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Recent Developments in the Theory of Critical Phenomena*

BERT SCHROER

Institut für Theoretische Physik, Freie Universität, Berlin†

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We review the work of Kadanoff, Wilson and Wegner, in the language of Euclidian field theory. In addition to Wilson's *renormalization group* method, which is based on the idea of eliminating short range fluctuations, we discuss the renormalization method of quantum field theory which, in the present context, we call *reparametrization* (in order to avoid confusion). A reparametrization which is of particular interest in the theory of critical phenomena is the one which leads to *scaling equations.* We derive new scaling equations which remain free of infrared divergences in two and three dimensions. Our method allows us to give a rather compact and unified discussion of Kadanoff's scaling laws and the related concept of global *scaling fields,* as well as the scale invariant correlation functions.

Revemos aqui o trabalho de Kadanoff, Wilson e Wegner, na linguagem da teoria euclidiana de campos. Além do método do *grupo de renormalização* de Wilson, que se baseia na idéia de eliminar as flutuações de curto alcance, discutimos o método de renormalização da teoria quântica de campos o qual, no presente contexto, denominamos *reparametrização,* a fim de evitar confusão. Uma reparametrização de especial interesse, na teoria dos fenômenos críticos, é a que conduz a *equações de escala.* Obtemos novas *equações de escala,* livres de divergências infravermelhas, em duas e três dimensões. Nosso método permite-nos apresentar uma discussão bastante compacta e unificada das leis de escala de Kadanoff como também do conceito relacionado de *campos de escala* globais, assim como das funções de correlação invariantes por escala.

Introductory Remarks

The material in these lecture notes was presented to an audience with some formal training in field theory at the University of São Paulo, in 1973/74. Most of it is standard, only the last section containing new results. These results on Kadanoff scaling equations and the field theoretical discussion of the Riedel-Wegner hypothesis of scaling fields are an elaboration of: Comment on a *New Approach to the Renormali-*

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†Postal address: 1, Berlin 33, Arnimallee 3, Germany.

zation Group, by *M*. Gomes and the present author. (University of São Paulo preprint, March 1973).

1. Basic Observations and Some Simple Formalism

In connection with first order phase transitions, which physicists have studied in liquid-gas systems, ferromagnets and in many other systems for over a century, there is the interesting phenomenon of *critical behavior* which one encounters at the high ternperature end of the phase coexistence curve.

The first phenomenological theory for phase transitions was that of Van der Waals, while further theoretical developgments in this century are associated with the names of Weiss, Ornstein and Zernicke, Landau and Ginsburg[1]. Those developments are all different versions of what is nowadays known as *mean field theory*.

In the late forties, the shadows of doubt were spreading. They originated, on the one hand, from Onsager's[2] result that critical indices, in the two-dimensional Lenz-Ising model, are different from mean field theory predictions and, on the other hand, from Guggenheim's[3] experiments on the liquid-gas transitions of many different substances.

Many subsequent measurements with refined experimental techniques have demonstrated the breakdown of the mean field theory description near to the critica1 point.

During the last couple of years, some new model-independent ideas have paved the way towards a new theoretical framework which we will sketch in the following Sections.

When lecturing to an audience with some background in quantum field theory, the question why a field- (or elementary particle –) theorist should be interested in critica1 phenomena naturally comes up. The answer is really very simple.

The conceptual idealizations and the accompanying mathematical formalism for critica1 phenomena are *very close* to those of relativistic quantum field theory. To be more specific, the probabilistic language of classical statistical mechanics is equivalent to the description in terms of euclidean field theory. In particular, lattice systems near the critica1 point "lose their memory" of the lattice distance and become identical to self-coupled euclidean theories with a local polynomial interaction. On the other hand, one knows from Schwinger's[4] and Symanzik's[5] work that one obtains an euclidean theory if one continues relativistic correlation functions to imaginary times.

Although the important "short distance problem" of relativistic theory is different from the "long distance problem" of critical behaviour, these problems can be formulated in such a way that they just correspond to two different "fixed points" of the same parametric scaling equation[6].

In short, such similarities between critical correlations and vacuum expectation values ,of local fields are more striking than the formal similarities between relativistic theories and the quantum theoretical many-body problems (in the framework of second quantization).

In the following, we briefly explain the language of critical phenomena in the case of a ferromagnetic phase transition. In many excellent articles, the reader may look up for the "translation key" which turns that language into that appropriate to other systems, e.g., liquid-gas transitions.

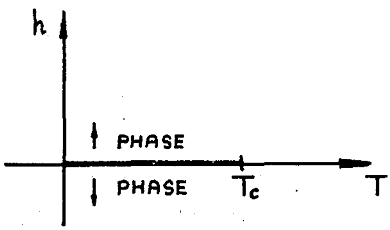


Fig. 1 - The h x T ferromagnetic phase diagram.

Ferromagnetic transitions only happen in a zero magnetic field (Fig. 1). The magnetization m, below the critical temperature T_c, has a jump, i.e., by letting the field h go to zero through positive values, one only reaches the points of the upper curve and, by doing the same for negative values of h, the limiting magnetization is given by the lower curve (Fig. 2).

The vanishing of an "order parameter" (m, in our case), as one approaches T_c from the ordered side, is a distinctive feature of a critical point.

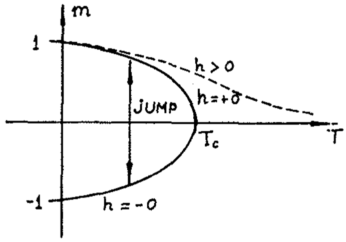


Fig. 2 - The m x T diagram with lines of constant magnetic field strength.

Experimental results and model considerations suggest the following parametrizations near the critical point:

magnetization: $m \sim (-t)^{\beta}$, with β defined only for $t = (T - T_c)/T_c < 0$;

susceptibility: $\chi \sim |t|^{-\gamma}$;

specific heat at constant field: $c_h \sim |t|^{-\alpha}$;

correlation length: $$\xi \sim |t|^{-\nu} \tag{1-1a}$$

In the above quantities, one has $h = 0$, while, on the critical isotherm ($t = 0$),

$$m = |h|^{1/\delta} \operatorname{sign} h \tag{1-1b}$$

holds.

There are, furthermore, indices for critical pair correlation functions which we will introduce later on.

In the above parametrizations, it is tacitly assumed (consistent with experimental facts) that a, γ and v are the same, independently of whether one approaches T_c from either the left or the right hand side.

Important experimental findings are:
i) critical indices are different from mean field indices;
ii) different critical systems form "universality classes". After subjecting the experimental parameters to a suitable transformation ("law of corresponding states"), the critical behaviour within each universality class is described by the same universal functions[7].

The characteristic features of a universality class are that the systems in one class, although possessing the same dimensionality, lattice symmetry and (perhaps) other "hidden" symmetries, have vastly different interactions.

We now turn to a (very schematic) discussion of the mathematical description for the special case of a Lenz-Ising system. Such a system belongs to a Hamiltonian

$$\mathscr{H} = -K \sum_{<n,n'>} \sigma_n \sigma_{n'} - h \sum_n \sigma_n , \tag{1-2}$$

where the σ_n are lattice spin variables which take only the values ± 1, while $K = J/kT$, J denoting the nearest neighbour exchange coupling, positive for ferromagnets and negative for antiferromagnets. The bracket under the sum stands for summation over nearest neighbours and $h = H/kT$ is the externa1 field in suitable units.

All thermodynamic quantities can be derived from the Gibbs free energy f which is introduced by means of the partition function. With N = number of lattice spins, we have

$$\begin{aligned} \exp(-Nf) &= Z = \mathrm{Tr} \exp(-\mathscr{H}) \\ &= \sum_{\{\sigma\}} \exp(-\mathscr{H}\{\sigma\}), \end{aligned} \tag{1-3}$$

in which the sum extends over all configurations (i.e., distributions of ± 1 over all lattice points).

The dynamical variable $\sigma(r)$ or functions thereof are called *operators*. Fields are the parameters in $\mathscr{H}$ which multiply operators, i.e., they are "thermodynamically conjugate" to the corresponding operators. If a function $\mathscr{O}(\sigma)$ only depends on the o's around one point, we call O a local operator. Because of translational invariance, the operators in $\mathscr{H}$ are global, i.e., they appear as sums over local operators. An important example of a local operator is the energy density,

$$E_n = \frac{1}{z} \sum_{n'(\mathrm{n,n,})n} \sigma_n \sigma_{n'} , \tag{1-4}$$

where the sum extends over nearest neighbours (n.n.); z denotes the number of nearest neighbours.

In terms of E_n, the Hamiltonian reads

$$\mathscr{H} = -K \sum_n E_n - h \sum_n \sigma_n . \tag{1-5}$$

Viewing

$$P\{o) = \frac{1}{Z} \exp(-\mathscr{H}\{o)) \tag{1-6}$$

as the probability for finding a particular configuration {o), we introduce expectation values

$$\langle \mathcal{O} \rangle = \sum_{\{\sigma\}} \mathcal{O}(\sigma)\, P\{\sigma\}. \tag{1-7}$$

The expectation values

$$\langle \sigma_{n_1} \cdots \sigma_{n_n} \rangle \tag{1-8}$$

are called n-point correlation functions of the magnetization o density. One may introduce correlation functions, involving composite variables, e.g.,

$$\langle E_n\, E_{n'} \rangle,$$

the 2-point correlation function of energy density.
It is a well known fact that thermodynamical quantities can always be written as sums (integrals) over correlation functions. The reader may convince himself, by a simple computation, of the validity of the following expressions:

$$m = \frac{\partial f}{\partial h} = \langle \sigma_n \rangle \equiv \langle \sigma \rangle, \tag{1-9}$$

$$\chi = \frac{\partial^2 f}{\partial h^2} = \sum_n \langle \sigma_0\, \sigma_n \rangle_c\,, \tag{1-10a}$$

$$c_h = \frac{\partial^2 f}{\partial K^2} = \sum_n \langle E_0\, E_n \rangle_c \tag{1-10b}$$

$$\chi_{K,h} = \frac{\partial^2 f}{\partial h\, \partial K} = \sum_n \langle \sigma_0\, E_n \rangle_c\,. \tag{1-10c}$$

Here the subscript c denotes the connected part of the correlation function, i.e.,

$$\langle \sigma_0\, \sigma_n \rangle_c = \langle \sigma_0\, \sigma_n \rangle - \langle \sigma \rangle^2$$

Since each spin variable o is bounded by 1, the only way that Eq. (1-10) can be divergent at the critical point is that the connected functions become "long ranged", i.e., that the sum diverges for large n.

From the study of the two-dimensional Lenz-Ising[8] model, one knows that the expressions

$$\langle \sigma_0 \, \sigma_n \rangle_{n\to\infty} = \frac{\text{const.}}{|n|^{2d_\sigma}} \tag{1-11a}$$

$$\langle E_0 \, E_n \rangle_{n\to\infty} = \frac{\text{const.}}{|n|^{2d_E}}, \tag{1-11b}$$

give the correct description at $t = 0$ and for *asymptotic separation.*

According to one's background, one uses either the indices η and ν defined by

$$d_\sigma = \frac{D-2}{3} + \frac{\eta}{2} = \frac{D-2}{2} + \gamma_\sigma , \tag{1-12a}$$

$$d_E = D - \frac{1}{\nu} = D - 2 + \gamma_E , \tag{1-12b}$$

where D stands for the space dimensionality of the model, or one chooses to talk about "anomalous dimensions" γ_σ, γ_E, defined on the right hand sides. of Eqs. (1-12).

For the D = 2 Lenz-Ising model[8], the values are $\eta = 1/4$, $\nu = 1$.

The mixed expectation value of σ and E do vanish. This is no surprise for a field theorist who is familiar with the close connection of scale invariance and conformal invariance[9]. According to a general theorem of conformal invariant theories, the 2-point function of two operators with different dimensions has to vanish.

Using a method developed, for the D = 2 Lenz-Ising model, by Kadanoff[10] and Ceva and Kadanoff[11], one may in principle compute the "long distance" dimensions of *any* composite fluctuation. In order to check the consistency of interpreting d_σ and d_E as "operator dimensions" (rather then numbers just showing up in the 2-point function), one may, by applying again the Kadanoff technique, prove that, at $t = 0$, for example, one has

$$\langle \sigma_{\lambda n_1} \dots \sigma_{\lambda n_N} \rangle_{\lambda\to\infty} \sim \lambda^{-N d_\sigma}. \tag{1-13}$$

We shall not, however, go into details of any model, since the framework described in the next Section, which is made more precise in the subsequent Sections, allows us to achieve a model-independent understanding.

2. The Phenomenological Kadanoff-Wilson-Wegner Framework

The first attempt to explain how the scale invariance of correlation functions and the thermodynamic scaling laws come about was given by Kadanoff[12] and is nowadays referred to as the Kadanoff "block-picture".

In order to supply a clear conceptual basis for Kadanoffs rough picture, Wilson[13] introduced the fundamental *Renormalization Group Transformation*, emphasizing the significance of fixed points.

Wegner[14] converted Wilson's ideas into a detailed and quantitative phenomenological description of critical phenomena.

