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An Approximate Cylindrically Symmetric Stationary Solution of the Einstein – Maxwell Equations with Quasi-Neutra1 Dust Sources

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An approximate solution of Einstein-Maxwell equations is obtained for cylindrically symmetric fields due to oppositely charged clusters of particles moving in circles in opposite directions. The solution eorresponds to a distribution of weakly charged particles in a limited region around the axis of symmetry.

Uma solução aproximada das equações de Einstein-Maxwell é obtida para campos, com simetria cilíndrica, devidos a aglomerados de partículas de cargas opostas, movendo-se em círculos em direções opostas. A solução corresponde a uma distribuição de particulas fracamente carregadas, em uma região limitada ao redor do eixo de simetria

1. Introduction

After Melvin[1] discovered his magnetic universe as a singularity-free solution to the Einstein-Maxwell equations in general relativity, Thorne[2] has considered the physical structure of such a universe having no matter anywhere. He showed that the Melvin universe is absolutely stable under radial perturbation. Som[3] has obtained explicit solutions where matter coexists with a cylindrically symmetric axial magnetic field. Buth the magtic field is source-free in this case too. Later, Banerji[4] introduced a source in the form of a conduction current in the azimuthal direction inside a perfectly conducting dust. Though the solution can be matched with an outside magnetic field solution due to Bonnor[5], the Melvin magnetic universe cannot be fitted with a dust distribution in this way.

In the present paper, we propose to study the case where the cylindrically symmetric axial magnetic field arises solely due to a steady motion of charged particles. By cylindrically symmetric system we understand here

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(the usual definition) that the system is invariant both under rotations about an axis (rotationally symmetric) and translations parallel to this axis. The distribution considere. here is in the form of two clusters of oppositely charged particles moving in circles in opposite directions, so that the net angular momentum is zero and the system as a whole is electrically neutral. In view of the difficulty in finding an exact solution, we have looked for an approximate one. For a particular choice of the azimuthal current, the approximate solution describes a cylindrically symmetric axial magnetic field wholly within the: bounded distribution, while outside the solution is again that of Marder[6].

2. Basic Equations

For regions in which there are both matter and an electromagnetic field, the Einstein-Maxwell equations are

$$R^{\mu}_{\nu} - \delta^{\mu}_{\nu} R/2 = -8\pi G(T^{\mu}_{\nu} + E^{\mu}_{\nu})/c^4, \tag{2.1}$$

with

$$T^{\mu\nu} = c^2 \sum_i \rho_{(i)} u^{\mu}_{(i)} u^{\nu}_{(i)} \tag{2.2}$$

and

$$E^{\lambda}_{\nu} = (F^{\lambda}{}_{\alpha} F^{\alpha}{}_{\nu} - \delta^{\lambda}_{\nu} F^{\alpha}{}_{\beta} F^{\beta}{}_{\alpha}/4)/(\mu_0 c^2), \tag{2.3}$$

where $\rho_{(i)}$ is the matter density of the i^{th} group of charged particles having velocity $u^{\mu}{}_{(i)}$ and μ_0 is the magnetic permeability. In the present case, there are only two groups of oppositely charged particles, so that i = 1, 2. The Maxwell equations are

$$(\sqrt{-g}\, F^{\lambda\nu})_{;\lambda} = \mu_0 c^2 \sqrt{-g}\, j^{\nu} \tag{2.4}$$

and

$$F_{[\mu\nu;\rho]} = 0; \tag{2.5}$$

where as usual

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.$$

For the cylindrically symmetric system, we number the coordinates ct, r, z, ϕ as 0, 1, 2, 3 respectively. So, for a purely axial magnetic field, the only surviving components of $F_{\mu\nu}$ are $F_{31} = -F_{13}$. Equation (2.5) is then automatically satisfied, since F_{31} is a function of the radial coordinate r only. Therefore, $E^1_1 = -E^2_2$ and $T^1_1 = T^2_2 = 0$, so that we obtain

$$R^0_0 + R^3_3 = 0, \tag{2.6}$$

and then the metric may be taken in Weyl's canonical form (Synge[7])

$$g_{\mu\nu} = \mathrm{diag}\,(e^{2\alpha},\ -e^{2\beta-2\alpha}\ \ -e^{2\beta-2\alpha},\ -r^2 e^{-2\alpha}), \tag{2.7}$$

where a and β are functions of r only. Let us define $u^{\mu}_{(i)}$ as

$$u^{\mu}_{(1)} = (u^0, 0, 0, \omega/c), \qquad u^{\mu}_{(2)} = (\mathrm{u}^0, 0, 0, -\omega/c), \tag{2.8}$$

with

$$\rho_{(1)} = \rho_{(2)} = \mathrm{pj2} \quad \text{and} \quad F^3,\ = -cB/r. \tag{2.9}$$

