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Application of the Basic Approximation of the K-Harmonics Method to the Alpha Particle*

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Use is made of the basic approximation of the K-harmonics method to obtain the wave function of the ground state of the alpha particle (^{4}He) and its binding energy. This wave function is then used to caiculate the charge radius and the form factors of the alpha particle. Calculations are made for seven two-body potentials commonly used in the literature.

Usa-se a aproximação básica do método dos K-harmônicos para se obter a função de onda do estado fundamental da partícula alfa e sua energia de ligação. Essa função de onda é então usada para obter o raio de carga e os fatores de forma da particula alfa. Os cálculos são feitos para sete potênciais de dois corpos comumente usados na literatura.

1. Introduction

The K-harmonia method, used for the quantum-mechanical treatment of the many-body problem, was proposed by Simonov and Badalyan[1,2,3] in 1966. The essence of the method consists in expanding the wave function, in the C. M. frame, of a system of A nucleons in terms of a complete system of hyperspherical and harmonic functions in the $3(A-1)$-dimensional vector space spanned by the position vectors of the nucleons relative to the center of mass of the system, analogously to the multipole expansion for one particle problems. These functions are the angular part of homogerieous and harmonic polynomials (usually called K-harmonics) which are the $3(A-1)$ generalization of the three dimensional solid harmonics. A method to construct such polynomials, taking into account the Pauli Principie, is proposed in Ref. 3.

In the construction of the K-harmonics use is made of single-particle states and a shell structure emerges which, although being quite analogous to that of the nuclear shell model, has a more universal character. The introduction of a hyperdistance in the 3(A – 1)-dimensional vector space

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allows us to treat the nucleons collectively, the overall interaction being represented by a kind of effective potential. Hence, the K-harmonics method has some features of an unified nuclear model.

The *K*-harmonics method has been succesfully used in the determination of the wave function of the ground state and the binding energy of the light nuclei[2,4-9] and there are evidences that it can be applied to heavy nuclei[10] and even to the nuclear matter[11].

In this paper the basic approximation of the K-harmonics method is used in order to obtain the wave function of the ground state of the alpha particle, as well as its binding energy, charge i-adius and form factors for the two-body potentials commonly used in the literature.

In Section 2, a summary of the method is given aiming to stablish the notation and to present the formulae which will be used in this paper, making it self-contained. In Section 3, the method is applied to obtain the wave function of the ground state of the alpha particle, whereas the form factors are studied in Section 4. In Section 5, an analysis of the results obtained is made.

2. Summary of the K-Harmonics Method

The Hamiltonian of a system of *A* nucleons of equal masses and interacting through a potential V is given by

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{A} \nabla^2_{\mathbf{r}(i)} + V(\mathbf{r}(1), \boldsymbol{\sigma}(1), \boldsymbol{\tau}(1), \ldots, \mathbf{r}(A), \boldsymbol{\sigma}(A), \boldsymbol{\tau}(A)), \quad (2.1)$$

where $\boldsymbol{\sigma}(i)$ and $\boldsymbol{\tau}(i)$ are the spin and the isospin operators acting on the particle *i*.

In the *K*-harmonics method, the intraction potential V in *(2.1)* is usually taken as a superposition of two-body potentials:

$$V = \sum_{i<j=1}^{A} U(\mathbf{r}(i) - \mathbf{r}(j)) \quad (2.2)$$

with

$$\begin{aligned} U(\mathbf{r}) = {} & U_{33}(\mathbf{r})\, P_\sigma^{(+)}\, P_\tau^{(+)} + U_{13}(\mathbf{r})\, P_\sigma^{(-)}\, P_\tau^{(+)} + \\ & + U_{31}(\mathbf{r})\, P_\sigma^{(+)}\, P_\tau^{(-)} + U_{11}(\mathbf{r})\, P_\sigma^{(-)}\, P_\tau^{(-)}. \end{aligned} \quad (2.3)$$

In (2.3), the $P_\sigma^{(\pm)}$ are spin projectors of triplet and singlet state of the relative spin of the pair of particles:

$$P_\sigma^{(\pm)}(i,j) = \frac{1}{2} \pm \frac{1}{4}[1 + \sigma(i) \cdot \sigma(j)].$$

Analogously, the $P_\tau^{(\pm)}$ are isospin projectors [For the treatment of two-body potentials containing tensor forces, spin-orbit forces, etc, see Ref. 12.]. The indices 1 and 3 in (2.3) stand for singlet and triplet, respectively.

In order to separate the relative motion from the center of mass motion, Jacobi coordinates $\xi(i)$ are introduced by the relations

$$\xi(i) = \frac{1}{\sqrt{i(i+1)}}\left[\sum_{j=1}^{i} \mathbf{r}(j) - i\mathbf{r}(i+1)\right], \quad i = 1, 2, \ldots, (A-1), \tag{2.4}$$

$$\xi(A) = \frac{1}{\sqrt{A}} \sum_{i=1}^{A} \mathbf{r}(i). \tag{2.5}$$

The coordinates $\xi(i)$ for i = 1, 2, . . . , (A – 1) are relative coordinates while $\xi(A)$ is proportional to the center of mass of the system

$$\xi(A) = \sqrt{A}\,\mathrm{R}, \qquad \text{with} \qquad \mathrm{R} = \frac{1}{\mathrm{A}} \sum_{i=1}^{A} \mathbf{r}(i). \tag{2.6}$$

The numerical factors in (2.4) and (2.5) were chosen such as to make the linear transformation relating the $\mathbf{r}(i)$ coordinates to the Jacobi coordinates $\xi(i)$ an orthogonal transformation, what simplifies the calculations. The kinetic energy operator in (2.1), for instance, splits into a relative (intrinsic) and a center of mass part.