Let $\mathscr{H}_0(\sigma_1 \ldots \sigma_N)$ be a Hamiltonian for a system of N spins, with a translational invariant interaction (imagine, for simplicity, periodic boundary conditions). The first step will consist in extending the system by doubling its linear dimensions but keeping the same interaction for the larger system. The Hamiltonian is $\mathscr{H}_0(\sigma_1 \ldots \sigma_{2^D N})$. Now divide the $2^{D}N$ lattice points into cells of size 2^D (i.e., each cell contains 2^D lattice spins). Introduce then, within each cell, the cell spin $S_i = \sum_{\text{cell } i} \sigma_i$, and $(2^D - 1)$ relative variables (spin differences in the ith cell) $\sigma'_{i,n(i)}$. The next step is to rewrite the Hamiltonian as

$$\mathscr{H}_0(S'_1 \ldots S'_N,\ \sigma'_{(i)} \ldots)$$

and perform the partial sum over the σ' (an integral if the original spins have a continuous distribution):

$$C \exp\left[-\mathscr{H}_1(S'_1 \ldots S'_N)\right] = \int_{\{\sigma'\}} \exp\left[-\mathscr{H}_0(S'_1 \ldots S'_N, \sigma'_{(i)} \ldots)\right] d[\sigma']. \quad (2\text{-}1)$$

(Note that the integration means summation for discrete spins). Here we have absorbed the c-number part (which is independent of the S's). After this elimination procedure for certain short range spin fluctuations, we rescale our spin variables and our length scale:

$$S_i = \sigma_i\, 2^{(D+2-\eta)/2}, \quad \mathbf{n} = 2\mathbf{m}, \qquad (2\text{-}2)$$

where $\mathbf{m}$ stands for integer vectors. The resulting Hamiltonian is called $\mathscr{H}_1(\sigma_1 \ldots \sigma_N)$ and the transformation

$$\mathscr{H}_0 \to \mathscr{H}_1$$

is called a (Wilson) Renormalization Group Transformation. The new Hamiltonian has different interactions than the original one. For all physical questions which do not depend on short range fluctuations, the new Hamiltonian should give the same answer.

In the probabilistic language, the Renormalization Group Transformation is clearly a transformation of the probability,

$$dP_0\{\sigma\} = \lim_{N\to\infty} \frac{1}{Z} \exp\left[-\mathscr{H}_0\{\sigma\}\right] d[\sigma]. \tag{2-3}$$

into a rescaled *conditional* probability

$$dP_1\{\sigma\} = \int_{\{\sigma'\}} dP_0\{\sigma, \text{a'}\}). \tag{2-3}$$

We call a Hamiltonian $\mathscr{H}_0$ *critica1* if we can adjust the parameter η so that the sequence of subsequent renormalization group transformations,

$$\mathscr{H}_0 \xrightarrow{T} \mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2 \xrightarrow{T} \ldots \xrightarrow{T} \mathscr{H}^*, \tag{2-4}$$

has a limit $\mathscr{H}^*$. Only Hamiltonians $\mathscr{H}_0$ in which the interaction parameters have suitable chosen values will have such a property.

In order to see why this must be so, let us imagine that the Renormalization Group Transformation T has a fixed point, i.e.

$$T\,\mathscr{H}^* = \mathscr{H}^* \tag{2-5}$$

and that we restrict our attention to Hamiltonians $\mathscr{H}_0$ which are infinitesimally close to $\mathscr{H}^*$. By definition of "infinitesimal", the transformation (for this, the number *1* of repeated applications T^l must be allowed to be continuous, an assumption which will be justified later)

$$\mathscr{H}^* + \delta\mathscr{H}_0 \xrightarrow{T} \mathscr{H}^* + \delta\mathscr{H}_1 \tag{2-6}$$

is linear on the interaction parameters ("fields") pertaining to $\delta\mathscr{H}_0$. This transformation may be represented by a matrix. To obtain it, let us imagine that we have a basis $\tilde{\mathscr{O}}_i$ of translational invariant operators (sums or integrals over local composite operators):

$$\delta\mathscr{H}_0 = \sum_i \tilde{\mu}_i \tilde{\mathscr{O}}_i \xrightarrow{T} \delta\mathscr{H}_1 = \sum_i \tilde{\mu}_i' \tilde{\mathscr{O}}_i, \tag{2-7}$$

with

$$\tilde{\mu}_i' = \sum_i \tilde{a}_{ij} \tilde{\mu}_j. \tag{2-8}$$

Let us then assume that the matrix $\boldsymbol{A} = (\tilde{a}_{ij})$ can be diagonalized (i.e., that we do not need associated eigenvectoss as occur for Jordan forms). As we wrote the rescaling in terms of powers of two, we now write, for the eigenvalues of $\boldsymbol{A}$,

$$\lambda_i = 2^{y_i}. \tag{2-9}$$

We shall denote, by $\mathcal{O}_i$, the basis operators on which the Renormalization Group acts diagonally. Hence,

$$\mathcal{H}_i \xrightarrow{T} 2^{y_i} \mathcal{O}_i \tag{2-10}$$

and

$$\delta\mathcal{H}_0 = \sum_1 \mu_i \mathcal{O}_i \xrightarrow{T} \sum_1 \mu_i 2^{y_i} \mathcal{O}_i. \tag{2-11}$$

Repeating, the Renormalization Group transformations, l times, we obtain

$$\delta\mathcal{H}_0 \xrightarrow{T^l} \sum_i \mu_i 2^{l y_i} \mathcal{O}_i. \tag{2-12}$$

One classifies the eigenoperators $\mathcal{O}_i$ according to the sign of y_i:

$y_i > 0$: relevant;
$y_i < 0$: irrelevant,
$y_i = 0$: marginal.

In the irrelevant case, the transformed fields μ_i are contracting each time by a factor $2^{-|y_i|}$. The relevant case leads to an increase of fields (so that the infinitesimal considerations become meaningless after a certain number of steps). The marginal case requires a more detailed discussion which we will come back to.

A necessary condition for the criticality of $\mathcal{H}_0 = \mathcal{H}^* + \delta\mathcal{H}_0$ is clearly that all relevant operators have zero fields. Note that in this language the identity operator $1 = \mathcal{O}_0$, which we absorbed into the constant C, would be a relevant operator with $y_0 = D$. To the free energy, it only contributes in an additive fashion. We obtain

$$Nf(\mathcal{H}_1) = 2^D Nf(\mathcal{H}_0)$$

or, repeating the process 1 times,

$$\mathbf{f}\{\mu_0, \mu_1, \ldots\} = 2^{-Dl} \mathbf{f}\{\mu_0 2^{Dl}, 2^{y_1 l}, \ldots\}. \tag{2-13}$$

The temperature t and the magnetic field h are expected to be among the relevant fields since their conjugate operators ΣE_n and $\Sigma\sigma_n$ are expected to have dimensions still near to their canonical (free field) values $D-2$ and $(D-2)/2$ (in mass units, i.e., inverse length units). So let us set $\mu_1 = t$, $\mu_2 = h$. We chose 1 to be so large that (assuming $t << 1$)

$$|t|\, 2^{y_E l} \simeq 1. \tag{2-14}$$

The part of the free energy f_s, after splitting of the (regular) contribution, coming from the unit operator, namely,

$$f\{\mu_0, t, h, \mu_3, \ldots\} = \mu_0 + f_s\{t, h, \ldots\},$$

fulfills the functional equation

$$f_s\{t, h, \ldots, \mu_i, \ldots) = |t|^{D/y_E} f_s\{\pm 1, h|t|^{-\Delta_h}, \ldots, \mu_i |t|^{-\Delta_i} \ldots), \tag{2-15}$$

with $\Delta_i = y_i/y_E$.

In the case of only two relevant fields t and h, the other arguments belonging to irrelevant fields drop out for very small t and we obtain the well known Kadanoff scaling law for the singular part of the free energy:

$$f_s(t, h) = |t|^{D/y_E} f(\pm 1, h|t|^{-\Delta_h}). \tag{2-16}$$

The critical exponents a, β, γ and 6, in terms of y_E and y_h, follow from the scaling law of the free energy. As an example, consider the specific heat. Differentiating f_s twice with respect to t and putting $h = 0$, we have

$$c_h(t) = |t|^{(D/y_E)-2} f(\pm 1, 0) + \text{less singular terms}; \tag{2-17}$$

hence a $= 2 - (D/y_E)$.

Similar considerations lead to scaling laws for correlation functions. We obtain, for example, for the connected spin correlation:

$$g_{\sigma\sigma}(r, t, h) = 2^{2(y_h - D)l} g_{\sigma\sigma}(r\, 2^{-l}, t\, 2^{y_E l}, h\, 2^{y_h l}). \tag{2-18}$$

The factor in front is just the rescaling of the spin (2-2). Since o is conjugate to h,

$$\eta = 2 + D - 2\, y_h.$$

The elimination of l, by using (2-14), leads to

$$g_{\sigma\sigma}(r, t, h) = |t|^{2(D-y_h)/y_E} g_{\sigma\sigma}(r\,|t|^{1/y_E}, \pm 1, h|t|^{-\Delta_h}). \tag{2-19}$$

We prefer to use the symbol r in the argument of the connected 2-point o-correlation instead of writing the lattice vector n. The reader may for himself derive the analogous scaling law for the energy correlation function g_{EE}.

In the K.W.W. phenomenological framework, all critical indices for ferromagnetic systems will be reduced to two basic numbers: y_E and y_h (Refs. 12, 13).

Later, we will show how, by a more quantitative discussion of the Wilson Renormalization Group, one can actually compute these numbers approximately.

A further important contribution to the critical phenomenology is the idea of "scaling fields" of Wegner[15] and Riedel and Wegner[16]. In the discussion up to now, the fields μ_i had to be infinitesimal, i.e., we do not have strictly speaking a global scaling law of the form (2-19). In order to obtain a useful global form, the authors proposed to make the following hypotheses:

There exist scaling fields g_i in terms of which the (non infinitesimal) μ_i's can be expanded:

$$\mu_i = g_i + \frac{1}{2} \sum_{jk} b_{ijk}\, g_j\, g_k + \dots . \tag{2-20}$$

In terms of the g_i's, the free energy and the correlation functions fulfill, in the typical case, global scaling laws, i.e.,

$$f_s\{g_i\} = e^{-Dl}\, f_s\{g_i\, 2^{y_i l}\}. \tag{2-21}$$

Criticality is now determined by the global condition of the vanishing of all relevant scaling fields. Note that to neglect irrelevant scaling fields, for large l, is only justified in the case that the free energy has a smooth limit for vanishing scaling fields with $y_i < 0$. Hence, a necessary condition is that the Hamiltonian remains bounded below for vanishing irrelevant scaling fields. There are cases for which this condition is not met (mean field theory for $D > 4$). A more detailed study shows that one has to take into account at least one relevant field[16].

The arguments in favour of the existence of scaling fields are basically consistency arguments.

In order to avoid clumsy notation and lengthy arguments, let us assume that instead of the discrete scale transformation $2^{y_i l}$, with l integer, we may use instead the continuous transformation $\exp(y_i l)$, with a continuous l (in Sec. 6, we will justify such an assumption). Infinitesimally, the old μ_i fields transform linearly,

$$\frac{\partial}{\partial l}\mu_i(l) = y_i\mu_i(l), \tag{2-22}$$

but, in higher orders, we have

$$\frac{\partial}{\partial l}\mu_i(l) = y_i\mu_i + \frac{1}{2}\sum a_{ijk}\,\mu_j\mu_k + \dots . \tag{2-23}$$

The Ansatz of scaling fields, (2-20), with

$$\frac{\partial}{\partial l}g_i(l) = y_i g_i, \quad \text{globally}, \tag{2-24}$$

is consistent with (2-23) if the b_{ijk} can be computed from the a_{ijk} and the y_i's by means of

$$(y_j + y_k - y_i)\,b_{ijk} = a_{ijk}. \tag{2-25}$$

If $y_j + y_k \neq y_i$, there is a unique solution. **A** similar consideration holds for all higher terms.

If however, the equality $y_i = y_j + y_k$ holds, we can only save the situation by working with l-dependent b's. Instead of (2-25), we have

$$\frac{\partial}{\partial l}b_{ijk} = a_{ijk}. \tag{2-26}$$

This leads to a linear l-dependence in b if there is no l-dependence in a. The quadratic term in (2-20) leads to an l-dependence (l_0 = integration constant)

$$\mu_i \sim e^{ly_i}(l + l_0), \tag{2-27}$$

which is now the leading term. Fixing l by the condition

$$|g_1|\,e^{y_i l} \simeq 1, \tag{2-28}$$

the ith argument of the free energy has now a logarithm, namely,

$$|g_1|^{-\Delta_i}\ln|g_1|.$$

If a marginal field occurs, one then obtains powers of logarithms. The reader is referred to the article of Riedel and Wegner[16] for a discussion of several special examples.

Thus, the consistency discussion of the hypothesis of scaling fields does not only yield the typical form of the global scaling law (2-21), but also leads to the exceptional logarithmic modifications.

It is often helpful to picture the condition of criticality in terms of scaling fields in a geometric fashion. Suppose we introduce a parameter space whose axes are the scaling fields. Then the critical surface $g_{i,relev.} = 0$ is a subspace of irrelevant (and, perhaps, marginal) coordinates only (Figs. 3 and 4). Each $\mathscr{H}_0$, whose pararneters lie in the critical subpace, belongs to a critical system and the repeated Renormalization Group transformations will transform $\mathscr{H}_0$, along a path, into $\mathscr{H}^*$.

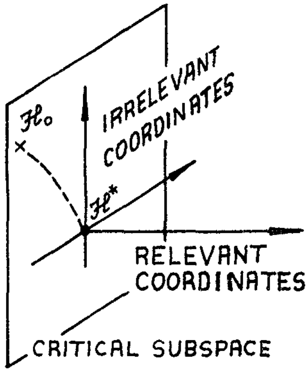


Fig. 3 - Critical surface in terms of "scaling-field" coordinates.

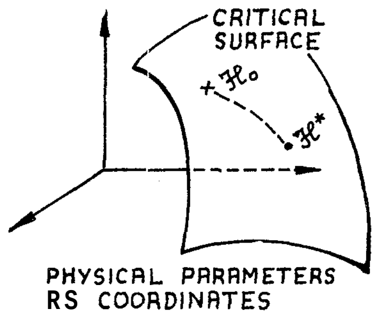


Fig. 4 - Critical surface in terms of "physical-parameter" coordinates.

The Lenz-Ising model, in two dimensions, in zero magnetic field is, for T = T_c, a point on the critical surface. It is known that this model has critical correlations which are only asymptotically invariant. We, therefore, only reach the point $\mathscr{H}^*$ if we leave the (too small) model space. On the other hand, the A^4 coupling modcl, which approximates the Lenz-Ising model to any degree of accuracy, has enough parameters (namely, the quadrilinear coupling g in addition to the temperature) to be able to reach $\mathscr{H}^*$ without enlarging the model space.

