \longrightarrow H^{\mu}{}_{(\alpha)} = R^{(\beta)}{}_{(\alpha)}\, h^{\mu}{}_{(\beta)}$ *and* $A^{(\alpha)} \longrightarrow (R^{-1})^{(\alpha)}{}_{(\beta)}\, A^{(\beta)} \equiv \mathbb{A}^{(\alpha)}$. *Note* that $\underline{A} = A^{(\alpha)}\, \underline{n}_{(\alpha)} = \mathbb{A}^{(\alpha)}\, \underline{N}_{(\alpha)}$.
29. *The label (A)* $\equiv$ *(1) or (2)* is convenient. *We could, of course, use any other labelling, say (A)= (0) or (1), etc.*
30. *Note* $\varepsilon^{(A)(B)}\, \varepsilon_{(A)(B)} = \varepsilon^{AB}\, \varepsilon_{AB}$ *and* $\varepsilon^{(A)(C)}\, \varepsilon_{(C)(B)} = -\delta^{(A)}_{(B)}$.
31. $h^{(A)}{}_{C}\, h^{C}{}_{(B)} = -\delta^{(A)}{}_{(B)} = -h^{(A)C}\, h_{(B)C}$, $h^{A}{}_{(C)}\, h_{B}{}^{(C)} = -\delta^{A}_{B} = -h^{(C)A}\, h_{(C)B}$.
32. *Like we expanded* A^{μ} *in terms of the linearly independent set* $h^{\mu}_{(\alpha)}$: $A^{\mu} = A^{(\alpha)}\, h^{\mu}{}_{(\alpha)} = A_{(\alpha)}\, h^{(\alpha)\mu}$.
33. *A similar discussion may be* made *for the 4-spinor space. However, we do not have a useful relation analogous to Eq. (4-32) for gamma matrices.*
34. $\gamma_5 = \gamma^0\, \gamma^1\, \gamma^2\, \gamma^3 = \frac{1}{4!}\, \varepsilon_{\mu\nu\lambda\rho}\, \gamma^{\mu}\, \gamma^{\nu}\, \gamma^{\lambda}\, \gamma^{\rho}$.

35. Due to the half angles, space rotations also have double valued representation, viz., $S(\theta)$ and $S(\theta + 2\pi)$ represent the same rotation in Minkowski space. There is no ambiguity of sign for pure Lorentz transformations (space-time rotations) due to the half angles.
36. In spinor representation of γ matrices, this amounts to det $S(\Lambda_s) =$ det $S(\Lambda_t) =$ $=$ det $S(\Lambda_{st}) = 1$.
37. These relations (as well as the expressions for $S(\Lambda_s)$ and $S(\Lambda_t)$) are derived using Eq. (2-18), $A\,\gamma^\mu A^{-1} = \gamma^{\mu\dagger}$ and the definition $\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$.
38. Thus, we need two kinds of 2-spinors to represent parity and time inversions, corresponding to inequivalent iepresentations $D(1/2, 0)$ and $D(0, 1/2)$. Under space-time inversion, $u'^A = i(ab) u^{\dot{A}}$, $v'_{\dot{A}} = -\,i(ab)\, v_{\dot{A}}$.
39. $(\varepsilon^{AB})' = (\varepsilon^{\dot{A}\dot{B}})'^* = a^2(\varepsilon_{AB})$, etc., and $u'_A = -\,(\varepsilon_{AB})'\, u'^B$, etc.
40. $\gamma'_5 = (\det\ \Lambda)\,\gamma_5$.
41. $\mathbb{C}$ must be antisymmetric just like B matrix.
42. $\mathbb{C}^{-1} = S(\Lambda)\ \mathbb{C}^{-1}\ S(\Lambda)^{\mathrm{T}}$; $\ S^\dagger(\Lambda)\ A\, S(\Lambda) = \boldsymbol{A}$.