The interaction potential (2.2), due to the fact that it depends on the spatial coordinates only through $\mathbf{r}(i) - \mathbf{r}(j)$, depends only on the relative Jacobi coordinates. So, the Hamiltonian (2.1), with the potential (2.2) splits into a relative and a center of mass part.. The intrinsic properties of the system of A nucleons depend only on the relative coordinates. Therefore, from now on, we shall focus our attention only to the relative (intrinsic) part of the Hamiltonian (2.1):

$$H_{\mathrm{intr}} = -\frac{\hbar^2}{2m} \sum_{i=1}^{A-1} \nabla_{\xi(i)}^2 + \sum_{i<j=1}^{A} U(r(i) - r(j)). \tag{2.7}$$

2.1 Hyperspherical coordinates

Let us consider the 3(A – 1)-dimensional vector space $E_{3(A-1)}$ spanned by the $3(A-1)$ components of the vectors $\xi(i)$, i = 1, 2, . . . , (A – 1) and introduce hyperspherical coordinates in this space.

The radial coordinate ρ, usually called hyperdistance, is defined by

$$\rho^2 = \sum_{i=1}^{A-1} \xi(i)^2 \tag{2.8}$$

Using Eqs. (2.4) and (2.5), it can be shown that the following identities hold

$$\rho^2 = \sum_{i=1}^{A} \rho(i)^2 = \sum_{i=1}^{A} \mathbf{r}(i)^2 - A\mathbf{R}^2 = \frac{1}{A} \sum_{i<j=1}^{A} [\mathbf{r}(i) - \mathbf{r}(j)]^2, \tag{2.9}$$

where $\boldsymbol{\rho}(i) = \mathbf{r}(i) - \mathrm{R}$ are the position vectors of the nucleons referred to the center of mass of the system.

Besides the hyperdistance, 3A-4, angular variables are introduced. The choice of these coordinates is obviously not unique. The choice which seems to us to be the more convinient is presented in the Appendix. We shall designate the angular coordinates colectively by Ω_ρ.

In the hyperspherical coordinates the intrinsic kinetic energy operator in (2.7) is written as[13]

$$-\frac{\hbar^2}{2m} \sum_{i=1}^{A-1} \nabla^2_{\xi(i)} = -\frac{\hbar^2}{2m} \left[\rho^{-(3A-4)} \frac{\partial}{\partial \rho} \left(\rho^{3A-4} \frac{\partial}{\partial \rho} - \frac{\mathscr{J}^{(2)}}{\rho^2} \right) \right]. \tag{2.10}$$

where $\mathscr{J}^{(2)}$ is an angular operator. This operator, which is the $3(A-1)$-dimensional generalization of the angular momentum squared, is a Casimir invariant of $R_{3(A-1)}$, the rotation group in $3(A-1)$ dimensions. Its linearly independent eigenfunctions belonging to a same eigenvalue $\lambda = K[K + 3(A-1)-2]$ carry a basis for the most degenerated irreducible representation [K] of $R_{3(A-1)}$. They are the angular part of harmonic and homogeneous polynomials of degree K in $3(A-1)$ variables. From (2.10), it follows that $\mathscr{J}^{(2)}$ is invariant under permutation of the coordinates of the nucleons. Then, we can contruct functions which are simultaneously eigenfunctions of $\mathscr{J}^{(2)}$ and antisymmetric under permutation of the coordinates of the nucleons. These functions are usually called K-harmonics. We shall designate them by $Y^K_\nu(\Omega_\rho, u(i), v(i))$, where $u(i)$ and $v(i)$ are SU(2) tensors of rank 1/2 designating the states of spin

and isospin of the i^{th} nucleon and the label v distinguish the different K-harmonics with the same K. They can be taken orthonormalized in the unity sphere of $E_{3(A-1)}$:

$$\int Y_{\nu}^{K}(\Omega_{\rho}, u_{m_i}(i), v_{n_i}(i))^{\dagger} \; Y_{\nu'}^{K'}(\Omega_{\rho}, u_{m'_i}(i), v_{n'_i}(i))\, d\Omega_{\rho} =$$

$$= \delta_{KK'}\, \delta_{\nu\nu'} \prod_{i=1}^{A} \delta_{m_i m'_i}\, \delta_{n_i n'_i} \qquad (2.11)$$

Now, the wave function of the intrinsic motion is expanded in terms of the K-harmonics as

$$\psi(\rho(i), u(i), v(i)) = \rho^{-(3A-4)/2} \sum_{K,\nu} \chi_{K,\nu}(\rho)\; Y_{\nu}^{K}(\Omega_{\rho}, u(i), v(i)) \qquad (2.12)$$

where the coefficients $\chi_{K,\nu}(\rho)$ are functions of ρ. To determine these functions we substitute the expansion *(2.12)* in the Schrodinger equation for $H_{,,,,}$, multiply from the left by $Y_{\nu'}^{K'}(\Omega_{\rho}, u(i), v(i))^{\dagger}$, use the orthogonality relations *(2.11)* and obtain the infinite set of coupled differential equations

$$\left\{\frac{d^2}{d\rho^2} - \frac{\mathscr{L}_K(\mathscr{L}_K+1)}{\rho^2} - \frac{2m}{\hbar^2}\left[W_{K'\nu'}^{K\nu}(\rho) - E\right]\right\} \chi_{K,\nu}(\rho) =$$

$$= \frac{2m}{\hbar^2} \sum_{K'\nu' \neq K\nu} W_{K\nu}^{K'\nu'}(\rho)\, \chi_{K'\nu'}(\rho), \qquad (2.13)$$

where

$$\mathscr{L}_K = K + 3(A-2)/2,$$

$$W_{K\nu}^{K'\nu'}(\rho) = \int Y_{\nu'}^{K'}(\Omega_{\rho}, u(i), v\, i)\,) \sum_{j<k=1}^{A} U(\mathbf{r}(j) - \mathbf{r}(k))\; Y_{\nu}^{K}(\Omega_{\rho}, u(i), v(i))\, d\Omega_{\rho}\,. \qquad (2.15)$$

The infinite set of differential equations (2.13), the expansion *(2.12)* in K-harmonics and the matrix elements *(2.15)* of the interaction potential V are exact and form the basis of the K-harmonics method.

