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Convolution Quotients in the Production of Heat in an Infinite Cylinder

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A solution of the problem of heat production in an infinite cylinder is considered by an appeal to the concept of convolution quotients[11] and finite Hankel transforms[13]. The result given by Erdèlyi[7] follows as a particular case of the result established here.

Obtem-se, neste trabalho, uma solução ao problema da produção de calor em um cilindro infinito que é obtida fazendo-se uso do conceito de quocientes de convolução[11] e das transformações de Hankel finitas[13]. O resultado obtido por Erdèlyi[7] decorre como caso particular daquele aqui estabelecido.

1. Introduction

In mathematical physics, one often encounters singular functions, such as Dirac's δ-function[15] which vanishes except for a single value of the independent variable t (which one usually takes at $t = 0$), is not defined at $t = 0$ and satisfies the shifting property,

$$\int_{-\infty}^{+\infty} \delta(t) f(t) dt = f(0), \tag{1-1}$$

for every continuous function $f(t)$.

When one uses such functions in problems in the physical sciences, one obtains tentative results which need verification by use of different, more exact, techniques. Here, if one uses Mikusinski's approach[11], which was later developed by Erdèlyi[7], then that insatisfactory state of affairs is redeemed. In Mikusinski's theory, the very basic concept of *function* is generalized in a way which is similar to the extension of the concept of *number* from integers to rationals. The abstract entities of this theory may be interpreted either as operators or as generalized functions and they include the operators of differentiation, integration, numbers, continuous functions and impulse functions, as well.

Here, in this paper, we use convolution quotients in a problem of production of heat in an infinite cylinder. The problems of production of heat in cylinders are important in various technical applications[6] and, consequently, have been a focus of attention for many authors (Refs. 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14).

The finite Hankel transform[13], defined as

$$\bar{f}(\eta_i) = \int_0^a r f(r) J_0(r\,\eta_i) dr, \tag{1-2}$$

where η_i is a 'root of the transcendental equation

$$J_0(a\,\eta_i) = 0 \tag{1-3}$$

is also used in our derivation. If f (r) satisfies Dirichlet's conditions, then, at each point of the interval at which f (r) is continuous,

$$f(r) = \frac{2}{a^2} \sum_i \bar{f}(\eta_i) \frac{J_0(r\,\eta_i)}{[J_1(a\,\eta_i)]^2},$$

the sum being taken over all positive roots of Eq. (1-3).

In the following Section, we discuss some basic concepts of Mikusinski's theory which will be needed in our problem of heat production, treated in Sec. 3.

2. Convolution Quotients

We denote a function by f or {f (t)), this meaning the function as a whole entity, while f *(t)* will denote the value (real or complex) of the function f at the point t. Next, let us consider the set C of all continuous functions and define two operations of addition and corivolution product, for any two elements of C, as

(i) f + g : the function whose value, at t, is $f(t) + g(t)$, (2-1)
(ii) f g : the function whose value, at t, is

$$\int_0^t f(x)\, g(t - x) dx. \tag{2-2}$$

Here, fg ($f * g$) is called a convolution (resultant, *Faltung*, or composition; ff will be denoted by f^2 and a similar notation will be employed for higher convolution powers.

We denote the constant function $\{1\}$ by h:

$$h(t) = 1, \tag{2-3}$$

$$h^2(t) = \int_0^t h(x)\, h(t-x)dx = t \tag{2-4}$$

and, similarly, by induction,

$$h^n = \left\{\frac{t^{n-1}}{(n-1)!}\right\}, \quad n = 1, 2, 3, \ldots . \tag{2-5}$$

More generally, we may define

$$h^\alpha = \left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}, \quad \mathrm{Re}\ \alpha > 0. \tag{2-6}$$

Of course, h E C. Now, for any **f** $\in$ C, we have

$$hf = \left\{\int_0^t f(u)du\right\}, \tag{2-7}$$

so that convolution of a continuous function with h effects an integration of that function with zero as fixed lower limit. Thus, h may be regarded as an operator of integration and, h^n, as the operator of n-times repeated integration; indeed. $h^\alpha f$ is the Riemann-Liouville fractional integral of order a. It is easy to verify that C is a commutative ring, the socalled "convolution ring". The zero element of the ring is the function $\{0\}$. There is no unit element in this ring.

The convolution ring has no divisors of zero[7] and, hence, can be embedded in a field F. Let us call, convolution quotients, (Distributions or Generalized Functions) the elements of this field. In F, all equations

$$f * \xi = g, \quad f, g \in C \tag{2-8}$$

are solvable, the unique solution being written as

$$\xi = g/f. \tag{2-9}$$

Here we are to understand that the convolution quotients of the functions **f** and g follow the same rules as rational numbers. It is easy to see that there are elements in F which do not belong to C.

As an example, we consider the equation

$$f e = f, \quad \forall f \in C,$$

or

$$\int_0^y f(y-x)e(x)dx = f(y),$$

which is not always possible; hence, $e \notin C$. In fact, e corresponds to a b-function, i.e., a δ-function is the unit element for convolution products.