There are models, e.g., the A^6 coupling in D = 3 (equivalent to a classical spin which can take on three values), which, ia addition to a symmetry breaking relevant field (in analogy to h), have two relevant non-symmetry breaking fields.

Such models where there is, in addition to the temperature field, another field, are called *tricritical.* The mixture of He^3 and superfluid He^4 is an example of a system described by a tricritical model. In fact, the above mentioned model gives a good quantitative description of that system. It is easy to exhibit, for D = 2, perturbative fixed points which have any wanted "degree" of criticality.

3. Euclidian Field Theory and Functional Integration

Consider a multicomponent classical field variable $\Phi_i(x)$ with a cut-off Fourier-Transforrn:

$$\Phi_i(x) = (2\pi)^{-D/2} \int_{|k|<\Lambda} e^{ikx}\, \tilde{\Phi}_i(k)\, d^D k. \tag{3-1}$$

Regard such a variable as a random variable in the sense of probability theory by assigning a differential probability via a Gibbs type formula:

$$P[\Phi] = \frac{1}{Z}\, e^{-\mathscr{H}[\Phi]}, \tag{3-2}$$

with

$$Z = \int d[\Phi]\, e^{-\mathscr{H}[\Phi]}$$

and

$$\mathscr{H}[\Phi] = \sum \frac{1}{n!} \int h_{i1\ldots in}(x_1 \ldots x_n)\, \Phi_{i1}(x_i) \ldots \Phi_{in}(x_n)\, d^D x_1\, d^D x_n . \tag{3-3}$$

Here the "functional integral" our Φ is defined in the "physicists'way": replace the field variable $\Phi_i(x)$ by a "periodic box field",

$$\Phi_i(x) \to \Phi_i^{(L)}(x) = L^{-D/2} \sum_{|k|<\Lambda} e^{ikx}\, \tilde{\Phi}_i(k), \tag{3-4}$$

with $k = \frac{2\pi}{L}$ n, n = vector of integers, and then form

$$Z_L = \int \prod_{i,\,|k|<\Lambda} d\Phi_i(k) \exp [\![-\mathscr{H}[\Phi^{(L)}]]\!], \tag{3-5}$$

where, because of the reality condition $\tilde{\Phi}_i(k) = \tilde{\Phi}_i(-k)$, the integration may be resctricted to a half-space by combining k and (– k):

$$\mathrm{Im} \int d\tilde{\Phi}_i(k) \int d\tilde{\Phi}_i(-k) = \int_{-\infty}^{+\infty} d\, \mathrm{Re}\, \tilde{\Phi}_i(k) \int_{-\infty}^{+\infty} d\, \mathrm{Im}\, \tilde{\Phi}_i(k). \tag{3-6}$$

In a formal sense these equations are generalizations of the sum (1-3) for the Lenz-Ising model. The main difference is that Φ may now have a continuous range instead of just having the discrete values ± 1. **As** in the Lenz-Ising model, we can expect Z_L for large L to have an exponential volume factor whereas the correlation functions,

$$\langle\Phi_{i1}(x_i)\dots\Phi_{i_N}(x_N)\rangle^{(L)} = \int\prod_{i,k} d\Phi_i^{(L)}(k)\,P[\Phi^{(L)}]\,\Phi_{i1}^{(L)}(x_1)\dots\Phi_{i_N}^{(L)}(x_N), \quad (3\text{-}7)$$

are expected to stay finite in the "thermodynamic limit" $L \to \infty$.

An explicit characterization of an optimal "interaction space" $\mathscr{H}[\Phi]$, which leads to a mathematical defined $L \to \infty$ limit measure, $P[\Phi]\,d[\Phi]$, is not presently known. There are detailed disciissions of special models on which we will comment later on. It is clear, from physical intuition, that $\mathscr{H}[\Phi]$ should be essentially bounded from below, i.e., those parts where $\mathscr{H}$, in Φ space, is not bounded from below should be of "measure zero".

The "parameter space" is defined as the set of all coefficient functions:

$$h = (h_i(x), h_{i1i2}(x_1, x_2), \dots). \quad (3\text{-}8)$$

Most physically interesting models are local, i.e., the Fourier-transformed functions,

$$h_{i1\dots in}(x_1 \dots x_n) = (2\pi)^{-D(n-1)}\int \tilde{h}_{i1\dots in}(k_1 \dots k_n)\, e^{i\Sigma k_i x_i}\,\delta(\Sigma k_i)\, d^D k_1 \dots d^D k_n \quad (3\text{-}9)$$

have the form

$$\tilde{h}_{i1\dots in}(k_1 \dots k_n) = \text{polynomial in } k_i. \quad (3\text{-}10)$$

Needless to say that the parameter functions h may, without loss of generality, be assumed to be symmetric:

$$\tilde{h}_{iP(1)\dots iP(n)}(k_{P(1)} \dots k_{P(n)}) = \tilde{h}_{i1\dots in}(k_1 \dots k_n). \quad (3\text{-}11)$$

Because of the reality of the fields, the parameter functions are real in x-space, i.e.,

$$\tilde{h}_{i1\dots in}(k_1 \dots k_n) = \bar{h}_{i1\dots in}(-k_1 \dots -k_n). \quad (3\text{-}12)$$

Of special interest to us are A-cut-off euclidean theories which, in addition to the translation al invariance, already insured by the momentum space δ-function in (3-9), are also rotational invariant in k-space. Lattice systems, as the Lenz-Ising model, do not have this invariance, but their renormalization group properties near fixed points are not expected to change if we "euclidianize" them. In any case (this will become clear later on), most of the techniques we are going to introduce in the following, can be carried out to lattice theories with Brillouin zones of any shape. If not stated otherwise, we will restrict our considerations from now on to euclidean tfieories.

The standard method consists in splitting off the bilinear part from $\mathscr{H}$:

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_I,$$

with

$$\mathscr{H}_0 = \frac{1}{2}\int |\Phi(k)|^2\, G_0^{-1}(k) = \frac{1}{2}(\Phi, G_0^{-1}\, \mathrm{Q}) \tag{3-13}$$

and to write the generating functional of the correlation functions as

$$Z\{J\} = C\int \exp\left[-\frac{1}{2}(\Phi, G_0^{-1}\,\Phi)\right] d[\Phi]\, \exp[-\mathscr{H}_I[\Phi] + (J,\Phi)]$$

$$= C\exp\left[-\mathscr{H}_I\left[\frac{\delta}{\delta J}\right]\right]\int \exp\left[-\frac{1}{2}(\Phi, G_0^{-1}\,\Phi) + (J,\Phi)\right] d[\Phi]. \tag{3-14}$$

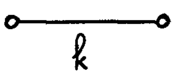
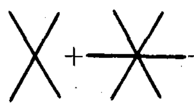
Here $(J,\Phi) = \int J(x)\,\Phi(x)\, d^D x$ and C contains the J-independent part, i.e., is determined by the requirement that, $Z\{0\} = 1$. The remaining integral is Gaussian and may be reduced to a limit of ordinary integrals using the methods described after (3-3). The result is

$$Z\{J\} = C'\exp\left[-\mathscr{H}_I\left[\frac{\delta}{\delta J}\right]\right] Z_0\{J\}, \tag{3-15}$$

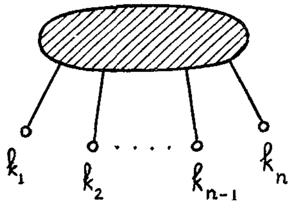
with $Z_0\{J\}\exp[(1/2)(J, G_0'\, J)] =$ free generating functional. By expanding the exponential of $\mathscr{H}_I$ in a power series, one obtains for the correlation functions,

$$\left.\frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)} Z\{J\}\right|_{J=0} \equiv \langle \Phi(x_1)\ldots\Phi(x_n)\rangle, \tag{3-16}$$

the Feynman rules

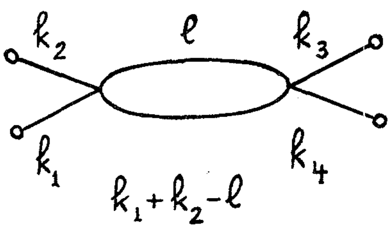
$G_0(k)$:  ; $\mathscr{H}_I$:  $+ \ldots$.

The number of the momentum-carrying lines emanating from each vertex (momentum conservation at each vertex!) agrees with the degree in Φ of each term in $\mathscr{H}_I$. If $\mathscr{H}_I[\Phi]$ contains x-space derivatives in Φ, the corresponding lines will represent powers in the momenta. The general Feynman graph for the n-point correlation function in mth order in $\mathscr{H}_I$ will be of the form ($k_n = -\sum_1^{n-1} k_i$):



where, inside the shaded region, one has m vertices from $\mathscr{H}_I$ connected to each other and to the external points by G_0-lines, the internal G_0-line being integrated over.

Example:



$\leftrightarrow$

$$\leftrightarrow \prod_{i=1}^{4} G_0(k_i) \int G_0(l)\, G_0(k_1 + k_2 - l)\, d^D l$$

Note that *all* the arguments of G_0 are restricted to the inside of the A-sphere.

The full combinatorics of Feynman diagrams (i.e., the combinatorical weight factors) are contained in the process of differentiation $\delta/\delta J$ with respect to the source J.

An important subclass of Hamiltonians, which under the Wilson renormalization group procedure will be transforrned into itself, is the class of interaction polynomials $\mathscr{H}_I$ of even degree. One or more component "classical" ferromagnets (i.e., Lenz-Ising models or classical limit of non-commutative $\mathcal{O}(N)$-Heisenberg models), in a zero magnetic field, for $T > T_c$, are belonging to this class. In Section 4, we will see that *all* known fixed points can be reached by doing renormalization group transformations in this class. It is customary to remove selfclosing loops as  by introducing Wick-products for the Gaussian theory $Z_0\{J\}$ by the standard recursion

$$\Phi_0(x) = \,:\Phi_0(x):, \; \Phi_0(x)\,\Phi_0(y) = \,:\Phi_0(x)\,\Phi_0(y): + \langle\Phi_0(x)\,\Phi_0(y)\rangle,$$
$$\Phi_0(x_1)\,\Phi_0(x_2)\,\Phi_0(x_3) = \,:\Phi_0(x_1)\,\Phi_0(x_2)\,\Phi_0(x_3): + \langle\Phi_0(x_1)\,\Phi_0(x_2)\rangle \times$$
$$\times\, \Phi_0(x_3) + \text{Permut.} \qquad (3\text{-}17)$$

and so on.

Without loss of generality, one may assume that $\mathscr{H}_I$ is given in the Wick-ordered form relative to $\mathscr{H}_0$. This just amounts to a simple reparametrization in the parameter-space of h's. For a Gaussian theory $\mathscr{H}_0$, one defines composite fields $:\Phi_0^n(x):$. The most general local function $\mathcal{O}(x)$ is a sum over Wick-products, at one point, involving x-space derivations. Note that the field

$$\Psi_0(x) = \int G_0^{-}{}'(\mathrm{x} - \mathrm{x}')\,\Phi_0(x')\,\mathrm{dx}'$$

belongs to this set if G_0^{-1} is a local expression, for example:

$$G_0^{-1}(x) = (-\, d,\; d,\; + m^2)\,\delta(x). \qquad (3\text{-}18)$$

In the case of a relativistic free field of mass m^2, the corresponding Ψ_0 would be zero. However, in the cut-off euclidean theory, Ψ_0 does not drop out from the basis of local functions, but it will be a *"short ranged" field.* For example,

$$\langle\Psi_0(x)\,\Phi_0(y)\rangle = \delta_\Lambda(x - y), \qquad (3\text{-}19)$$

where we have indicated that the 8-function has a cut off A in momentum space, i.e., is a short ranged function. Therefore, our set of local composite fields consists of *"normal" composite fields* as, e.g., $:\Phi_0^n(x):$ and *short ranged* composite fields as $:\Phi_0^n(x)\,\Psi_0(x):$. These features remain essentially preserved if we go from the free Gaussian theory to the interacting theory. By a straightforward computation, one verifies that $Z\{J\}$ obeys the Schwinger functional differential equation:

$$\left\{\int dx'\; G_0^{-1}(x - x')\,\frac{6}{8J(x')} - \mathscr{H}_{I,x}\left[\frac{\delta}{\delta J}\right] + J(x)\right\} Z\{J\} = 0, \qquad (3\text{-}20)$$

with $\mathscr{H}_{I,x}[\Phi] = \dfrac{\delta \mathscr{H}_I[\Phi]}{\delta J(x)}$.