2.2 Construction of the K-harmonics

The Pauli principle requires that the wave function of a system of A identical nucleons (fermions) be totally antisymmetric under the interchange of all variables of any pair of nucleons. As it is well known, the simplest way to obtain totally antisymmetric functions from single-particle functions $\psi_j(\rho(i), u(i), v(i)), (j = 1, 2, \ldots\ldots\ldots, A)$ is to construct the Slater determinant

$$P = \begin{vmatrix} \psi_1(1) & \psi_1(2) \ldots & \psi_1(A) \\ \psi_2(1) & \psi_2(2) \ldots & \psi_2(A) \\ \ldots\ldots\ldots\ldots\ldots & & \\ \psi_A(1) & \psi_A(2) \ldots & \psi_A(A) \end{vmatrix}, \qquad (2.16)$$

where use was made of the notation $\psi_j(i) = \psi_j(\rho(i), u(i), v(i), (j = 1, 2, \ldots, A)$.

The single particle functions used to feed the determinant (2.16) are, to some extent, arbitrary. As the K-harmonic must be the angular part of harmonic and homogeneous polynomials of degree K, it is convenient to take the single-particle functions as homogeneous polynomials in $\rho(i)$. Simonov and colaborators[2,14] suggest the functions

$$\psi_j(i) = \psi_{n_j l_j m_j p_j q_j}(\rho(i), u(i), v(i)) = |\rho(i)|^{2n_j + l_j} Y^{l_j}_{m_j}(\hat{\rho}(i)) u_{p_j}(i) v_{q_j}(i), \qquad (2.17)$$

which are homogeneous polynomials of degree $K_j = 2n_j + l_j$. [$Y^l_m(\hat{\mathbf{x}})$ is here the usual spherical harmonic.].

Filling the determinant *(2.16)* with the functions (2.17), we obtain homogeneous polynomials of degrees $K = K_{\min}, K_{\min} + 1, K_{\min} + 2, \ldots, K_{\min}$ being a function of A.

Of particular interest in what follows is lhe polynomials *(2.16)* of degree $K_{\min}$. The construction of these polynomials is given in Ref. 2. Suffice here to say that these polynomials, as well as the polynomials of degree $K_{\min} + 1$, are automatically harmonic and for nuclei such that Z and $N = A - Z$ are of the form

$$\frac{1}{3}(n + 1)(n + 2)(n + 3), \qquad n = 0, 1, 2, \ldots \qquad (2.18)$$

there is only one polynomial *(2.16)* of degree $K_{\min}$.

2.3 The Basic Approximation

When the non-diagonal matrix elements $W^{K'\nu'}_{K\nu}(\rho)$ of V in *(2.13)* do not vanish, the system *(2.13)* is an infinite system and there is no hope of solving it, even numerically. Then, what is usually done is to retain in the expansion *(2.12)* only the K-harmonia with K smaller and equal to some fixed value $K_{\max}$ and truncate the poteritial V by considering only its matrix elements between K-harmonia with $K \leq K_{\max}$, what implies in taking $W^{K'\nu'}_{K\nu} = 0$ if K or K' is greater than $K_{\max}$. Within this approximation, *(2.13)* becomes a *finite* set of *coupled* differential equations which we then try so solve.

The simplest of those approximations is to take $K_{max} = K_{min}$. This it the ***basic*** approximation, also called the drastic approximation. Besides the mathematical simplicity, there is a ***a posteriori*** reason for studying the basic approximation: the experience with the application of the K-harmonics method to light nuclei[8,9,14] have shown that the main contribution to the wave function and to the energy of the low-lying nuclear states is due to the first terms of the expansion (2.12), i.e., to those with $K = K_{min}$.

The basic approximation is specially convenient for the double closed nuclei (nuclei such that A and N are of the form (2.18)), since for each of these nuclei there is only one K-harmonic with $K = K_{min}$. In this case, the system (2.13) reduces to a single Schrodinger-type equation

$$\left\{\frac{d^2}{d\rho^2} - \frac{\mathscr{L}_K(\mathscr{L}_K + 1)}{\rho^2} + \frac{2m}{\hbar^2}[E - W(\rho)]\right\}\chi(\rho) = 0, \tag{2.19}$$

with

$$W(\rho) = \int Y^{K_{min}}(\Omega_\rho, u(i), v(i))^* \sum_{j<k=1}^{A} U(\mathbf{r}(j) - \mathbf{r}(k))\, Y^{K_{min}}(\Omega_\rho, u(i), v(i))\, d\Omega_\rho . \tag{2.20}$$

The physical reason to believe that for these nuclei this is a good approximation is that they have the most symmetric, spherical **and** compact spatial configuration.