Then, from Eqs. (2.2), (2.3), (2.8) and (2.9), one obtains

$$T^{\mu}_{\nu} = \mathrm{diag}[\mathrm{p}(\mathrm{c}^2 + \omega^2 r^2\, e^{-2\alpha}),\ 0,\ 0,\ -\rho\omega^2 r^2\, e^{-2\alpha}] \tag{2.10}$$

and

$$E^{\mu}_{\nu} = (2\mu_0)^{-1}\, e^{-2\beta}\, \mathrm{B}^2\ \mathrm{diag}(1,\ -1,\ 1,\ -1). \tag{2.11}$$

From (2.4), one obtains

$$j^3 = -(c\mu_0)^{-1}\, e^{2\alpha-2\beta}\, B_1/r, \tag{2.12}$$

where the subscript 1 means d/dr. Since the axial magnetic field is considered to be only due to the circular motions of oppositely charged particles in opposite directions, the azimuthal current j^3 must satisfy the condition

$$j^3 = \gamma\omega\rho/c, \tag{2.13}$$

where $\gamma = q/m$ is the specific charge of each particle.

3. Solutions of the Field Equations

From Eqs. (2.1), (2.7), (2.10) and (2.11) one can write the field equations explicitly as

$$(-\alpha_1^2 + \beta_1/r)\, e^{2\alpha-2\beta} = [87\mathrm{cG}/(2\mathrm{p}_0\mathrm{c}^4)]\ \mathrm{B}^2\, e^{-2\beta}, \tag{3.1}$$

$$(\alpha_1^2 + \beta_{11})\, e^{2\alpha-2\beta} = 8\pi G[\rho\omega^2 r^2 e^{-2\alpha} + B^2 e^{-2\beta}/(2\mu_0)]/c^4 \tag{3.2}$$

$$(-2\alpha_{11} + \alpha_1^2 - 2\alpha_1/r + \beta_{11})\, e^{2\alpha-2\beta}$$
$$= -87\mathrm{cG}[\mathrm{pe}^2 + \rho\omega^2 r^2 e^{-2\alpha} + B^2 e^{-2\beta}/(2\mu_0)]/c^4; \tag{3.3}$$

and, from (2.12) and (2.13), we write

$$\gamma\omega\rho = -e^{2\alpha-2\beta}\, B_1/(\mu_0 r). \tag{3.4}$$

Combining (3.1) and (3.3), one gets

$$\rho = (4\pi G/c^2)^{-1}\,(\mathrm{a},, -\alpha_1^2 + \alpha_1/r - \beta_{11})\, e^{2\alpha-2\beta}, \tag{3.5}$$

and, from (3.2),

$$B^2 = \mu_0 c^4(-\alpha_1^2 + \beta_1/r)\, e^{2\alpha}/(4\pi G). \tag{3.6}$$

Now eliminating ρ and β from the set *(3.1)* to (3.4), one obtains

$$[e^{2\alpha}(\alpha_{11} + \alpha_1/r) - \lambda B^2][\alpha_1(e^{2\alpha} + 2\omega^2 r^2/c^2) - \omega^2 r/c^2 - \gamma\omega B r/c^2] = 0 \tag{3.7}$$

and

$$\gamma\omega r[e^{2\alpha}(\alpha_{11} + \alpha_1/r) - \lambda B^2] = -\lambda c^2(e^{2\alpha} + 2\omega^2 r^2/c^2)\, B_1 , \tag{3.8}$$

where $\lambda = 4\pi G/(\mu_0 c^4)$.

The system of Eqs. (3.7)–(3.8) gives rise to two possible cases: the first one,

$$\begin{cases} e^{2\alpha}(\alpha_{11} + \alpha_1/r) - \lambda B^2 = 0, \\ B_1 = 0, \end{cases}$$

gives, on integration,

$$\begin{cases} B = \text{const}, \\ e^{2\alpha} = (\lambda B^2/4)(r/L)^2[(r/r_1)^L + (r_1/r)^L]^2, \end{cases}$$

and, from *(3.6)* and (3.5), we get

$$e^{2\beta} = (r/r_2)^{2(1+L^2)}[(r/r_1)^L + (r_1/r)^L]^4$$

and

$$\rho = 0;$$

in these expressions r_1, L and r_2 are constants of integration. The solution corresponds to the exterior field previously obtained by Ghosh and Sengupta[8].