By taking the n^{th} functional derivative and putting $\mathbf{J} = 0$, we obtain the "field equation" for correlation functions. Specializing, for the moment, to $\tilde{G}_0^{-1}(k) = k^2 + m^2$ and

$$\mathscr{H}_I = \frac{\mu_0}{4!}\int :\Phi^4(x): d^D x,$$

we see that

$$\Psi_0 \equiv (-\partial^2 + m^2)\,\Phi + \frac{\mu_0}{3!} :\Phi^3(x): \tag{3-21}$$

is a short ranged field. Here the $:\Phi^3(x):$ etc. are interacting composite fields corresponding to free fields whose expeclation values are defined, e.g., by

$$(:\Phi^3(x): X) = C \int :\Phi_0^3(x): X_{(0)} \exp[\![-\mathscr{H}_I[\Phi_0]]\!]\, dP[\Phi_0], \tag{3-22}$$

with

$$dP[\Phi_0] = \exp\left[-\frac{1}{2}(\Phi_0, G_0^{-1}\Phi_0)\right] d[\Phi_0],$$

$$X_{(0)} = \prod_{i=1}^{n} \Phi_0(x_i).$$

Up to now we have emphasized the probabilities, i.e., the measure theoretic aspect of our framework. We may equivalently describe our correlation functions as vacuum expectation values of a commuting set of euclidean field operators[17]. Consider, for example, the Gaussian theory. Introduce an euclidean Fock-space, $\mathscr{H}_E$, via creation and annihilation operators, which satisfy

$$\begin{aligned} [A(k), A(k')] &= 0 = [A^\dagger(k), A^\dagger(k')], \\ [A(k), A^\dagger(k')] &= \delta^{(D)}(k - k'), \end{aligned} \tag{3-23}$$

and the euclidean "free vacuum" $|\Phi_{E,0}\rangle$ with

$$A(k)\,|\Phi_{E,0}\rangle = 0, \tag{3-24}$$

by defining

$$\mathscr{H}_E = \overline{\{\text{polynomial}\,(A^\dagger)|\,\Phi_{E,0}\rangle\}}.$$

The field,

$$A_0(x) = (2\pi)^{-D/2} \int_{|k|<\Lambda} \{e^{-ikx} A^\dagger(k) + + \text{h.c.}\} \frac{d^D k}{\sqrt{k^2 + m^2}}, \tag{3-25}$$

leads to the desired two-point function

$$\langle \Phi_{E,0} | A_0(x) A_0(y) | \Phi_{E,0} \rangle = G_0(x-y). \tag{3-26}$$

The field $A_0(x)$, successively applied to the vacuum, defines the cut-off Hilbert-space $\mathscr{H}_{E,\Lambda} \subset \mathscr{H}_E$; it generates a maximal abelian set of operators in A?„, with the euclidean vacuum being a cyclic state. In order to obtain a complete (irreducible) set of operators, it is convenient to introduce a "canonical conjugate" of the form

$$\Pi(x) = \frac{i}{2} \int_{|k|<\Lambda} \{e^{-ikx} A^{\dagger}(k) - \text{h.c.}\} \sqrt{k^2 + m^2}\, d^D k,$$

which is such that: $[A(x), \Pi(y)] = i\delta(x-y)$, $[\Pi(x), \Pi(y)] = 0$ and

$$i \int e^{ikx} A(k) \sqrt{k^2 + m^2}\, d^D k = i\, \Pi(x) + \frac{1}{2}(-\partial_\mu \partial_\mu + m^2) A(x),$$

so that the euclidean Hamiltonian (generator of time translations in $\mathscr{H}_{E,\Lambda}$) is

$$H = -\frac{i}{2} \int : \Pi(x) \overleftrightarrow{\partial_t} A(x) : \mathrm{d}^D x. \tag{3-27}$$

As expected, one can not write euclidean generators of symmetry transformations in $\mathscr{H}_E$ solubly in terms of commuting fields $A(x)$. Note that the integrand is not the component of a conserved Noether current. The integral over $\partial_\mu : \Pi \overleftrightarrow{\partial_\mu} A :$ is, however, zero since it just gives boundary terms at infinity and, due to the fall off of G_0 in all directions, those boundaries do not contribute

In the euclidean operator language the functional expressions for the n-point correlation function

$$\langle X \rangle = C \int X_0 \exp[\![- \mathscr{H}_I[\Phi_0]]\!]\, dP[\Phi_0], \tag{3-28}$$

with

$$C^{-1} = \int \exp[\![- \mathscr{H}_I[\Phi_0]]\!]\, \mathrm{d}P[\Phi_0], \tag{3-29}$$

corresponds to

$$\begin{aligned} \langle E, 0 | A(x_1) \ldots A(x_n) | E, 0 \rangle &= C \langle \Phi_{E,0} | A_0(x_1) \ldots A_0(x_n) \exp[\![- \mathscr{H}_I[A_0]]\!] | \Phi_{E,0} \rangle \\ &= \langle \Phi_{E,0} | A_0(x_1) \ldots A_0(x_n) \exp[\![- \mathscr{H}_I[A_0]]\!] | \Phi_{E,0} \rangle \otimes, \end{aligned}$$

where $\otimes$ = omission of vacuum bubbles.

This is most easily seen by expanding the exponentials in power series and then using the correspondence of Gaussian and euclidean theories:

$$\langle \mathscr{H}_I[\Phi_0] \ldots \mathscr{H}_I[\Phi_0]\, X[\Phi_0] \rangle = \langle \Phi_{E,0} | \mathscr{H}_I[A_0] \ldots \mathscr{H}_I[A_0]\, X[A_0] | \, | \Phi_{E,0} \rangle. \qquad (3\text{-}30)$$

Therefore, the probabilistic versions and the euclidean field version are just different mathematical formulations of the same theory. The first formulation leads to a very refined and powerful mathematics, whereas the euclidean language is formally closer to relativistic quantum field theory. The formula (3-29), for example, is nothing but the euclidean version of the famous Gell-Mann and Low formula which, in most text books, is the starting point for the Feynman perturbation theory of the relativistic time-ordered functions. Note that the statistical Hamiltonian $\mathscr{H}$ corresponds to the Lagrangian $\mathscr{L}$ of relativistic QFT.

The cut off Λ is essential in order to be able to talk about euclidean operators, in interacting theories. Unlike in relativistic QFT, where through "wave function renormalization" and re-parametrization in terms of more "physical" masses and coupling constants (Secs. 5, 6), one obtains "operator-valued distributions" for $\Lambda \to \infty$, the euclidean theory allows one to talk about smeared out operators only if the two-point functions of these objects are not too singular. For example, the operator $:A_0^2:$, i.e. the Wick-ordered square of a free field, exists as an operator valued distribution in the Minkowski-version. The corresponding euclidean cut-off version has, however, only a limit $\Lambda \to \infty$ as a bilinear form, not as a smeared out field operator. Whereas the euclidean norm $\| :A_{0,\Lambda}^2(f): | \Phi_{E,0} \rangle \|$ ceases to exist in the limit $\Lambda \to \infty$, the correlation functions behave as

$$\langle \Phi_{E,0} | :A_{0,\Lambda}^2:(f_1) \ldots :A_{0,\Lambda}^2:(f_n) | \Phi_{E,0} \rangle \xrightarrow[\Lambda \to \infty]{} \text{finite limit},$$

if $f_1 \ldots f_n$ are non-overlapping test functions. Only, in $D = 2$, the field and all its non-derivative powers and, in $D = 3$, the field A and its square $:A^2:$, survive the $\Lambda \to \infty$ limit as operator-valued distributions. A satisfying framework, for bilinear forms and their products for non-overlapping arguments, does (in the opinion of the author) not exist at the present time. The lack of euclidean operator-concepts is a severe handicap in the formulation of certain properties of the scale invariant theory of Sec. 2, which because of its infinite cor'relation length looses all "memory" of any cut-off[18]. Here, even the restriction to $D = 2$ does not, in general, bypass this difficulty since the anomalous dimensions of A^n, $n = 1, 2, \ldots,$ may be quite large. There are, essentially, two ways out. One, advocated by Mack[19], is to view operator-properties as, for example, the Kadanoff-Wilson[20] operator expansion in the $\mathscr{H}^*$ theory, as a mere statement on the correlation function. The other possibility, which we will use in this review, is to affiliate with the statistical mechanics correlation of the $\mathscr{H}^*$ theory the corresponding scale invariant relativistic theory. It has been emphasized by Wilson and Kogut[21] that this is a useful construction even if one's prime interest is the understanding of critica1 behaviour.

Let us finally look at lattice theories. For a lattice system of classical spins σ_n, for which the values at each lattice point are distributed according to the function

$$\exp[\![-\mathscr{H}_I[\sigma]]\!],$$

the generating functional $Z\{J\}$ is

$$Z\{J\} = C\int\prod_n d\sigma_n \exp[\![-\mathscr{H}_I[\sigma] + J_n\,\sigma_n]\!]\exp[\sum_n K_{nm}\,\sigma_n\,\sigma_m]. \tag{3-31}$$

Here the n's are D-dimensional vectors with integer components (lattice-vectors).

Without loss of generality, we may assume that $K_{ii} = 0$ (the diagonal part may be absorbed in $\mathscr{H}_I$).

If the coupling matrix, K_{ij}, between different sites, has only non negative elements, the system is called ferromagnetic. Consider now the special case of a nearest ferromagnetic coupling. On functions f_n of the lattice, K acts in the following way (use translation invariance):

$$(Kf)_n = \sum K_{n-m}\, f_m\ , \tag{3-32}$$

with

$$K_n = \begin{cases} 0 & \text{for} \quad |\mathbf{n}| > 1, \quad n = 0, \\ K, & |\mathbf{n}| = 1. \end{cases}$$

Therefore,

$$K_n = (2\pi)^{-D}\int_{-\pi}^{+\pi} \tilde{K}(k)\, e^{ikn}\, d^D k, \tag{3-33}$$

with

$$\tilde{K}(k) = 2K\sum_{i=1}^{D} \cos k_i$$

The propagator in this theory has, according to (3-14), the form:

$$G_0(k) = -[2K \sum_i \cos k_i - h]^{-1}, \tag{3-34}$$

where h appears in the bilinear part of $\mathscr{H}_I$,

$$\mathscr{H}_I = h \sum \sigma_i^2 + \ldots .$$

Near the origin of k-space, $G_0(k)$ agrees, up to a factor K^{-1}, with (3-18). We will absorb this factor in the integration variable o. K, therefore, appears as a multiplication factor of the h's in $\mathscr{H}_I$ as well as of the source ($\sqrt{K}$ with every o). The new propagator has the form

$$G_0(k) = -[2 \sum_i \cos k_i - h]^{-1},$$

where because of K $\sim T^{-1}$, h' is a linear function of the temperature. For large wave-length (which are unaffected by the renormalization group procedure of Sec. 2), the lattice propagator behaves as an euclidean propagator

$$G_0(k) = [k^2 + m_B^2]^{-1}, \tag{3-35}$$

with $m_B^2 =$ linear function of T.

For reasons which become obvious in Sec. 5, it is important to know that discrete lattice spin theories, as the Lenz-Ising model, can be approximated by continuous "lattice fields". With

$$\sum_{n,n'} \sigma_n K_{nn'} \sigma_{n'} = \sigma K \sigma, \qquad \sum_n J_n \sigma_n = J\sigma, \tag{3-36}$$

we have

$$Z_{\triangle\mathrm{L.I.}}\{J\} = C \int \prod_n \delta(\sigma_n^2 - 1) d\sigma_n \exp\left(\frac{1}{2} \sigma K \sigma + J\sigma\right)$$

$$= \lim_{u_0 \to \infty} C_{u_0} \int \prod_n d\sigma_n \exp[-u_0(1 - \sigma_n^2)^2] \exp\left(\frac{1}{2} \sigma K \sigma + J\sigma\right) \tag{3-37}$$

Here, we used $\delta(x^2 - 1) = \lim_{u_0 \to \infty} (u_0/\pi) \exp[-u_0(x^2 - 1)^2]$ and C_{u_0} is the normalization factor of the fourth-degree polynomial theory:

$$\mathscr{H}[\sigma] = -\frac{1}{2} \sigma K \sigma + \sum_n u_0(1 - \sigma_n^2)^2$$

$$= \frac{K}{2} \int \tilde{\sigma}(k) [\sum_i 2 \cos k_i] \tilde{\sigma}(-k) + \text{polynomial}. \tag{3-38}$$

This "approximation" statement is, for our later discussions, more important than the statement that the Lenz-Ising model can be converted (by Laplace transformation of the spin distribution) into a non-polynomial model with a continuous range of the dynamical variable. Since this last possibility plays a role in various important papers on critical phenomena[22], we briefly discuss it in the following.

The use of

$$\exp\left(\frac{1}{2}\sigma K\sigma\right) = \pi^{-N/2} D^{-1/2} \int \prod_n d\phi_n \exp\left(-\frac{1}{2}\phi K^{-1}\phi + \sigma\phi\right), \quad (3\text{-}39)$$

where N is the number of lattices and D = det K (note that the factors in front are independent of the externa1 source and, therefore, may be absorbed in C) in Eq. (3-36) leads to a linear o-dependence so that the o-integration can be performed. The result is

$$Z_{\text{L.I.}}\{J\} = C \int \prod_n d\phi_n \exp\left(-\frac{1}{2}\phi K^{-1}\phi\right) \exp\left[\sum_n \ln \cosh(\phi_n + J_n)\right]. \quad (3\text{-}40)$$

Note that, in this description, the dependence on the source J is fairly complicated. The correlation functions, at different lattice points, are expectation values of tanh ϕ, i.e., for n $\neq$ $\mathbf{n}_i$, one has

$$\frac{\delta^n Z_{\text{L.I.}}}{\delta J_{n1} \dots \delta J_{nn}} = \langle \tanh \phi_{n1} \dots \tanh \phi_{nn} \rangle. \quad (3\text{-}41)$$

4. Wilson's Form of Renormalization Group Transformation

The qualitative idea of constructing renormalization group transformations, which "wipe out" the short-range fluctuation but retain the long range fluctuation unmodified, can be quantitatively formulated in different ways. All these different transformations should lead to the same number of fixed points. The totality of composite operators of various dimensions, around each fixed point, should be isomorphic for corresponding fixed points of different renormalization group transformations. With other words, different renormalization group transformations are expected to lead to different "coordinate" descriptions of the intrinsically identical "fixed point physics" for corresponding fixed points. A step towards a fixed point equivalence theorem, in this sense, has been recently made by Wegner[23]. We will return to this point later on.

A particular renormalization group transformation, which implements the idea of elimination of short range fluctuation, was given by Wilson[13]. It is most conveniently constructed by using the probabilistic language of euclidean fields, explained in the last section.