3. Ground State of the Alpha Particle

The alpha particle is the only bound system of four nucleons since ^{4}n and ^{4}p = ^{4}B do not exist and ^{4}H and ^{4}Li are unbond. The ground state is an S state with $J^\pi = 0^+$. Its binding energy is 28.3 MeV, which gives the value of 4.7 MeV of binding energy per nucleon pair, the greatest value of this quantity to be found among the entire range of nuclides. From this fact we may infer that the nucleons mus lit very close together, a supposition corrobarated by the strinkingly low value of the charge radius

$$R_{\alpha} = 1.63 \pm 0.04 \text{ fm}. \tag{3.1}$$

Since the alpha particle is a double closed shell nucleus, in the basic

approximation of the K-harmonics method, its wave function (Eq. *2.12)* is given *by*

$$\psi = \rho^{-4}\, \chi(\rho)\, Y^{K_{\min}}(\Omega_\rho\,, u(i), v(i)), \tag{3.2}$$

with

$$Y^{K_{\min}}(\Omega_\rho\,, u(i), v(i)) = \rho^{-K_{\min}}\, P^{K_{\min}}(\rho(i), u(i), v(i)).$$

For the alpha particle $K_{\min} = 0$, therefore

$$P^{K_{\min}}(\rho(i), u(i), v(i)) = P^0(\rho(i), u(i), v(i)) = \\ = \mathcal{N} \begin{vmatrix} u_+(1)\, v_+(1) \ldots\ldots u_+(4)\, v_+(4) \\ u_+(1)\, v_-(i) \ldots\ldots u_+(4)\, v_-(4) \\ u_-(1)\, v_+(1) \ldots\ldots u_-(4)\, v_+(4) \\ u_-(1)\, v_-(1) \ldots\ldots u_-(4) v_-(4) \end{vmatrix}. \tag{3.3}$$

[We shall use the following convention: $v_-(i)$ designates a proton state and $v_+(i)$ a neutron state].

Taking $\mathcal{N} = (4!\Omega_\rho)^{-1/2} = \sqrt{35}/(16\pi^2)$, P^0 turns out to be normalized in the unity sphere of E,. This reduces the normalization condition for ψ,

$$\int \psi(\rho, \Omega_\rho\,, u(i), v(i))^* \, \psi(\rho, \Omega_\rho\,, u(i), v(i))\, dV_9 = 1, \tag{3.4}$$

simply to

$$\int_0^\infty \chi^2(\rho)\, d\rho = 1. \tag{3.5}$$

Expanding the determinant *(3.3)* we have

$$P^0(\rho(i), u(i), v(i)) = \frac{\sqrt{105}}{8\pi^2} \times \\ \times \left[\{[u(1), u(2)]_1\,, [u(3), u(4)]_1\}_{00}\, \{[v(1), v(2)]_0\,, [v(3), v(4)]_0\}_{00} - \right. \\ \left. - \{[u(1), u(2)]_0\,, [u(3), u(4)]_0\}_{00}\, \{[v(1), v(2)]_1\,, [v(3), v(4)]_1\}_{00}\right], \tag{3.6}$$

where

$$[T^{(k)}, T^{(k')}]_{k''m''} = \{T^{(k)}, T^{(k')}\}_{k''m''} = \sum_{mm'} \langle km\, k'm' \mid k''m'' \rangle\, T^{(k)}_m\, T^{(k')}_{m'}\,, \tag{3.7}$$

for any two tensors $T^{(k)}$, $T^{(k')}$ of SU(2).

The interest in gathering the particle in pairs as in *(3.6)* is because the interaction potential we use is a two-body potentiai. The coupling of the spin and isospin pairs in singlets and triplets makes obvious the action

of the projectors $P_\sigma^{(\pm)}$ and $P_\tau^{(\pm)}$ and shows explicitly that P^0 has total spin and isospin equal to zero. Since P^0 is of degree zero, it has positive parity. Therefore P^0 has built in the spin, isospin and parity characteristics of the ground state of the alpha particle.

The "effective potential" **(2.20)** assumes now the form

$$W(\rho) = \frac{105}{64\pi^4} \sum_{i<j=1}^{4} \int [U_{13}(\mathbf{r}(i)-\mathbf{r}(j)) + U_{31}(\mathbf{r}(i)-\mathbf{r}(j))]\, d\Omega_\rho , \qquad (3.8)$$

where use was made of the explicit expression of P^0, Eq. (3.6).

Using the angular variables given in the Appendix, $W(\rho)$ takes the form

$$W(\rho) = \frac{315}{8} \int_0^1 [U_{13}(\sqrt{2}\,\rho x) + U_{31}(\sqrt{2}\,\rho x)]\,(1-x^2)^2\, x^2\, dx, \qquad (3.9)$$

for two-body nuclear central potentials of the form (2.3). In the present paper we used two-body nuclear central potentials of the form

$$U(r) = U_s(r)\, P_\sigma^{(-)} + U_t(r)\, P_\sigma^{(+)}, \qquad \mathrm{r} = |\mathbf{r}(i)-\mathbf{r}(j)|, \qquad \textbf{(3.10)}$$

where

$$U_s(r) = \sum_{k=1}^{3} \alpha_k\, f(r/\beta_k) \qquad (3.11)$$

and similarly for $U_t(r)$. For these potentials, the expression of $W(\rho)$ is the same as (3.9) with the replacements $U_{13} \to U_s$ and $U_{31} \to U_t$. The potentials we studied have the parameters a,, β_k and the shape functions f given in Table 1 and the "effective potentials" (3.9) they lead are plotted in Fig. 1.