We have seen that h is the operator of integratiori and h" that of n-times repeated integration. We now introduce the inverse of h,

$$s = 1/h = \frac{\{\delta(x)\}}{\{h(x)\}} \in F,$$

i.e.,

$$s = h^{-1} = h/h^2.$$

Further, we have for this operator

$$s^0 = 1, \; s^\alpha = h^{-\alpha}, \; s^\alpha s^\beta = s^{\alpha+\beta}.$$

If a function $a = \{a(t)\}$ possesses a locally integrable derivative of order n, we then define the extended derivative of order n as

$$s^n a = a^{(n)} + a^{(n-1)}(O) + a^{(n-2)}(0)s + \ldots + a(0)s^{n-1}; \qquad (2\text{-}10)$$

for n = 1,

$$sa = s' + a(0)\delta \qquad (2\text{-}11)$$

and, for a function such that $a(0) = \ldots = a^{(n-1)}(0) = 0$, we have $s^n a = a^{(n)}$, i.e., the extended derivative reduces to the ordinary n^{th} derivative of the function.

We now consider functions involving the operator s. We have

$$s\,\{e^u(')) = \{a\, e^u(')) + 1,$$

or

$$(s - a)\,(e^{a\,'''}) = 1,$$

which gives

$$\{e^{\alpha(t)}\} = \frac{1}{s-\alpha},$$

and, similarly, we have

$$(s-\alpha)^{-n} = \left\{\frac{t^{n-1}\, e^{\alpha t}}{(n-1)!}\right\}, \quad n = 1, 2, \ldots .$$

3. Solution of the Heat Equation

Let us consider now the production of heat in an infinite cylinder of radius a. If we set that, initial and boundary conditions do not contain the φ coordinate, then the flow of heat takes place in planes through the z-axis and the equation of heat conduction is

$$\frac{1}{k}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} + A\delta(t)\,\delta(z), \tag{3-1}$$

where k is the diffusivity of the material; the temperature is a function of r, z, t, and the sources of heat are represented by $A\delta(t)\delta(z)$, δ denoting the unit element of the convolution field F (or Dirac's 6-function). We set, as initial condition (t = 0),

$$\theta(r, z, 0) = f(r)\, g(z), \tag{3-2}$$

where f is a function of r alone and, g, a function which depends only on z; $\theta(a, z, t)$ denotes the temperature at the outer cylindrical surface; also, we assume that $\theta \to 0$ as $z \to \pm\infty$. Multiplying both sides of Eq. (3-1) by $rJ_0(r\,\eta_i)$ and integrating from 0 to a, we obtain

$$\frac{1}{k}\frac{\partial\bar{\theta}}{\partial t} = a\eta_i\,\theta(a, z, t)\,J_1(a\eta_i) - \eta_i^2\,\bar{\theta} + \frac{\partial^2\bar{\theta}}{\partial z^2} + \frac{aA}{\eta_i}J_1(a\eta_i)\,\delta(t)\,\delta(z), \tag{3-3}$$

(s. Ref. 13), where

$$\bar{\theta}(\eta_i, z, t) = \int_0^a r\,\theta(r, z, t)\,J_0(r\,\eta_i)\,dr, \tag{3-4}$$

the η_i being the positive roots of Eq. (1-3). By an appeal to the concept of extended derivative, Eq. (2-11), we now write the operational form of Eq. (3-3):

$$s\bar{\theta} - \bar{f}(\eta_i)\,g(z) = ka\eta_i\,\theta(a, z, t)\,J_1(a\eta_i) - k\eta_i^2\bar{\theta} + k\frac{\partial^2\bar{\theta}}{\partial z^2} + \frac{aAk}{\eta_i}J_1(a\eta_i)\,\delta(t)\,\delta(z), \tag{3-5}$$

or

$$\frac{\partial^2 \bar{\theta}}{\partial z^2} - (s + k\eta_i^2)\ \frac{\bar{\theta}}{k} = -\ \frac{1}{k}\bar{f}(\eta_i)\, g(z) - a\eta_i\, \theta(a,\ z,\ t)\, J_1(a\eta_i)$$
$$-\ \frac{aA}{\eta_i}\, J_1(a\eta_i)\, \delta(z). \tag{3-6}$$

Eq. (3-6) is of the form

$$\frac{\partial^2 \bar{\theta}}{\partial z^2} - \omega^2 \bar{\theta} = F(\eta_i,\ z), \tag{3-7}$$

which has the particular solution[7]

$$\bar{\theta} = -\ \frac{1}{2\omega} \int_{z_1}^{z} e^{(y-z)\omega}\, F(y) dy - \frac{1}{2\omega} \int_{z}^{z_2} e^{(z-y)\omega}\, F(y) dy. \tag{3-8}$$