Let $\mathscr{H}$ be a Λ-cut-off Hainiltonian, written in terms of the fields $\Phi_i^{(L)}$, within the periodic box L^D. For notational convenience, we will suppress the index L as well as the internal index i. We define a transformed Hamiltonian $\mathscr{H}'$ by:

$$\exp[-\mathscr{H}' - L^D E_0] = \int d\Phi\left[\frac{\Lambda}{s} < k' < \Lambda\right] e^{-\mathscr{H}}\Big|_{\Phi(k)\to\alpha_s\Phi(s,k)}, \tag{4-1}$$

$$d\Phi\left[\frac{\Lambda}{s} < k' < \Lambda\right] = \prod_{\Lambda/s<k'<\Lambda} d\Phi(k'). \tag{4-2}$$

The integration variables are the (discrete) $\Phi(k')$ within the "shell" $\left[\frac{\Lambda}{s}, \Lambda\right]$. After the integration, the necessary "rescaling" of momenta and O are performed. The additive contribution $L^D E_0$, in the exponent, is just the value of the right hand side for O = 0. In other words, we want to define $\mathscr{H}'$ in such a way that it contains no constant (O-independent) term. $\mathscr{H}'$ is a Hamiltonian with cut-off A but it is written in terms of fields $\Phi(k)$ with a lesser number of k-values: since the shell has been wiped out (the subsequent rescaling does not change the total number of k-values), we have $S^{-D} L^D$ k-values. Writing $\mathscr{H}'$ in terms of interaction parameters h, one should keep in mind that the Φ is really a $\Phi^{(1/s)L}$. The subsequent "transition" to $\Phi^{(L)}$, before repeating the momentum-shell "wipe-out", is the trivial extension procedure of Sec. 2. The L is superfluous if we view the renormalization group transformation as a transition from the (infinite volume) probability measure $d\mu = \frac{1}{Z} e^{-\mathscr{H}} d[\Phi]$, in the space of the Φ's, to the "rescaled conditional measure":

$$d\mu_{Ts} = \int_{\substack{\Phi(k)\\ in\ shell}} d\mu[\Phi]\Big|_{\Phi(k)\to\alpha_s\Phi(sk)} \tag{4-3}$$

The rescaling allows to view the conditional measure again as a measure over the **original** probability space. Since our measures are always written in the exponential Hamiltonian form, we obtain a transformation in parameter-space

$$h' = T_s h, \tag{4-4}$$

which only enjoys semi-group properties if the rescaling factor a,, for $s \leq 0$, fulfills:

$$\alpha_{s'} \alpha_s = \alpha_{s'.s}, \quad \text{i.e.,} \quad \alpha_s = s^{1-1/2\eta} \tag{4-5}$$

Here, η is the quantity already introduced in Sec. 2.

Wilson's renormalization group transformation (4-4) is still too complicated for the explicit determination of fixed points. Some simplification is reached if one works with $\frac{d}{ds} T_s |_{s=0}$. Using this infinitesimal version, Wegner and Houghton[24] have shown how to find, by perturbation theory in $\varepsilon = 4 - D$, the non-guassian fixed point which, using a different method, was already studied before by Wilson and Fischer[25]. (The Wilson-Fischer method was restricted to first order perturbation theory in ε). For models, with N-component fields, one can also obtain this non-gaussian fixed point by $(1/N)$ expansions[26].

From a practical point of view, a simplified vèrsion of the renormalization group, the so called "approximate renormalization group transformation" of Wilson[13], has been most useful. Here, the word "approximate" does not necessarily mean that the operator-properties around the fixed points (i.e., their dimensions) are not correctly described. It rather means that, in addition to the "wiping out" procedure for large momenta fluctuations, some other more or less plausible simplifying assumptions are made. In the present state of affairs, it is not known how "far" we may deviate from the "orthodox" formulation (4-12), without wrecking the intrinsic physics of fixed points. In particular, it is not clear whether the "approximate renormalization group" is an exact or approximate description of the intrinsic fixed point physics. However, the perturbative ε-expansions, of the exact and the approximate R.G., are known to be different[27].

Let us consider the class of Landau Ginsberg Hamiltonians:

$$\mathscr{H} = \frac{1}{2} \int d^D x \, (\partial_\mu \Phi(x))^2 + \int d^D x \, P[\Phi(x)], \tag{4-6}$$

where P is a local polynomial, resp., an infinite power series in $\Phi(x)$ (without constant terms). We imagine that we rescaled our momentum space cut-off to be $\Lambda = 1$. We then decompose $\Phi(x)$ into a long range fluctuation Φ_0 and a short range (rapidly varying) part:

$$\Phi(x) = \Phi_0(x) + \Phi'(x), \tag{4-7}$$

with

$$\Phi_0(x) = (2\pi)^{-D/2} \int d^D k\, \tilde{\Phi}(k)\, e^{ikx}, \qquad |k_i| \leq b^{-1}. \tag{4-8}$$

Note that, in this consideration, we think in terms of square Brillouin cut-off's instead of a rotationally invariant cut-off. Imagine, now, $\Phi_0(x)$ as being affiliated with a square lattice of lattice length $b \cdot \pi$. The small wave length part, $\Phi'(x)$, has the density $(1/\pi)^D - (1/b\pi)^D$, so that we can formally relate the variable with a lattice of lattice distance $a = \pi(1-b^{-D})^{-1/D}$. Therefore, a description of Φ' in terms of real localized wave functions $\Psi(x)$, with an effective localization region of size a, as expressed by the formula

$$\Phi'(x) = \sum_n \Phi'_n \Psi(x - x_n),$$

with

$$\int \Psi(x - x_n)\Psi(x - x_{n'})\, d^D x = \delta_{nn'}\,. \tag{4-10}$$

does not seem to be totally unreasonable.

The gradient of such wave packets is expected to behave as

$$\int \partial_\mu \Psi(x - x_n)\, \partial_\mu \Psi(x - x_{n'}) = \bar{p}^2\, \delta_{nn'}\,,$$

where $\bar{p}^2$ is a mean value of momenta in the shell, i.e.,

$$b^{-2} \leq \bar{p}^2 \leq 1.$$

With these assumptions we are able to separate the Φ_0 and Φ' variables in the gradient part of $\mathscr{L}$. The additional assumption is that $\Phi_0(x)$ varies slowly within one a-cell and that Ψ may be approximated, in half of the cell, by the constant $a^{-D/2}$ and, in the other half, by $(-a^{-D/2})$. Then, $\mathscr{H}$ has the form,

$$\mathscr{H} = \frac{1}{2}\int d^D x [\partial_\mu \Phi_0(x)]^2 + \frac{1}{2}\bar{p}^2 \sum_n \Phi_n'^2$$
$$+ \frac{1}{2} a^D \sum_n \{P[\Phi_0(x_n) + a^{-D/2}\Phi'_n] + P[\ldots - \ldots]\}. \tag{4-11}$$

We are now able to perform the integration over the Φ'_m. With

$$P_T(\Phi) = -a^D \bar{p}^2 \ln \int dy \exp\left\{-y^2 - \frac{a^D}{2} P\left[b^{-(D/2)+1}\Phi + \frac{\bar{p}^2 a^D b^{-D+2}}{2} y\right] - \frac{a^D}{2} P[\ldots - \ldots]\right\}, \tag{4-12}$$

we obtain for the new Hamiltonian, in which the Φ' fluctuations have been integrated out according to (4-1),

$$\mathcal{H}_T = \frac{1}{2}\int (\partial_\mu \Phi)^2 \, d^D x + \int P_T(\Phi(x))\, d^D x. \tag{4-13}$$

It is again of the original form, only the local interaction, i.e., the coupling constants in form of the n^{th} degree local polynomial,

$$P(\Phi) = \sum_m h_m \frac{1}{m!} \Phi^m, \tag{4-14}$$

has changed.

Repeating this process many times, we arrive at a Hamiltonian $\mathcal{H}_{T^n}$ which is again of the form (4-6), i.e., we obtain a recursion relation for the $T^n h_m$. This relation may be studied for various classes of input Hamiltonians on a computer. For example[13], in the class $P(\Phi) = r\Phi^2 + u\Phi^4$ (as input), one finds for a particular value of r, a convergent sequence $P_{T^n}(\Phi)$.

For analytic computation, one has to make further simplifying assumptions. Neglecting all terms which contain higher powers than quadratic terms in r and u, one easily obtains:

$$\begin{aligned} r_{n+1} &= b^2(r_n + 3u_n - 3u_n r_n - q u_n^2), \\ u_{n+1} &= b^{4-D}(u_n - q u_n^2). \end{aligned} \tag{4-15}$$

A trivial solution is $u^* = 0$, $r^* = 0$. This solution is unstable within the above class, i.e., the $\mathcal{H}^*$ cannot be viewed as a point in a critical surface (line!) within the two parametric (r, u)-space. One can show that such a critical surface exists for $D = 3$, if one enlarges the class to consist of three parameters r, u and v, by adding the interaction $v\Phi^6$.

This Hamiltonian has been used by Riedel and Wegner for[2*] a description of the tricritical $He^3 - He^4$ mixture. We will come back to this problem in our field theoretical part. For $D < 4$, there exists another non-trivial solution of the quadratic approximation to the recursion relation (4-15):

$$u^* = \frac{b^{4-D}-1}{9} \tag{4-16}$$

The neglect of terms higher than quadratic is only justified for small $\varepsilon = 4-D$. For infinitesimal ε, we obtain

$$\begin{aligned} u^* &= \frac{1}{9}\,\varepsilon \ln b + \mathcal{O}(\varepsilon^2), \\ r^* &= -\frac{\varepsilon \ln b}{3(b^2-1)} + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{4-17}$$

This perturbative method does not tell us anyth ng directly for D = 3 and 2. For continuity reasons, it is plausible that the infinitesimally established fixed point should not get lost. There is up to date, however, no rigorous analytical derivation of non-canonical fixed points. The perturbative method in lowest order can also be used for the determination of eigenoperatoi-s O, and their dimensions.

5. Legendre Transforms and Re-Parametrizatioa

The Renormalization Group Transformation, of last Section, starts from a physical Hamiltonian which is local apart from a cut-off A. The intermediate Hamiltonians are generally non-local Hamiltonians but the fixed point $\mathcal{H}^*$ is (as a scale-invariant Hamiltonian) again a local one. One may, therefore, expect that methods of relativistic local quantum field theory allow one to link the given local Hamiltonian in a more direct way with the scale invariant fixed point Hamiltonian, *without leaving* the set of *local Hamiltonians*. The interpolation is done using Gell-Mann Low[29] type of parametric differential equations, which also have been called Renormalization Group equations. In order to avoid confusion, we will just simply talk about parametric differential-equations and, for reasons explained later, we will call the field theoretical "renormalization procedure" a "re-parametrization". In order to develop these techniques, we need some more definitions and formalism.

The generating functional for connected Green's functions is defined by:

$$Z\{J\} = \exp G\{J\}. \tag{5-1}$$

Generalizing the wellknown technique of Legendre transforms for relating different thermodynamic potentials to functionals[30], one defines a new "source" by:

$$\mathscr{A}(x) = \frac{\delta G}{\delta J(x)} \cdot \tag{5-2}$$

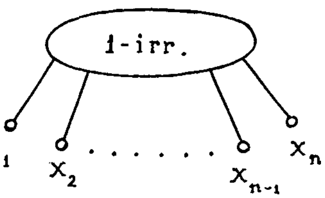
The $\mathscr{A}(x)$ is the field induced by the external source. In the language of ferromagnetic systems, $J(x)$ is the external magnetic field and, $\mathscr{A}(x)$, the induced magnetization. The vertex functional (in statistical language, the "Helmholtz potential") is defined by the Legendre transformation:

$$\Gamma\{\mathscr{A}\} = \int \mathscr{A}(x)\, J(x)\, \mathrm{d}^{\mathrm{D}}\mathrm{x} - G\{J\}. \tag{5-3}$$

Graphically this functional is known to generate 1-line irreducible (called 1-particle irreducible in relativistic QFT) Feynman graphs. Representing the verte-functions:

$$\left.\frac{\delta^n \Gamma}{\delta\mathscr{A}(x_1)\ldots\delta\mathscr{A}(x_n)}\right|_{\mathscr{A}=0} = \Gamma^{(n)}(x_1 \ldots \mathrm{x},) \tag{5-4}$$

by



it follows, from (5-3), that the graphs representing $G^{(n)}(x_1 \ldots \mathrm{x},)$ are "trees" in terms of the 1-irr. graphs.

For the two-point function, one obtains, by insertion of $\frac{\delta G}{\delta J}$ into the left hand side and functional differentiation with respect to **J**:

$$\iint G^{(2)}(x-x')\,\Gamma^{(2)}(x',y')\,G^{(2)}(y'-y)\,d^D x'\,d^D y' = G^{(2)}(x-y),$$

i.e.,

$$\Gamma^{(2)} = [G^{(2)}]^{-1}. \tag{5-5}$$

Note that with the minus sign in the Legendre transform, the lowest order contribution to $\Gamma^{(4)}$ will be identical to the positive coupling constant g.

The introduction of the Γ's is helpful in problems of re-parametrization. In a theory without symmetry-breaking, as the $\mathbf{A^4}$ theory, the parameters u_0 and m_0 of the bare Hamiltonian are inconvenient for the study of critical behaviour. Since the criticality can only be reached if the inverse correlation length, i.e., the physical mass, goes to zero, we will introduce a renormalized mass m. The introduction of a renormalized coupling constant also simplifies the discussion of criticality, and it is of particular importance if we want to construct the scale invariant $\mathscr{H}^*$ theory as a limit of perturbatively constructed theories.

There are many possibilities for introducing convenient parameters. Let us mention two of them:

"mass-shell" parameters:

$$\Gamma^{(2)}\big|_{p^2=-m^2} = 0, \tag{5-6a}$$

$$\Gamma^{(4)}\big|_{s.p.-m^2} = u, \tag{5-6b}$$

with s.p. μ^2 given by $p_i p_j = \frac{1}{3}(4\delta_{ij} - 1)p^2$;

"intermediate" parameters:

$$\Gamma^{(2)}\big|_{p=0} = m^2, \tag{5-7a}$$

$$\Gamma^{(4)}\big|_{p=0} = u. \tag{5-7b}$$

Besides re-parametrizing the theory, one also finds it convenient to change the normalization of the field:

$$A_r(x) = Z^{-1/2} A(x). \tag{5-8}$$

The Z-factor is fixed via normalization properties of the A_r two-point function:

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}\bigg|_{p^2=-m^2} = 1, \text{ for scheme (5-6),} \tag{5-6c}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}\bigg|_{p=0} = 1, \text{ for scheme (5-7).} \tag{5-7c}$$

Because of the multiplicative change (5-8), the transition from (m_0, u_0, A) to (m, u, A_r) is usually called "re-normalization". However, the mere change of normalization properties of the field should not be confused with the (Wilson-) "Renormalization Group Transformation" which is a mapping T obtained by eliminating long range fluctuations. For this reason, we will deviate from the usual terminology of Q.F.T. and call (5-6, 5-7), including (5-6c, 5-7c), a "*re-parametrization*".