Potential	$f(x)$	triplet (t) singlet (s)	α_1(MeV)	β_1(MeV)	α_2(MeV)	β_2(MeV)	α_3(MeV)	β_3(MeV)
$V_1^{(21)}$	$\exp(-x^2)$	$t = s$	–83.34	1.60	144.86	0.82	–	–
$V_2^{(22)}$	$\exp(-x^2)$	$t = s$	–140.6	1.40	389.5	0.7	–	–
$V_3^{(23)}$	$\exp(-x^2)$	$t = s$	51.50	1.60	–	–	–	–
$V_4^{(24)}$	$\exp(-x^2)$	t	1000.0	0.5773	– 326.69	0.9759	– 43.0	1.2909
		s	1000.0	0.5773	– 166.00	1.118	– 23.0	1.5811
$V_5^{(25)}$	$\exp(-x^2)$	t	600.0	0.4264	– 69.79	1.4142	– 27.59	1.6222
		s	880.0	0.4303	– 69.79	1.25	– 21.0	1.4433
$V_6^{(26)}$	$x^{-1}\exp(-x)$	$t = s$	–29.0276	1.58	–	–	–	–
$V_7^{(27)}$	$x^{-1}\exp(-x)$	$t = s$	4545.182	0.3215	– 900.739	0.6431	–	–

Table 1 - Parameters and shapes of the potentials (See Eq. 3.11) used in this paper. The superscripts indicate the references.

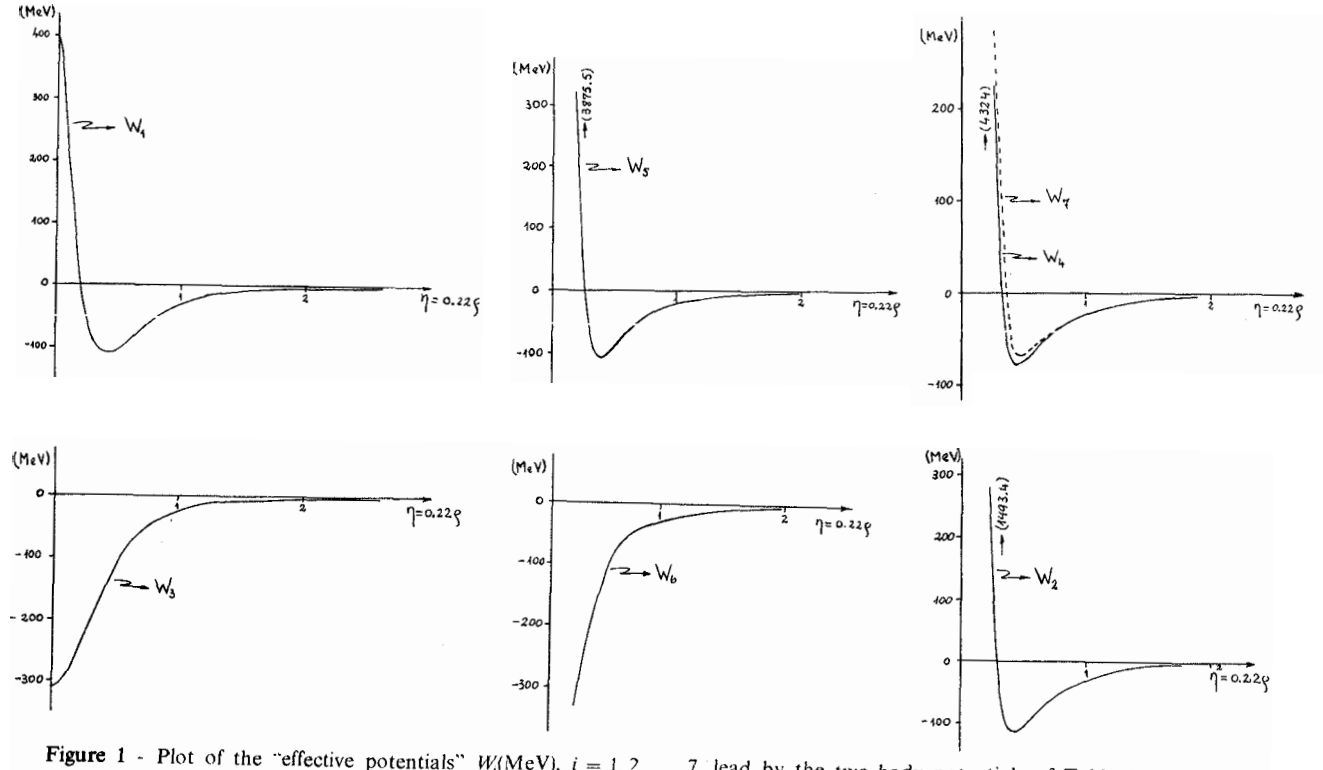

Figure 1 - Plot of the "effective potentials" W_i(MeV), $i = 1, 2, \ldots, 7$, lead by the two-body potentials of Table 1.

Since the alpha particle has two protons, besides the nuclear interaction we have also to take into account the Coulomb repulsion between the two protons. To this end we need to calculate the "effective potential" $W_{\text{Coul}}(\rho)$ due to this interaction. The Coulomb potential between nucleons i and j is given by

$$U_{\text{Coul}}(\mathbf{r}(i)-\mathbf{r}(j)) = \left[\frac{1}{2}-\tau_3(i)\right]\left[\frac{1}{2}-\tau_3(j)\right]\frac{e^2}{|\mathbf{r}(i)-\mathbf{r}(j)|}, \tag{3.12}$$

where $\tau_3(i)$ is the z-component of the spin operator of the nucleon i. Now we use ***(3.12)*** and ***(3.3)*** in ***(2.20)*** and obtain

$$W_{\text{Coul}}(\rho) = \int P^0(u(i), v(i))^{\dagger} \sum_{j<k=1}^{4} U_{\text{Coul}}(\mathbf{r}(j)-\mathbf{r}(k))\, P^0(u(i), v(i))\, d\Omega_\rho = \\ = \frac{35\, e^2}{16\sqrt{2}\,\rho} = \frac{2.23 \text{ MeV fm}}{\rho} \tag{3.13}$$

where use was made of the experimental result $e^2 = 1.44$ MeV fm.