4. Approximate Solution with Source

In the present section, we shall construct the solution which represents the axial magnetic field due to the distribution we are considering. In this case, our system of equations is

$$\begin{cases} \alpha_1(e^{2\alpha} + 2\omega^2 r^2/c^2) - \omega^2 r/c^2 - \gamma\omega B r/c^2 = 0, \\ \gamma\omega r[e^{2\alpha}(\alpha_{11} + \alpha_1/r) - \lambda B^2] = \lambda c^2(e^{2\alpha} + 2\omega^2 r^2/c^2) B_1 \end{cases}$$

Now for $\omega = \text{const}$, this system reduces to

$$(e^{2\alpha} + 2\omega^2 r^2/c^2)[(e^{2\alpha} + 2\omega^2 r^2/c^2)\alpha_1/r]_1 - r[\omega^2/c^2 - (e^{2\alpha} + 2\omega^2 r^2/c^2)\alpha_1/r]^2$$
$$= -\kappa\omega^2(r\alpha_{11} + \alpha_1)\, e^{2\alpha}/c^2, \qquad (4.1)$$

where $\kappa = \gamma^2/(\lambda c^2)$ is a dimensionless constant.

In view of difficulty in finding an exact solution, we try an approximate solution in powers of $r\omega/c \ll 1$, which implies that $r = r,$ defines the range of the distribution in such a way that $\rho(r > r,) = 0$. Let us put $v = \omega^2 r^2/c^2$. Then *(4.1)* reduces to

$$2(2^{2\alpha} + 2v)[2\alpha_1(e^{2\alpha} + 2v)]_1 - [1 - 2\alpha_1(e^{2\alpha} + 2v)]^2 +$$
$$+ 2\kappa\,(2\alpha_1 + 2\alpha_{11}v)\, e^{2\alpha} = 0, \qquad (4.2)$$

where ths subscript *1* denotes differentiation with respect to v. Now, let us suppose

$$2\alpha = av + bv^2 + O(v^3); \qquad (4.3)$$

an additive constant term is unnecessary since for $\xi = \text{const}$ we find that $\xi e^{2\alpha}$ is also a solution of *(4.2)* if we substitute v by $z = \xi v$ and reinterpret the subscript 1 as d/dz.

Substituting *(4.3)* in *(4.1)* and collecting terms independent of v, one obtains for $o = \text{const}$

$$a(a + 6 + 2\kappa) + 4b = 1. \qquad (4.4)$$

Thus the equation for B reduces to

$$B = \omega[a - 1 + v(a^2 + 2a + 2b)]/\gamma + O(v^2). \qquad (4.5)$$

Since our solution should correspond to a solution $B = 0$ when $y = 0$, i.e., when the particles are uncharged, we put

$$a - 1 = e\kappa + O(\kappa^2) \qquad (4.6)$$

and

$$a^2 + 2a + ab = gic + O(\kappa^2), \qquad (4.7)$$

where e and g are constants independent of κ. Then *(4.5)* reduces to

$$B = \gamma\mu_0\, c^2\omega(e + gv)/(4\pi G) + O(v^2) + O(\gamma^3).$$

Now from *(3.1)* one gets

$$d\beta/dv = e^2\kappa/2 + (1 + 2e\kappa)v/2 + O(v^2) + O(\kappa^2),$$

which on integration gives

$$e^{2\beta} = h[1 + v^2/2 + \kappa e v(e + v)] + O(v^3) + O(\kappa^2); \qquad (4.9)$$

we take constant h of integration as 1 in order to have $g_{11} = -1$ on the axis.

An expression for ρ is obtained by adding (3.2) and (3.3), namely

$$\rho = \omega^2[1 - 7v + \kappa(e + 2gv)]/(2\pi G) + O(v^2) + O(\kappa^2). \qquad (4.10)$$

For $r \to r_0$, one must have

$$B \to \mu_0\gamma\rho\omega r_0(r_0 - r) \simeq \gamma\mu_0 c^2\omega(r_0^2\omega^2/c^2 - r^2\omega^2/c^2)/(4\pi G), \qquad (4.11)$$

so by comparison with (4.8), we deduce

$$e = r_0^2\omega^2/c^2 \quad \text{and} \quad g = -1; \qquad (4.12)$$

the corresponding values of a and b satisfy (4.4).

The expressions for g_{00}, g_{11}, ρ and B correct up to the order of $r\omega/c$ which appears in the lowest order in κ, are thus

$$g_{00} = 1 + (r\omega/c)^2 - (r\omega/c)^4 + \kappa r^2(r_0^2 - r^2/2)(\omega/c)^4, \qquad (4.13)$$

$$g_{11} = -1 + (r\omega/c)^2 - 5(r\omega/c)^4/2 + \kappa r^2(r_0^2 - r^2/2)(\omega/c)^4, \qquad (4.14)$$

$$\rho = [1 - 7(r\omega/c)^2 + \kappa(r_0^2 - 2r^2)(\omega/c)^2]\,\omega^2/(2\pi G), \qquad (4.15)$$

$$B = \gamma\mu_0\omega^3(r_0^2 - r^2)/(4\pi G). \qquad (4.16)$$

For $\gamma = 0$, the solution goes over to that of Teixeira and Som[9] for uncharged distributions.

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