In order to have smooth transition for $m \to 0$, of the massive correlation function into mass-less functions, it is necessary to introduce a normalization spot μ which is to be distinguished from the mass m which enters the free Hamiltonian $\mathscr{H}_0$ (i.e., Feynman rules). Historically, Gell-Mann and Low[29] first introduced such a μ via a normalization spot in momentum space. However, for application of QFT to critical behaviour, it is much more convenient to introduce μ as a "mass-normalization spot" and keep the momenta, as in (5-6), at $p = 0$. This new normalization scheme leads to scaling equations in which the "scaling mass", m, plays the role of the temperature. We will discuss this renormalization scheme in a separate section (Sec. 6).

Re-parametrizations are conveniently done in the Bogoljubov-Parasiuk counter-term formalism[31]. The Hamiltonian $\mathscr{H}$, in the original parametrization,

$$\mathscr{H} = \frac{1}{2}\int\left(\partial_\mu A \partial_\mu A + \frac{m_0^2}{2} A^2 + \frac{u_0}{4!} A^4\right) d^D x, \tag{5-10}$$

is written as

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_{\text{int}}, \tag{5-10}$$

with

$$\mathscr{H}_0 = \int\left(\frac{1}{2}\,\partial_\mu A_r\,\partial_\mu A_r + \frac{m^2}{2} A_r^2\right) d^D x \tag{5-11}$$

and

$$\mathscr{H}_{\text{int}} = \int\left(\frac{u}{4!} A_r^4 + \frac{a}{2} A_r^2 + \frac{b}{2}\,\partial_\mu A_r\,\partial_\mu A_r + \frac{c}{4!} A_r^4\right) d^D x. \tag{5-12}$$

The a, b and c counterterms are determined by the normalizations (5-6, 5-7).

One first evaluates the Gell-Mann and Low formula for the correlation functions:

$$\langle X^{(N)}\rangle = \frac{1}{C}\left\langle \Phi_0 \left| X_0^{(N)} \exp\left[-\int \mathscr{H}_{\text{int}}(A_0)\, d^D x\right]\right| \Phi_0\right\rangle, \tag{5-13}$$

with

$$C = \left\langle \Phi_0 \left| \exp\left[-\int \mathscr{H}_{\text{int}}(A_0)\, d^D x\right]\right| \Phi_0\right\rangle, \tag{5-14}$$

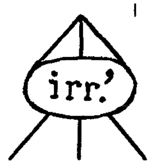
and Φ_0 = free (euclidean) vacuum, A_0 = free (euclidean) field with mass m,

$$X_0^{(N)} = \prod_{i=1}^{N} A_0(x_i), \qquad X^{(N)} = \prod_{i=1}^{N} A(x_i).$$

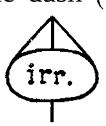
The Legendre-transform (5-3) leads to the I- functions which are represented by one-line irreducible Feynman graphs. The Feynman graphs are constructed from the massive propagator $\frac{1}{p^2 + m^2}$ and the u, a, b, c-interaction vertices. The normalization conditions (5-6), in terms of graphs, read (we restrict our discussion to the intermediate re-parametrization):

$$\Gamma^{(2)}\Big|_{p=0} = \Big\{p^2 + m^2 + a + bp^2 +$$  $$\Big\}\Big|_{p=0} = m^2, \qquad \text{(5-15a)}$$

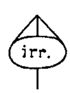
$$\frac{}{\partial p^2}\Gamma^{(2)}\Big|_{p=0} = \{\mathrm{i} + b \quad a \quad \Big\}\Big|_{p=0} = 1, \qquad \text{(5-15}b\text{)}$$

$$\Gamma^{(4)}\Big|_{p=0} = \Big\{u + c +$$  $$\Big\}\Big|_{p=0} = 0, \qquad \text{(5-15c)}$$

the dash (′) indicating the omission of the no-loop contribution.

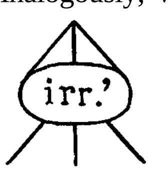


indicates the set of *all* graphs which start with one u-interaction vertex and end on one line. Written in terms of operators:



$$= \frac{u}{3!}\langle A_r^3(0)\,\tilde{A}_r(p)\rangle^{\text{prop}}. \qquad \text{(5-16)}$$

Analogously, we have:



$$= \frac{u}{3!}\Big\{\langle A_r^3(0)\,\tilde{A}_r(p_2)\,\tilde{A}_r(p_3))\tilde{A}_r(p_4)\rangle^{\text{prop}} - \textit{zero order}\Big\}. \qquad \text{(5-17)}$$

$p_2 \quad p_3 \quad p_4$

We obtain implicit (since higher order graphs contain in turn these counterterms) formulas for a, b and c in terms of higher order graphs. By an explicit low order computation, the reader may convince himself that the determinant condition for the perturbative solubility in terms of $a = \sum_n a_n u^n$, etc., is fulfilled.

The intuitively plausible formulas may be put on a more solid mathematical basis by introducing composite fields via a Gell-Mann and Low formula:

$$\langle A_r^n(x)\, X^{(N)}\rangle = \frac{1}{C}\left\langle \Phi_0 \middle| A_0^n(x)\, X_0^{(N)} \exp\left[-\int \mathscr{H}_{\text{int}}(A_0)\, d^D x\right] \middle| \Phi_0 \right\rangle. \quad (5\text{-}18)$$

This may be easily generalized to arbitrary composite fields. The "proper" functions $\langle\ \ \rangle^{\text{prop}}$ are 1-line irreducible with respect to X^N lines. They are defined via Legendre transformations of the corresponding generating functionals (for brevity we omit from now on the index r on the fields):

$$G_{A^n}(x, J) = \sum \frac{1}{m!} \int \langle A^n(x)\, A(x_1) \ldots A(x_n)\rangle_c\, J(x_1) \ldots J(x_m)\, dx_1 \ldots dx_m\,, \quad (5\text{-}20)$$

$$\Gamma_{A^n}(x, \mathscr{A}) = \sum \frac{1}{m!} \int \langle A^n(x)\, A(x_1) \ldots A(x_m)\rangle^{\text{prop}}\, \mathscr{A}(x_1) \ldots \mathscr{A}(x_m)\, dx_1 \ldots dx_m\,.$$

They are related according to

$$G_{A^n}(x, J) = \Gamma_{A^n}(x, \mathscr{A}), \quad (5\text{-}21)$$

where the connection between $\mathscr{A}$ and J is given by (5-2).

The relations (5-15) can be formally derived by a Legendre transformation of the field equations for the correlation functions (3-20).

All manipulations are to be done for finite euclidean cut-off A (resp. finite lattice distance). Our re-parametrization of correlation functions, involving the basic field only, already assures the finiteness for $A \to \infty$. This is particularly easy to see for the A^4 coupling in D = 3 and 2. If we write the interaction part $\mathscr{H}_{\text{int}}(A_0)$, (5-12), as a Wick-product of free fields (we saw in Sec. 3 that this amounts to a trivial re-parametrization), then, for D = 2, there are no divergent Feynman graphs. For D = 3, one finds terms which are logarithmically divergent for $A \to \infty$. These terms appear in the bare mass m_0. However, these terms are dropping out if one applies the usual Dyson momentum-space Taylor operator. The important point is that, in this situation, we have *no overlapping divergencies*. The proof of finiteness, for $A \to \infty$, of 4-dimensional correlation functions, normalized according to (5-6) or (5-7), is more complicated: the reason is the occurrence of overlapping divergencies. Here, the Bogoliubov-Parasiuk-Hepp[31] method leads to the desired result. The intermediate re-parametrization (5-7) is particularly simple to deal with in the BPH- renormalization. In order to construct a composite field A^2, which stays finite for $\Lambda \to \infty$, one makes the following Ansatz:

$$N[A^2](x) = Z(A^2) : A^2(x) :$$

$Z(A^2)$ is to be determined from the intermediate normalization condition:

$$\langle \tfrac{1}{2} N[A^2](0)\, \tilde{A}(p_1)\, \tilde{A}(p_2)\rangle^{\text{prop}}|_{p=0} = 1. \tag{5-21}$$

For the definition of a $(\Lambda \to \infty)$ – finite A^4 composite field we write

$$N[A^4](x) = Z(A^4) : A^4(x) : + Z(A^4, (\partial A)^2) : (\partial A(x))^2 : \\ + Z(A^4, A\partial^2 A) : A(x)\, \partial^2 A(x) : + Z(A^4, A^2) : A^2(x) : . \tag{5.22}$$

The normalizations are:

$$\frac{1}{4!} \langle N[A^4](0)\, \tilde{A}(p_1) \ldots \tilde{A}(p_4)\rangle^{\text{prop}}|_{p_i=0} = 1, \tag{5-23a}$$

$$\langle N[A^4](0)\, \partial\tilde{A}(p_1)\, \partial\tilde{A}(p_2)\rangle^{\text{prop}}|_{p_i=0} = 0, \tag{5-23b}$$

$$\langle N[A^4](0)\, \tilde{A}(p_1)\, \partial^2\tilde{A}(p_2)\rangle^{\text{prop}}|_{p_i=0} = 0, \tag{5-23c}$$

$$\langle N[A^4](0)\, \tilde{A}(p_1)\, \tilde{A}(p_2)\rangle^{\text{prop}} = 0. \tag{5-23d}$$

The symbol N stands for "normal product" which is a generalization of the Wick-product. The proof of finiteness of

$$\langle N[A^2](x)\, X^{(N)}\rangle \quad \text{and} \quad \langle N[A^4](x)\, X^{(N)}\rangle$$

can be given in the BPH-framework. The generalization to the construction of normal products $N[\mathcal{O}](x)$ of more complicated field monomials $\mathcal{O}$ is straightforward: write $N[\mathcal{O}](x)$ as a superposition of all (Wick-products) monomials in A and derivatives of A which have the same transformation properties (including discrete symmetries) as $\mathcal{O}$ and whose canonical dimension is not bigger than that of $\mathcal{O}$. This general Ansatz is only necessary for renormalizable couplings, e.g., A^4-interaction in $D = 4$.

Zimmermann[32] showed that one can avoid counter-terms and Z-factors which diverge in the limit $\Lambda \to \infty$, by giving directly a prescription for the re-parametrized correlation functions respectively the composite fields $N[\mathcal{O}]$. The Zimmermann formula is:

$$\langle X^{(N)}\rangle = \text{F.P.}\langle\Phi_0|X_0^{(N)}\exp[-\hat{\mathscr{H}}_{\text{int}}(A_0)]|\Phi_0\rangle\,\text{o}. \qquad (5\text{-}24)$$

The $\otimes$ indicates the omission of "vacuum bubbles" (i.e., division by C in (5-13)).The finite part operation, F.P., is defined in terms of Taylor-operators acting on the Feynman integrand and converting it into an absolute convergent expression. The counter-terms in $\hat{\mathscr{H}}_{\text{int}}$ are now finite in the limit A $\to \infty$. In particular, for the intermediate re-parametrization, there are no counter-terms. The Zimmermann F.P. operation may be generalized to the definition of normal products. We will not discuss this formulation any further, since our main interest here will be the field theoretical description of superrenormalizable couplings (e.g., A^4 in 2 and 3 dimensions) in which overlapping divergencies are absent. For a proof of the existence of the A $\to \infty$ limit, in the case of a three dimensional A^6 coupling, this formalism would be useful.

The change of parameters is particularly important for the discussion of symmetry-breakings in the "spontaneously broken limit". As an illustration, consider a two-component scalar model with $\mathcal{O}_2$ symmetry which is broken by a linear term:

$$\mathscr{H}(x) = \frac{1}{2}(\partial_\mu \mathbf{A}^2(x))^2 + \frac{m_0^2}{2}\mathbf{A}^2(x) + \frac{u_0}{4!}(\mathbf{A}^2(x))^2 + hA_2(x). \quad (5\text{-}25)$$

The re-parametrization $(m_0, u_0, \text{h}) \to (m, u, \text{h})$, (where the h $=$ 0 part of $\mathscr{H}$ is treated as before), is not such a good procedure. The dependence in h, around h $=$ 0, is expected to be non-analytic since the Goldstone behaviour contradicts the existence of a power series in h. The necessary re-parametrization is the transition from the field h to the magnetization $\langle A(x)\rangle$ it induces. The technique is the "loopwise resummation"[33], which we briefly sketch in the following. Denoting the generating functional of the connected correlation functions, for the symmetric h $=$ 0 theory, by $G_s\{J\}$, the generating functional of (5-25) is obtained by shifting the source. With h $=$ (0, h), we have:

$$G\{\mathbf{J}\} = G_s\{\mathbf{J} + \mathbf{h}\} - G_s(\mathbf{h}). \qquad (5\text{-}26)$$

The Legendre transformation

$$\Gamma\{\mathscr{A}\} = \int \mathscr{A}(x)\,\mathbf{J}(x)\,d^Dx - G\{\mathbf{J}\}, \qquad \mathscr{A} = \frac{\delta G}{\delta \mathbf{J}},$$

leads to

$$\Gamma\{\mathscr{A}\} = \Gamma_s\{\mathscr{A} + \mathbf{F}\} - \Gamma_s\{\mathbf{F}\}, \qquad (5\text{-}27)$$

with

$$\mathbf{F} = (0, F), \quad F = \left.\frac{\delta G_s\{\mathbf{J}\}}{\delta J_2(x)}\right|_{J=h}. \tag{5-28}$$