Now that we have the K_{min}-harmonic $Y^0 \equiv P^0$, the nuclear $W(\rho)$ and Coulomb $W_{\text{Coul}}(\rho)$ "effective potentials", the next step is to integrate Eq. ***(2.19)*** to determine the ground state energy and the corresponding radial part $R(\rho) = \chi(\rho)/\rho^4$ of the wave-function.

First of all, we put ***(2.19)*** in a dimensionless form by defining the dimensionless variable

$$\eta = \sqrt{\frac{2m \text{ MeV}}{\hbar^2}}\,\rho = \frac{0.22\rho}{\text{fm}} \tag{3.14}$$

In this variable, Eq. ***(2.19)*** reads

$$\left\{\frac{d^2}{d\eta^2} - \frac{12}{\eta^2} - \left[W(\rho = \frac{\text{fm}}{0.22}\eta)/\text{MeV} + \frac{0.49}{\eta} - E/\text{MeV}\right]\right\}\chi(\eta) = 0. \tag{3.15}$$

The integration of Eq. ***(3.15)*** was done by the Matching Method[15,16] using the Numerov formula[17]. The results obtained for the ground state energy of the alpha particle for the potentials of Table *1* are given in Table *2*. The radial parts of the wave-functions are plotted in Fig. *2*.

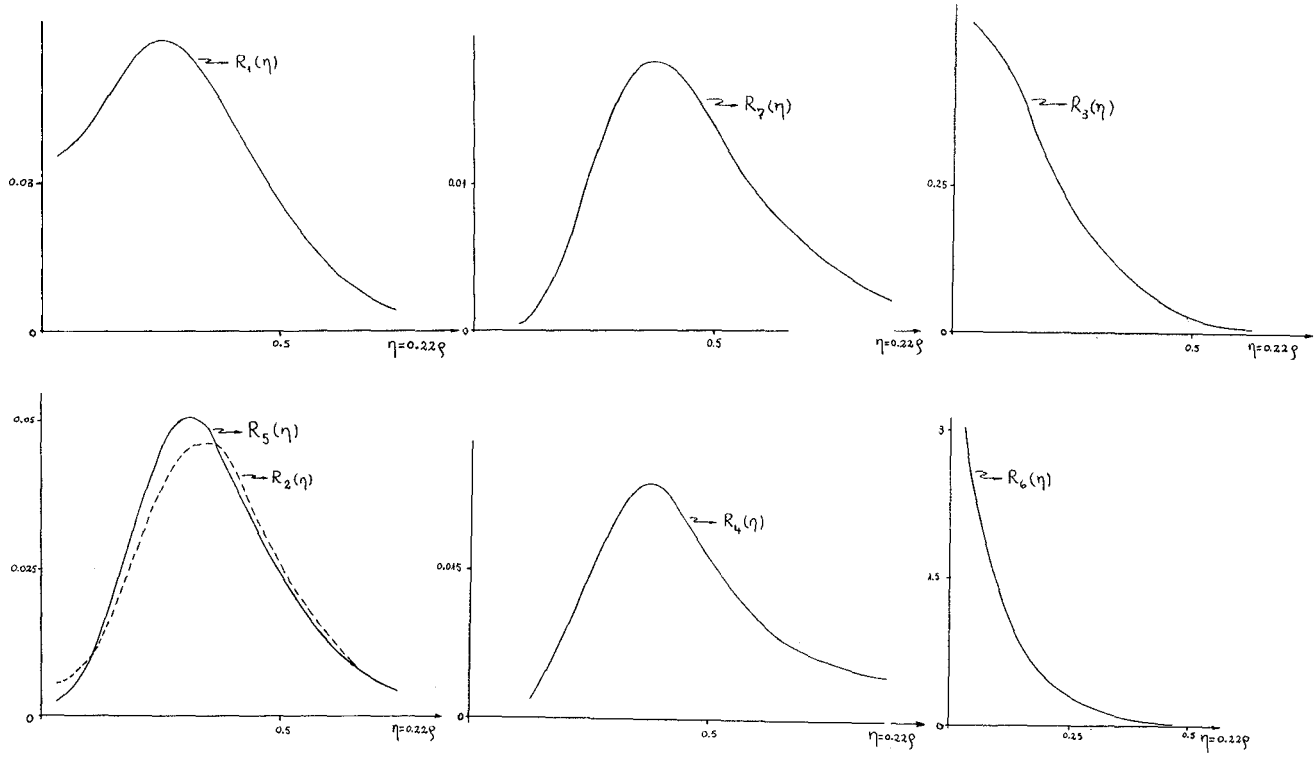

Figure 2 - Plot of the radial wave functions $R_i(\eta)$, $i = 1, 2, \ldots, 7$ of the ground state of the alpha particle corresponding to the potentials given in Table 1.

4. From Factors of the Alpha Particle

The charge form factor of a nucleus of charge Ze is defined as the Fourier transform of the charge density $\rho(\mathbf{r})$:

$$F_{\mathrm{Ch}}(\mathbf{q}) = \frac{1}{Ze}\int \exp(i\mathbf{q}\cdot\mathbf{r})\,\rho(\mathbf{r})\,d\mathbf{r}, \tag{4.1}$$

where $\hbar\mathbf{q}$ is the momentum transfer. It can be directly related[18] to the square root of the ratio of the differential elastic-scattering cross section to the Mott scattering. The charge density is given by

$$\rho(\mathbf{r}) = \int [\rho_p(\mathbf{r}-\mathbf{r}')\prod\nolimits_-(\mathbf{r}') + \rho_n(\mathbf{r}-\mathbf{r}')\prod\nolimits_+(\mathbf{r}')]\,d\mathbf{r}' \tag{4.2}$$

where $\rho_p(\mathbf{r})$ and $\rho_n(\mathbf{r})$ are the proton and neutron charge densities and $\Pi+(\mathrm{r})$are the expection values, relative to the ground state, of the operators

$$\prod\nolimits_\pm(\mathbf{r}') = \sum_{i=1}^{A} \delta[\mathbf{r}'-\rho(i)]\,[\tfrac{1}{2} \pm \tau_3(i)], \tag{4.3}$$

which give the density of probability of finding either a neutron or a proton, respectively, at a definite point r' in the space.