If $z_1 \rightarrow -\ \infty$ and $z_2 \rightarrow co$, then a particular solution of (3-6) is

$$\bar{\theta} = -\ \frac{\sqrt{k}}{2\sqrt{s + k\eta_i^2}} \int_{-\infty}^{z} \exp\left[-\ (z - y)\left(\frac{s + k\eta_i^2}{k}\right)^{1/2}\right]\left\{-\ \frac{1}{k} f(\eta_i)\ g(y)\right.$$
$$\left. -\ a\eta_i \theta(a,\ y)\, J_1(a\eta_i) - \frac{aA}{\eta_i}\, J_1(a\eta_i)\, \delta(y)\right\} dy$$
$$-\ \frac{\sqrt{k}}{2\sqrt{s + k\eta_i^2}} \int_{z}^{\infty} \exp\left[-\ (y - z)\left(\frac{s + k\eta_i^2}{k}\right)^{1/2}\right]\left\{-\ \frac{1}{k} f(\eta_i)\ g(y)\right.$$
$$\left. -\ a\eta_i \theta(a,\ y)\, J_1(a\eta_i) - \frac{aA}{\eta_i}\, J_1(a\eta_i)\, \delta(y)\right\} dy, \tag{3-9}$$

which can also be written as

$$\bar{\theta} = \frac{\sqrt{k}}{2\sqrt{s + k\eta_i^2}} \int_{-\infty}^{+\infty} \exp\left[-\ |z - y|\left(\frac{s + k\eta_i^2}{k}\right)^{1/2}\right]\left\{\frac{1}{k} f(\eta_i)\ g(y)\right.$$
$$\left. +\ a\eta_i \theta(a,\ y)\, J_1(a\eta_i) + \frac{aA}{\eta_i}\, J_1(a\eta_i)\, \delta(y)\right\} dy. \tag{3-10}$$

If we define the function

$$\Phi[(z - y)/\sqrt{k},\ t] = (\pi t)^{-1/2}\ \exp\ [-\ (z - y)^2/4kt], \tag{3-11}$$

then we can write

$$\frac{1}{2}\left(\frac{k}{s+k\eta_i^2}\right)\exp\left[-(z-y)\left(\frac{s+k\eta_i^2}{k}\right)^{1/2}\right]=$$

$$=\frac{\sqrt{k}}{?}\exp\,(-\eta_i^2kt)\,R[(z-y)/\sqrt{k},\ \mathrm{r}], \qquad (3\text{-}12)$$

After a trifle manipulation and making use of Ref. 7, relation (3-10) may be appropriately written as

$$\bar{\theta}(\eta_i,\ \mathrm{z},\ \mathrm{t}) = I\text{-}\quad \exp\,(-\eta_i^2kt)\,f(\eta_i)\int_{-\infty}^{+\infty}\exp\,(-\mathrm{u}^2)\,\mathrm{g}(\mathrm{z}+2u\sqrt{kt})du$$

$$+\ ka\pi^{-1/2}\exp\,(-\eta_i^2kt)\eta_iJ_1(a\eta_i)\int_{-\infty}^{+\infty}\exp\,(-u^2)\theta(a,z+2u\sqrt{kt})du$$

$$+\ \mathrm{ka}\,A\pi^{-1/2}\,\eta_i^{-1}J_1(a\eta_i)\,\exp\,[-(\eta_i^2kt)-\mathrm{z}^2/4?\mathrm{tkt}]\,. \qquad (3\text{-}13)$$

Applying the inversion theorem for finite Hankel transforms, Eq. (1-4), we obtain, for the temperature function, the expression

$$\theta(r,\ z,\ t)\ =\ \frac{2}{a^2}\sum_i\bar{\theta}(\eta_i,\ z,\ t)\,\frac{J_0(r\eta_i)}{[J_1(a\eta_i)]^2} \qquad (3\text{-}14)$$

In the limiting case $\mathrm{r}\to 0$, when the temperature is a function of z and t only, our solution reduces to the one given by Erdèlyi[7].

4. A Particular Case

If we set

$$f(r) = \mathrm{a}^2 - \mathrm{r}^2, \qquad (4\text{-}1)$$

then (Ref. 13)

$$\bar{f}(\eta_i)\ =\ 4a\eta_i^{-3}\,J_1(a\eta_i). \qquad (4\text{-}2)$$

Taking %(az, t) = 0 and $g(z)=\lambda\exp\,(-|\mathrm{z}|)$, the solution of our problem becomes

$$\theta(r,\ \mathrm{z},\ t)\ =\ \frac{8\lambda}{\mathrm{a}}\exp\,(kt)\,\exp\,(-z)\sum_{\mathrm{r}}\frac{J_0(r\eta_i)}{\eta_i^3J_1(a\eta_i)}\exp\,(-\eta_i^2kt)$$

$$+\ \frac{2kA}{a}\,\pi^{-1/2}\sum_i\frac{J_0(r\eta_i)}{\eta_iJ_1(a\eta_i)}\exp\,[-(\eta_i^2kt)-z^2/4kt]$$

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