The "loopwise summation" of the Γ_s-functions leads to a new perturbation theory in which the new "zero order" Feynman propagators of A_1 and A_2 have different masses m_1 and m_2. As the new parameters, one may use m_1, m_2 and $u \sim \frac{m_2^2 - m_1^2}{F^2}$. Recently, one has been able to reexpress this rather involved technique in terms of a re-parametrization formalism which may be directly formulated in terms of $\mathscr{H}$ (Ref. 34). The main idea is to make the translation

$$\mathbf{A} = \hat{\mathbf{A}} + \mathbf{F}, \quad \langle \mathbf{A} \rangle = \mathbf{F}, \tag{5-29}$$

directly in $\mathscr{H}$:

$$\mathscr{H} = \frac{1}{2}(\partial_\mu \hat{\mathbf{A}})^2 + \frac{m_1^2}{2}\hat{A}_1^2 + \frac{m_2^2}{2}\hat{A}_2^2 + \frac{u}{4!}(\hat{\mathbf{A}}^2)^2$$

$$+ \frac{u \cdot F}{3!}\hat{\mathbf{A}}^2 \cdot \hat{A}_2 + (h + F m_1^2)\hat{A}_2 + \textit{counter terms}. \tag{5-30}$$

Here,

$$m_1^2 = m^2 + \frac{uF^2}{3!} \quad \text{and} \quad m_2^2 = m_1^2 + \frac{uF^2}{3!}, \tag{5-31}$$

are taken as new parameters. Since

$$3u \cdot F^2 = m_2^2 - m_1^2, \tag{5-32}$$

one can either take $u(F \sim u^{-1/2})$ or $F(u \sim F^{-2})$ as the third parameter. "Perturbation theory" will be a power series decomposition in powers of u, resp., $\frac{1}{F}$. The coefficient of the linear terin is determined via a Gell-Mann and Low formula, by the requirement

$$\langle \hat{\mathbf{A}} \rangle = 0. \tag{5-33}$$

The most important point is now the following. If we choose the counter-term structure in (5-30) to be what we obtain by the translation applied to the a, b and c counter-terms of (5-12) and if, in addition, we *do not Wick-order* $\mathscr{H}_{\text{int}}$, then all of the classical relations (i.e., those obtained by formal manipulations) are fulfilled in the quantized theory[35]. In particular the Ward-Takahashi identities for the correlation functions are valid.

The most convenient normalization conditions, which are consistent with the "translated counter-term structure, are:

$$\Gamma^{(2)}\Big|_{p=0,\,m_2^2-m_1^2=0} = m_1^2, \tag{5-34a}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}\Big|_{p=0,\,m_2^2-m_1^2=0} = 1, \tag{5-34b}$$

$$\Gamma^{(4)}\Big|_{p=0,\,m_2^2-m_1^2=0} = u. \tag{5-34c}$$

In terms of Taylor-operations, the u-parametrization leads to the "soft quantization scheme" of Gomes, Lowenstein and Zimmermann[34]. The main point is that the Taylor-operations do not only act on externa1 momenta of subgraphs, but also on certain parameters. A detailed discussion of these interesting and recent developments in quantum field theory would go beyond the main purpose of the review. This method applied to 4-dimensional theories allows to discuss spontaneous symmetry breaking in perturbation theory. For $D \leq 4$, the spontaneous limit $m_1 \to 0\,(h \to 0)$ generates infrared-divergencies which increase in increasing perturbation order. Therefore, the difficulties in dealing with 1st order phase transitions, corresponding to continously broken symmetries, are similar to the infrared problems one encounters in studying critical behaviour. For the latter case, we develop a non-perturbative discussion based on certain (non-perturbative) properties of parametric differential equations. Similar ideas may be applied to phase transitions of first order and we will return to this problem in a future publication.

The main reason why re-parametrizations are so useful in the discussion of phase transitions is that they allow a very simple description of what happens under scale changes $x \to \lambda x$. To compensate such a scale change, by a change of physical parameters, leads, in general, to complicated and therefore useless transformation proprerties. Therefore, one should always attempt to change the (accidental) parameters, in terms of which $\mathscr{H}$ is given, to "intrinsic" parameters in terms of which physical properties (in our case, properties near the critica1 point) are simple. This remark sets the stage for the discussion in the next section.

6. Derivation of Scaling Equations

According to a hypothesis of Wegner[15], the discussion of infinitesimal variations, around a fixed point $\mathscr{H}^*$, can be strengthened by assuming the existence of global scaling fields. Wegner's arguments, which we presented in Sec. 2, are, basically, consistency ai-guments. So, it is desirable to see how close onè can come to derive this hypothesis in the more restricted but, at the same time more manageable, framework of (euclidean) local quantum field theory. For the correlation functions

$$(X) = G^{(N)}(x_1 \ldots x_N \,|\, g_1, g_2 \ldots),$$

with $(X) = \prod_1^N \phi(x_i)$, the scaling field hypothesis leads to the global Kadanoff scaling law:

$$G^{(N)}(x_1 \ldots x_N | g_1, g_2 \ldots) = \lambda^{N d_\phi} G^{(N)}(\lambda x_1 \ldots \lambda x_N | \lambda^{y_1} g_1, \lambda^{y_2} g_2 \ldots), \qquad (6\text{-}1)$$

which is equivalent to the following parametric differential equation:

$$\left\{ \sum_1^N x_{i\mu} \frac{\partial}{\partial x_{i\mu}} + \sum_1^N y_i g_i \frac{\partial}{\partial g_i} \right\} G^{(N)} = -N d_\phi G^{(N)}. \qquad (6\text{-}2)$$

For reasons of simplicity, we have assumed that we are in a situation where there are no logarithmic modifications of these relations. In the following, we will show that, by a suitable parametrization of the (two-parametric) A^4 theory, via normalizatiori conditions, we can obtain the scaling equations[36]

$$\left\{ \sum x_{i\mu} \frac{\partial}{\partial x_{i\mu}} + 2(\delta - 1) m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} \right\} G^{(N)} = -(d_N + N_{\gamma\phi}) \cdot G^{(N)}, \quad (6\text{-}3)$$

where β and y depend only on g, whereas 6 has the representation[36]

$$\delta = \delta_1(g) + \frac{\mu^2}{m^2} \delta_2(g). \qquad (6\text{-}4)$$

Here, μ is a parameter which enters only via certain normalization conditions, i.e., it is not an intrinsic physical parameter of the theory. The meaning of the symbols is the following: $dN =$ can. dim. of $G^{(N)}$ in mass units, $\gamma_\phi =$ "would be" anomalous dimension of ϕ, $2\delta_1 = \gamma_\phi{}^2 =$ = "would be" anomalous dimension of ϕ^2.

Furthermore, δ_2 is not independent since we have the relation

$$\delta_1 + \delta_2 = \gamma_\phi\,. \qquad (6\text{-}5)$$

The terminology "would be" referes to the statement that $d_\phi = \frac{D-2}{2}$ $+ \gamma_\phi$ and $d_{\phi^2} = D - 2 + \gamma_{\phi^2}$ are the operator dimensions in a theory for which the coupling constant g is equal to a zero ("eigenvalue") of the functions $\beta(g)$.

The global scaling law, following from (6-3) by the method of characteristics[38], is

$$G^{(N)}(x_1 \ldots x_N \,|\, m_1 g) = \lambda^{-dN} a^{-N}(g, \lambda)\, G^{(N)}(\lambda^{-1} x_1 \ldots \lambda^{-1} x_N \,|\, \mathrm{m}, \bar{g}), \quad (6\text{-}6)$$

with $\bar{g}$ defined by

$$\ln \lambda = \int_g^{\bar{g}} \frac{1}{\beta(g')}\, dg' \quad (6\text{-}7a)$$

and

$$a(g, \lambda) = \exp\int_g^{\bar{g}} \frac{\gamma}{\beta}\, dg', \quad (6\text{-}7b)$$

$$\frac{d\bar{m}}{d\lambda} = 2[\delta_1(\bar{g}) - 1]\, \bar{m}^2 + 2\mu^2\, \delta_2(\bar{g}),$$

i.e.,

$$\bar{m}^2 = \mu^2 \int_g^{\bar{g}} dg' \frac{\delta_2}{\beta} \cdot \exp \int_g^{\bar{g}} \frac{\delta_1 - 1}{\beta}\, dg'' + m^2 \exp \int_g^{\bar{g}} \frac{\delta_1 - 1}{\beta}\, \mathrm{dg}'. \quad (6\text{-}7c)$$

The reader who is not familiar with the method of characteristics may easily verify that

$$3 \frac{\mathrm{d}}{d\lambda} (\text{right hand side of (6-6)}) = 0$$

is just the differential equation (6-3) with g and m replaced by $\bar{g}$ and m.

In the case of existence of a zero g, with

$$\beta(g_c) = 0 \qquad \text{and} \qquad \omega \equiv \beta'(g_c) > 0,$$

Eq. (6-6) yields the (zero magnetic field) Kadanoff scaling law including the "w-corrections":

$$G^{(N)}(x_1 \ldots x_N \,|\, m_1 \mathrm{g}) = 3^{-\mathrm{N}\left[d_\varphi + \frac{\gamma_c'(g - g_c)}{\omega} \frac{1}{-\ln\lambda} + \quad\right]},$$

$$G^{(N)}(\lambda^{-1} x_1 \ldots \lambda^{-1} x_N \,|\, m\lambda^{-1 + \delta_c + \frac{\delta_c'(g - g_c)}{\omega} \frac{1}{\ln\lambda} +} \quad , g_c + (g - g_c)\lambda + \ldots). \quad (6\text{-}8)$$

$i >> 1$ and the form of the asymptotic effective-mass results from the second part of (6-7c) which, for 6, $= \delta(g_c) < 1$, is dominating in the limit $\lambda \to 0$.

Interpreting the "scaling mass" m as a quantity proportional to $\tau = \frac{T - T_c}{T_c}$, Eq. (6-8) is the Kadanoff scaling law with the critical indices (in Kadanoff's terminology)

$$x_\phi = d_\phi = \frac{D-2}{2} + \gamma_\phi\,, \quad x_t = 2(1-\delta_1) = \frac{1}{\nu}\,, \quad \eta = 2\gamma_\phi\,, \quad \gamma = (2-\eta)\,\nu. \tag{6-9}$$

The interpretation of $m \sim \tau$ will be a necessary consequence of our derivation.

For the derivation of the parametric differential equation, we start from the Hamiltonian

$$\begin{aligned} \mathscr{H} &= \mathscr{H}_0 + \mathscr{H}_I\,, \\ \mathscr{H}_0 &= \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2, \\ \mathscr{H}_I &= \frac{1}{4!}u\phi^4 + \frac{b}{2}(\partial_\mu\phi)^2 + \frac{1}{2}a_1 m^2\phi^2 + \frac{1}{2}a_2\phi^2 + \frac{\hat{c}}{4!}\phi^4. \end{aligned} \tag{6-10}$$

Instead of u and $\hat{c}$, we prefer to work with dimensionless parameters:

$$u = \mu^{4-D} g, \qquad \hat{c} = {}^{4-D} c,$$

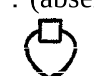
where μ is the already mentioned renormalization spot which enters the theory via renormalization conditions:

$$\Gamma^{(2)}\big|_{p=0,\, m=\mu} = \mu^2, \tag{6-11a}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}\big|_{p=0,\, m=\mu} = 1, \tag{6-11b}$$

$$\frac{\partial}{\partial m^2}\Gamma^{(2)}\big|_{p=0,\, m=\mu} = 1, \tag{6-11c}$$

$$\Gamma^{(4)}\big|_{p=0,\, m=\mu} = g\mu^{4-D} = u. \tag{6-11d}$$

Choosing the couinter-terms, a_1, a_2, b and c, iridependent of m, the four equations (6-11a)-(6-11d) will lead to their recursive perturbative determination. With the understanding that the interaction in (6-10) is either Wick-ordered : : (absence of  loops) or "triple dot ordered" ⋮ ⋮ (absence of  , as well), we obtain for D = 3 the wellknown logarithmic dependence of a_2 on A, i.e., $a_2 \sim \mu^2 \ln^n \frac{A}{\mu}$, whereas a_1, b and c approach already a finite limit. For D = 2, all counter-terms will stay finite for $A \to \infty$. In case of D = 4, we have a quadratic divergence in a_2 whereas a_1, b and c remain logarithmic divergent. In this particular case, it would have been simpler to have, instead of (6-11a), the normalization[35]

$$\Gamma^{(2)}\big|_{p=0,\, m=\mu} = 0. \tag{6-12}$$

For **D** < 4, this normalization is known to introduce infrared-divergences into the correlation functions[39]. Comparing the above Hamiltonian (6-10) with its unrenormalized form:

$$\mathscr{H} = \frac{1}{2}(\partial_\mu \phi_0)^2 + \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{2} \delta m_0^2 \phi_0^2 + \frac{1}{4!} u_0 \phi_0^4, \quad \text{(6-13a)}$$

we obtain, with $\phi = Z^{-1/2} \phi_0$ and the requirement that $m \to 0$ implies $m_0 \to 0$ (with δm_0^2 remaining $\neq 0$), the identifications:

$$Z = 1 + b, \qquad m_0^2 = m^2 \frac{1+a}{Z}, \qquad \delta m_0^2 = \frac{-a_2}{Z}, \qquad g_0 = \frac{g+c}{Z^2}. \quad \text{(6-13b)}$$

Note that $m_B^2 = m_0^2 - \delta m_0^2$ is the true "bare mass". The statistical mechanics origin of the Hamiltonian (Sec. 3) tells us that m~has a T-dependence. On the other hand, m_0^2 is just the difference between the bare mass and its "critical value":

$$m_0^2 = m_B^2 - m_{Bc}^2, \qquad \delta m_0^2 = m_{Bc}^2. \quad \text{(6-14)}$$

Therefore, m_0^2 and hence m^2 are proportional to $\tau = \frac{T - T_c}{T_c}$. From the form of the Lagrangian (6-10), we obtain immediately:

$$\frac{d}{dm^2}(X) = -\frac{1}{2}(1 + a_1) \int \langle \phi^2(x)\, X \rangle \, d^D x. \quad \text{(6-15)}$$