Potential	E (MeV)	R (fm)
V_1	27.94	1.72
V_2	28.66	0.799
V_3	38.38	1.44
V_4	6.49	0.81
V_5	17.73	0.80
V_6	20.06	0.791
V_7	6.18	0.82

Table 2 - Values of the binding energy and the charge radius of the alpha particle for the potentials of Table 1.

Substituting (4.2) in (4.1) and using the convolution theorem we get

$$F_{\mathrm{Ch}}(\mathbf{q}) = \mathrm{f}+(\mathbf{q})\,F+(\mathbf{q}) + \mathrm{f}-(\mathbf{q})\,F-(\mathbf{q}), \tag{4.4}$$

where

$$f_\pm(\mathbf{q}) = \int e^{i\mathbf{q}\cdot\mathbf{r}}\,\rho_\pm(\mathbf{r})\,d\mathbf{r}, \tag{4.5}$$

$$F_\pm(\mathbf{q}) = \frac{1}{Ze}\int e^{i\mathbf{q}\cdot\mathbf{r}}\prod\nolimits_\pm(\mathbf{r})\,d\mathbf{r}. \tag{4.6}$$

The functions $f_{\pm}(\mathbf{q})$, which are the charge form factors of the neutron and the proton, are due to the finite size of the nucleons. They were obtained experimentally by Janssens *et al.*[19] fitting the parameters of expressions for $f_{\pm}(q^2)$ obtained in the three poles approximation of the dispersion theory. Their explicit expressions are

$$f_+(q^2) = \frac{1.25}{1 + q^2/15.7} - \frac{0.8}{1 + q^2/26.7} - \frac{0.58}{1 + q^2/8.19} + 0.13 \qquad (4.7)$$

$$f_-(q^2) = \frac{1.25}{1 + q^2/15.7} - \frac{0.8}{1 + q^2/26.7} + \frac{0.58}{1 + q^2/8.19} - 0.03 \qquad (4.8)$$

where q^2 in given in fm^{-2}.

4.1 Charge Form Factor of the Alpha Particle

The expectation value of the operators (4.3) relative to ground state of the alpha particle, Eq. (3.2), gives

$$\prod_+(\mathbf{r}) = \prod_-(\mathbf{r}) = \frac{105}{16\pi^4} \int \chi^2(\rho)\, \delta(\mathbf{r} + \sqrt{3}\, \xi(3)/2)\, dV_9. \qquad (4.9)$$

Substituting (4.9) in (4.6) changing the order of integration and integrating in r we have

$$F_+(\mathbf{q}) = F_-(\mathbf{q}) = F(\mathbf{q}) = \frac{105}{32\mathrm{n}^4} \int \exp(-i\sqrt{3}\, \mathbf{q} \,.\, \xi(3)/2)\, \chi^2(\rho)\, dV_9 \quad (4.10)$$

[$F(\mathrm{q})$ is usually called *body* form factor.].

Now we use the expansion of the plane wave in spherical components

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{l=0}^{\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_m^{l*}(\hat{\mathbf{r}})\, Y_m^l(\hat{\mathbf{k}}) \qquad (4.11)$$

and observe that in the integration in (4.10) only the $1 = 0$ component contribute. [$j_l(kr)$ in (4.11) is the spherical. Bessel function.]. After a little manipulation we finally get for the alpha particle body form factor the expression

$$F(q) = 105 \int_0^{\infty} \frac{\chi^2(\rho)\, j_3(\sqrt{3}\, q\rho/2)}{(\sqrt{3}\, q\rho/2)^3}\, d\rho \qquad (4.12)$$

Since $\chi(\rho)$ was obtained only numerically, the same happens to $F(q)$ and $F_{Ch}(q)$. The values obtained for $F_{Ch}(q)$ are plotted in Fig. 3.

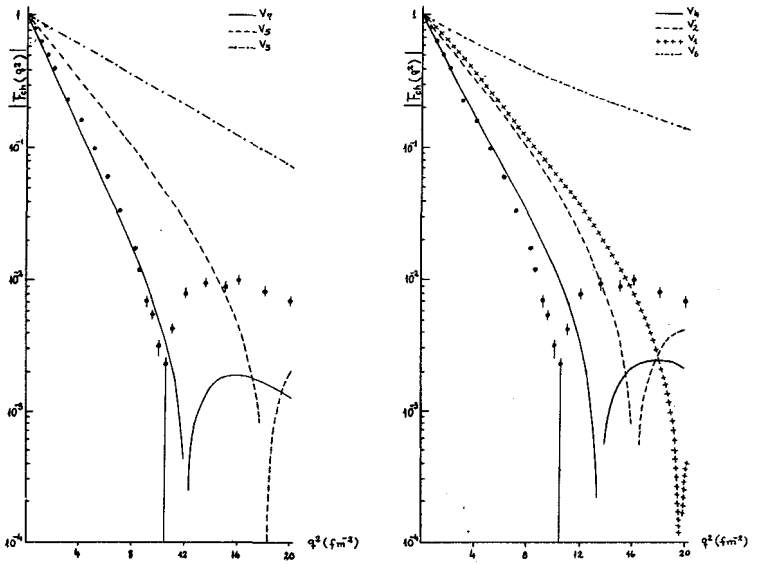

Figure 3 - Charge form factors of the alpha particle lead by the potentials of Tabie 1. The black circles are the experimental values given by Janssens *et al.*[19].

From the charge form factor we obtain the mean squared radius of the charge distribution through the relation.

$$\langle r^2 \rangle_{Ch} = -6 \frac{\partial F_{Ch}(\mathbf{q})}{\partial q^2}\bigg|_{q^2=0} \tag{4.13}$$

Using Eqs. (4.4), (4.7), *(4.8)* and (4.12), Eq. *(4.13)* reduces to

$$\langle r^2 \rangle_{Ch} = \frac{1}{4} \int_0^\infty \rho^2 \, \chi^2(\rho) \, d\rho + 0.6 \, \mathrm{fm}^2.$$

The values of $\sqrt{\langle r^2 \rangle_{Ch}}$ for the potentials of Table *1* are found in Table 2.

5. Discussion of the Results

From Table *2,* one can see that V_1 and V_2 give excellent results for the binding energy, while V_3 and V_6 give reasonable results and the remaining give bad results.

It was quite reasonable to expect that V_1 and V_2 would give good results for the binding energy since they were obtained by fitting the experimental binding energy and the dimension of the alpha particle in calculations by the Hartree-Fock method. But besides that, the values of Table 1 for V_1 and V, show that for the alpha particle, the K = 0 harmonic gives the major contribution to the binding energy, as it already happened in the three nucleon systems[8,9]. In view of that, we should abandon V_4 and V_7 in the higher approximations of the K-karmonics method to the alpha particle since they will give a very low value for the binding energy. We have also to abandon V_3 since it clearly overbinds the system. Potentials V, and V_6 can give good results in higher approximations. Unfortunalely, potential V_5 has no physical interest since it makes the nuclear matter to collapse[20].

About the charge radius, V_1 and V, give reasonable results, while the remaining give values much smaller than the experimental one As the higher approximations tend to spread the system to higher values of ρ, they certainly will raise these values.

About the charge form factor, we see from Fig. 3 that the best results are given by V_4 and V_7, although they are not very good. Unfortunately, these potentials give very low binding energy. Potentials V_1 and V_2, which give the best results for the binding energy, give charge form factors which are not extremely bad and we hope that the higher approximations will improve these results. Comparing Figs. 1 and 3 we see that the more repulsive is the core of W_i, the best is the corresponding charge form factor.

We plan to repeat these calculations for the higher approximations of the K-harmonics in order to study the convergence of the method for the alpha particle and check the conjecture, suggested by the values of the binding energy given by V_1 and V,, that the major contribution to the binding energy comes from the K = 0 terrn of the potential.

Appendix: Hyperspherical Coordinates in $E_{3(A-1)}$

The radial coordinate p is defined as in (2.8), namely

$$\rho^2 = \sum_{i=1}^{A-1} \xi(i)^2. \tag{A.1}$$

The 3A – 4 angular coordinates may be define in several different ways We choose the following definition:

i) $2(A-1)$ angles are taken as the polar and azimuthal angles of the vectors $\xi(i)$, defined in the usual way

$$\begin{aligned}
&\xi(i)_1 = |\xi(i)| \sin\theta_i \cos\phi_i , \\
&\xi(i)_2 = |\xi(i)| \sin\theta_i \sin\phi_i , \\
&\xi(i)_3 = |\xi(i)| \cos\theta_i , \\
&0 \le \theta_i < \pi, \quad 0 \le \phi_i < 2\pi, \quad i = 1, 2, \ldots, \quad (A-1).
\end{aligned} \tag{A.2}$$

ii) $(A-2)$ angles are taken as the interconexion angles among the $E_3(i)$ and $E_{3(A-1)}$, defined by

$$\begin{aligned}
&|\xi(1)| = \rho \cos\lambda_1 , \\
&|\xi(2)| = \rho \sin\lambda_1 \cos\lambda_2 , \\
&|\xi(3)| = \rho \sin\lambda_1 \sin\lambda_2 \cos\lambda_3 , \\
&\quad\vdots \\
&|\xi(A-3)| = \rho \sin\lambda_1 \sin\lambda_2 \ldots \sin\lambda_{A-4} \cos\lambda_{A-3} , \\
&|\xi(A-2)| = \rho \sin\lambda_1 \sin\lambda_2 \ldots \sin\lambda_{A-3} \cos\rho , \\
&|\xi(A-1)| = \rho \sin\lambda_1 \sin\lambda_2 \ldots \sin\lambda_{A-4} \cos\lambda_{A-3} \sin\mu , \\
&0 \le \lambda_i , \mu < \pi/2 , \qquad i = 1, 2, \ldots, (A-3).
\end{aligned}$$

The volume element is

$$dV_{3(A-1)} = \rho^{3A-4} \sin^2\mu \cos^2\mu \, d\rho \, d\mu \prod_{i=1}^{A-3} \cos^2\lambda_i (\sin\lambda_i)^{3A-4-3i} \, d\lambda_i \, d\Omega_{|\xi(i)|} ,$$

from which we obtain the solid angle element

$$d\Omega_\rho = \sin^2\mu \cos^2\mu \, d\mu \prod_{i=1}^{A-3} \cos^2\lambda_i (\sin\lambda_i)^{3A-4-3i} \, d\lambda_i \, d\Omega_{|\xi(i)|} ,$$

where $d\Omega_{|\xi(i)|}$ is the usual three-dimensional solid angle element of the E, of the vector $\xi(i)$.

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