Writing $\phi^2(x) = Z_1 (\phi^2)_R(x)$, with an m-independent renormalization factor Z_1, which is determined by the normalization condition

$$\Gamma^{(2,1)}(p_1 - p; 0 \,|\, m, \mu, g)\big|_{p=0, m=\mu} = -1, \quad \text{(6-16)}$$

we obtain, for the Legendre-transformed version of (6-15),

$$\frac{\partial}{\partial m^2} \Gamma^{(N)}(p_1 \ldots p_N; 0 \,|\, m, \mu, g) = -\Gamma^{(N,1)}(p_1 \ldots p_N; 0 \,|\, m, \mu, g). \quad \text{(6-17)}$$

That the constants on the right hand side combine to one, is most easily seen from (6-16), with (6-11c). We call, (6-17), the inhomogeneous parametric differential equation. The homogeneous equation has a more tricky derivation. Consider the unrenormalized vertex function $\Gamma_0^{(N)}$ which are obtained by Feynman rules based on the form of (6-13a), i.e., the mass in the propagator is m_0^2. The Γ_0's do depend naturally on m_0^2, u_0 and A. But they also have a μ-dependence through δm_0^2, (6-13b). Hence,

$$2\mu^2 \frac{\partial}{\partial \mu_2}\bigg|_{m_0, u_0, \Lambda} \Gamma_0^{(N)} = -2\mu^2 \left(\frac{\partial\, \delta m_0^2}{\partial \mu^2}\right)\bigg|_{m_0, u_0, \Lambda} \Gamma_0^{(N,1)}(p_1 \ldots p_N, 0 \,|\, m_0, \mu, u_0).$$

(6-18)

Rewriting this statement, in terms of the renormalized Γ's:

$$\Gamma^{(N)} = Z^{N/2}\,\Gamma_0^{(N)}, \qquad \Gamma^{(N,1)} = Z^{N/2} Z\, Z_1\, \Gamma_0(N,1),$$

and changing the parameters to m, g and A, we obtain:

$$\left\{2\mu^2 \frac{\partial}{\partial\mu^2} + 2\,\delta m^2 \frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial g} - N\gamma_\phi\right\}\Gamma^{(N)} = 0 \tag{6-19}$$

with

$$\beta = 2\mu^2 \frac{\partial g}{\partial\mu^2}\bigg|_{m_0,u_0,\Lambda}, \tag{6-20a}$$

$$\gamma_\phi = \mu^2 \frac{\partial \ln Z}{\partial\mu^2}\bigg|_{m_0,u_0,\Lambda}, \tag{6-20b}$$

$$\delta = \delta_1 + \frac{\mu^2}{m^2}\delta_2\,, \tag{6-20c}$$

$$\delta_1 = \frac{\mu^2}{m^2}\,\frac{\partial m^2}{\partial\mu^2}\bigg|_{m_0,u_0,\Lambda}, \tag{6-20d}$$

$$\delta_2 = -\frac{\partial\delta m_0^2}{\partial\mu^2}\bigg|_{m_0,u_0,\Lambda} Z\,Z_1\,. \tag{6-20e}$$

For D = 3 and 4, one has to demonstrate that, in addition, to the renormalized vertex-function, also the β, γ and 6 approach a finite limit for A $\to \infty$. This is most concisely done with the help of the "Normal Product Algorithm" which we described briefly in Sec. 5. For the derivation of the parametric differential Eqs. (6-17), (6-19), solely on the basis of normal products, we refer to Ref. 35.

The generalized Kadanoff scaling law (6-6), in order to be useful for the description of critical phenomena, has to be supplemented by the statement that the G's, resp., Γ's, have "zero mass" limits m $\to$ 0. In our treatment, the existence of this limit as well as the interpretation of m = 0 as zero mass, i.e.,

$$\Gamma^{(2)}\big|_{p=0} \xrightarrow{m\to 0} 0 \tag{6-21}$$

is inexorably linked with the assumption that β has a long distance zero g,. The vertex scaling equation, corresponding to (6-6), can be read directly off from (6-19):

$$\Gamma^{(N)}(\lambda p_1 \ldots \lambda p_N\,; 0\,|\,m,\mu,g\Lambda) = \lambda^{\,D - N\frac{d-2}{2}}$$
$$a^{-N}\,\Gamma^{(N)}\left(p_1 \ldots p_N\,; 0\,|\,\bar{m},\mu,\bar{g},\frac{\Lambda}{\lambda}\right). \tag{6-22}$$

We obtain, for small λ, an effective cut off which is driven towards infinity. In the differential equation,

$$\beta\left(g, \frac{\Lambda}{\mu}\right), \quad \delta_1\left(g, \frac{\Lambda}{\mu}\right), \quad \delta_2\left(g, \frac{\Lambda}{\mu}\right) \quad \text{and} \quad \gamma\left(g, \frac{\Lambda}{\mu}\right)$$

have a dependence on the cut-off which is lost in the limit of the effective cut-off approaching infinity (i.e., the limit for small λ). Hence, by setting $A = \infty$ from the outset, we have a euclidean version of a local cut-off independent quantum field theory which correctly describes the asymptotic behaviour. The argument, that the next to leading behaviour, in which ω as well as the derivatives γ_c', δ_c' enter, is still A-independent, is more involved and will not be given here. Let us from now on set $A = \infty$ and omit it as an argument of Γ.

The above scaling law (6-20) leads, for $p_i = 0$ in the presence of a "long distance eigenvalue" g_i, to

$$\Gamma^{(N)}(0\ldots 0; 0\,|\,m, \mu, g) \xrightarrow{m\to 0} \left(\frac{m^2}{M^2}\right)\frac{Nd_\phi - D}{2(\delta_c - 1)}\,\Gamma^{(N)}(0\ldots 0; 0\,|\,M, \mu, g_c), \tag{6-23}$$

with computable corrections. This equation tells us, in particular, that $m \to 0$ (for $\delta_c < 1$) is equivalent to zero mass (6.21). If the limit $m \to 0$ could be performed for finite (nonexceptional) momenta $p_1 \ldots p_N$, then we conclude from (6-22) that the zero mass functions $\Gamma^{(N)}(\lambda p_1 \ldots \lambda p_N$; $(0, \mu, g)$ will approach, for $\lambda \to \infty$, those of a canonical (free field) theory since $g \to 0$. However, for $R \to 0$, the same vertex functions will approach a noncanonical scale-invariant limit with $\dim \phi = d_\phi$. **A** slight generalization of these considerations to $\Gamma^{(N,L)}$, which we will discuss more explicitly later on, leads to

$$\dim \phi^2 = D - 2 + 2\delta_1\,.$$

The important remaining problem is therefore the existence of the $m \to 0$ limit for the vertex functions. In order to discuss this, we need the integrated inhomogeneous differential equation:

$$\Gamma^{(N)}(p_1 \ldots p_N\,; 0\,|\,m, \mu, g) = \Gamma^{(N)}(p_1 \ldots p_N\,; 0\,|\,M, \mu, g) -$$

$$- \quad \Gamma^{(N,1)}(p_1 \ldots p_N\,; 0\,|\,m', \mu, g)\, dm'^2 \tag{6-24}$$

For $N > 2$, the first term on the right hand side approaches zero for $M \to \infty$ (for $N = 2$ this only happens for the derivative $\frac{\partial}{\partial p^2}\Gamma^{(2)}$. The scaling law of the integrand in (6-24) follows from the differential equation for $\Gamma^{(N,1)}$:

$$\left\{2\mu^2 \frac{\partial}{\partial \mu^2} + 2\delta m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} - N\gamma_\phi + \gamma_{\phi^2}\right\} \Gamma^{(N,1)} = 0.$$

The derivation of this equation parallels (6-18)-(6-20e). The only difference is that the renormalized $\Gamma^{(N,1)}$ in terms of $\Gamma_0^{(N,1)}$ has in addition a $(Z Z_1)$ factor whose p-derivative gives rise to the γ_{ϕ^2} contribution.

E

From the normalization (6-16), one obtains:

$$2(\delta - 1)\, m^2 \frac{\partial}{\partial m^2} \Gamma^{(2,1)}\big|_{p=0,m=\mu} = (2\gamma_\phi - \gamma_{\phi^2})\, \Gamma^{(2,1)}\big|_{p=0,m=\mu}$$

Comparison with the $\frac{\partial}{\partial m^2}$ differentiated equation,

$$\frac{\partial}{\partial m^2}\left\{2\mu^2 \frac{\partial}{\partial \mu^2} + 2\delta m^2 \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} - 2\gamma\right\} \Gamma^{(2)}\big|_{p=0,m=\mu} = 0,$$

which, in view of the normalization condition (6-11a) and due to (6-17) for N = 2, leads to

$$-2(\delta - 1)\, m^2 \frac{\partial}{\partial m^2} \Gamma^{(2,1)}\big|_{p=0,m=\mu} + 2\delta_1 - 2\gamma_{\phi^2} = 0,$$

finally yields:

$$2\delta_1 = \gamma_{\phi^2}\,. \tag{6-25}$$

The global scaling behaviour, obtained from (6-19), is

$$\Gamma^{(N,1)}(p_1 \dots p_N\,; 0\,|\, m', \mu, g) = \lambda^{D - N\frac{D-2}{2} - 2} \exp 2\int_g^{\bar g} \frac{\delta_1}{\beta}\, dg' \cdot$$

$$\cdot\, a^{-N}\, \Gamma^{(N,1)}\left(\frac{p_1}{\lambda} \dots \frac{p_N}{\lambda}\,; 0\,|\, \bar m', \mu, \bar g\right) \tag{6-26}$$

It is convenient to perform a change of variables:

$$m'^3 = \mu^2 \exp 2\int_g^{\bar g} \frac{1 - \beta\delta_1}{\beta}\, dg', \quad dm'^2 = 2(1-\delta_1)\, m'^2\, \lambda^{-1}\, d\lambda.$$

For $\lambda \to \infty$, the coupling constant approaches zero and the right hand side can be estimated in lowest order perturbation theory. This yields the convergence of the integrand at the upper integration limit. For small λ, the integrand is

$$\sim \lambda^{D - N\frac{D-2}{2} - 2}\; a^{-N}\, \Gamma^{(N,1)}\left(\frac{p_1}{\lambda} \dots \frac{p_N}{\lambda}\,; 0\,|\, \mu, \mu, \bar g\right) d\lambda. \tag{6-27}$$

So we are facing a large momentum problem, at a coupling constant which is practically g,. Here we follow an idea by Symanzik[40] and use the operator short-distance expansion to determine the leading part:

$$\Gamma^{(N,1)}\left(\frac{p_1}{\lambda}\dots\frac{p_N}{\lambda};0\,|\,\mu,\mu,g\right)=\Gamma^{(N+2,0)}\left(\frac{p_1}{\lambda}\dots\frac{p_N}{\lambda};0,0\,;0\,|\,\mu,\mu,\bar{g}\right)\cdot$$
$$\cdot\,\Gamma^{(2,2)}(0\,0\,;0\,0\,|\,\mu,\mu,\bar{g})+\text{non leading terms.}\qquad(6\text{-}28)$$

The homogeneous equation for $\Gamma^{(N+2,0)}$ is

$$\left\{-\sum p_{i\mu}\frac{\partial}{\partial p_{i\mu}}+2(\delta-1)m^2\frac{\partial}{\partial m^2}+D-(N+2)\frac{D-2}{2}-(N+2)\gamma_\phi\right\}\Gamma^{(N+2,0)}=0.\qquad(6\text{-}29)$$

The asymptotic contribution from the second term may be estimated according to

$$m^2\frac{\partial}{\partial m^2}\Gamma^{(N+2,0)}(p_1\dots p_N,0,0;0\,|\,m,\mu,g)\,|_{m=\mu}$$
$$=-\mu^2\,\Gamma^{(N+2,1)}(p_1\dots p_N,0,0;0\,|\,\mu,\mu,g)\qquad(6\text{-}30)$$
$$\simeq\mu^2\,\Gamma^{(N+2)}(p_1\dots p_N,0,0;0\,|\,\mu,\mu,g)\,\Gamma^{(2,2)}(0,0;0,0\,|\,\mu,\mu,g),\qquad(6\text{-}31)$$

with

$$2(\delta-1)\,\mu^2\,\Gamma^{(2,2)}(0,0;0,0\,|\,\mu,\mu,g)=2\gamma_\phi-2\delta_1\,,\qquad(6\text{-}31)$$

which follows from taking the μ-derivative of the normalization condition (6-16):

$$0=2\mu^2\frac{\partial}{\partial\mu^2}\Gamma^{(2,1)}\,|_{m=\mu}=-2(\delta-1)\,\mu^2\frac{\partial}{\partial m^2}\Gamma^{(2,1)}\,|_{m=\mu}-2\gamma_\phi+2\delta.$$

We finally obtain:

$$\left\{-\sum p_{i\mu}\frac{\partial}{\partial p_{i\mu}}-2\delta_1+2\gamma_\phi+\beta\frac{\partial}{\partial g}+D-(N+2)\frac{D-2}{2}-(N+2)\,\gamma_\phi\right\}\Gamma^{(N)}_{\text{as}}=0.\qquad(6\text{-}32)$$

Therefore, the integrand (6-27) behaves, for $\lambda\to 0$ as

$$\sim\lambda^{D-4}\exp 2\int_g^{\bar{g}}\frac{\delta_1}{\beta}\,dg'\cdot\Gamma^{(N+2)}(p_1\dots p_N,0,0\,|\,\mu,\mu,\bar{g}).\qquad(6\text{-}33)$$

The condition for convergence is

$$D - 4 + 2\delta_c > -1$$

or, with $\delta_1 = \gamma_{\phi^2}$, $\gamma_{\phi^2}(g_c) > 1 + \varepsilon$.

The inequalities (6-34) and γ_{ϕ^2} are consistency requirements on the dimensions of ϕ^2.

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