Revista Brasileira de Física, Vo. 4, N.º 1, 1974

Relaxation Rate Spectrum of the Linearized Boltzmann Equation for Hard Spheres: Cases $l = 2$ and $l = 3$.

C. C. YAN

Instituto de Física, Universidade Federal do Rio de Janeiro, Rio de Janeiro GB*

Recebido em **30** de Agosto de 1973

The solutions of the linearized Boltzmann collision operator for hard spheres can be separated into spherical harmonics and investigated respectiveiy ior each angular index l. The fact that an infinite sequence of discrete eigenvalues exists below a continuum has been established elsewhere for the cases $l = 0$ and $l = 1$. For the case $l = 2$, however, the single gas and the foreign gas problems have markedly different spectral profiles below the continuum. For the latter, there is no discrete eigenvalue in that region while there is still an infinite sequence for the former. For the $l = 3$ case, no discrete eigenvalue exists below the continuum for both problems.

Para esferas duras, as soluções do operador de colisão de Boltzmann linearizado podem ser separadas em harmônicos esféricos e investigadas separadamente para cada valor de l. Já foi mostrado que existe, para $l = 0$ e $l = 1$, uma seguência infinita de autovalores discretos abaixo do continuo. Para $l = 2$, todavia, os gases sem e com perturbação apresentam perfis espectrais marcadamente diferentes abaixo do continuo. Para o gás com perturbacão, não há autovalores discretos nessa região, ao passo que para o gás sem perturbação, há uma sequência infinita de tais autovalores. Para $l = 3$, não há autovalores discretos abaixo do continuo em nenhum dos casos.

I. Introduction

In a previous paper[1], we have indicated the difference of the spectral profile between the $l = 2$ case and the $l = 0$ or $l = 1$ case. We conjectured that the discrete relaxation constants or the $l = 2$ case are infinite in number for the single gas problem while only finite for the foreign gas problem and suggested that the method of Kuščer and Williams be tried to resolve the question. In this paper, we shall carry out the actual computation following that suggestion. Furthermore, we shall extend the calculation to include the case of $l = 3$.

*Postal address: Ilha do Fundão, 20.000 -- Rio de Janeiro GB.

2. Existence of Discrete Relaxation Constants

(a) Case $l = 2$

For a spatially uniform hard sphere gas without applied field, the linearized Boltzmann equation is given by[1,3]

$$\frac{\partial}{\partial t}\phi(\mathbf{v}, t) = -V(v)\,\phi(\mathbf{v}, t) + \int K(\mathbf{v}, \mathbf{v}')\,\phi(\mathbf{v}', t)\,d^3v', \tag{1}$$

with

$$V(v) = \pi^{-1/2}\left[\exp(-v^2) + \left(2v + \frac{1}{v}\right)\int_0^v \exp(-x^2)\,dx\right] \tag{2}$$

and

$$K(\mathbf{v}, \mathbf{v}') = \pi^{-3/2}\exp(-v'^2)\left[\frac{2^\gamma}{|\mathbf{v} - \mathbf{v}'|}\exp\left\{\left(\frac{|\mathbf{v}\times\mathbf{v}'|}{|\mathbf{v}-\mathbf{v}'|}\right)^2\right\} - \frac{1}{2}\left[1-(-1)^\gamma\right]|\mathbf{v}-\mathbf{v}'|\right], \tag{3}$$

where v stands for $(m/2kT)^{1/2}$ v, t for $4\pi a^2\rho(2kT/m)^{1/2}\,t$ and γ for the following:

$$\begin{cases} \gamma = 1 & \text{for the single gas problem,} \\ \gamma = 0 & \text{for the foreign gas problem.} \end{cases}$$

Assuming exponentially decaying solution; and decomposing them into spherical harmonics,

$$\phi(\mathbf{v}, t) = \phi_l(v)\,P_l(\cos\theta)\,e^{-\lambda t},$$

we are led to the equation for $l = 2$ (Refs. 1, 4):

$$\begin{aligned}[][V(v) - \lambda]\,\phi_2(v) = &\int_0^v K_2(v, u)\,\phi_2(u)\exp(-u^2)\,u^2\,du \\ &+ \int_v^\infty K_2(u, v)\,\phi_2(u)\exp(-u^2)\,u^2\,du,\end{aligned} \tag{4}$$

with

$$\pi^2 K_2(v,u) = 2^\gamma \Bigg[\frac{u}{(vu)^3}(-3v^2 + 3u^2 - 18)$$

$$+ \frac{1}{(vu)^3}\{3v^2 - 2(vu)^2 + 6u^4 - 15u^2 + 18\} \exp(u^2) \int_0^u \exp(-x^2)\,dx \Bigg]$$

$$+ \{1-(-1)^\gamma\}\left(\frac{2}{15}\frac{u^2}{v} - \frac{2}{35}\frac{u^4}{v^3}\right) \tag{5}$$

Both the single gas and foreign gas problems contain the continuum $2\pi^{-1/2} = \lambda^* \leq \lambda < \infty$, covered by the values of $V(v)$, as a part of their spectra[5,6]. However, we are interested in the discrete spectra below this continuum.

By making the following transformation,

$$\psi_2(v) = v \exp(-v^2/2)[V(v)-\lambda]^{1/2}\,\phi_2(v),$$

we obtain from (4) the equation

$$C^\gamma(\lambda)\,\psi_2(v) = \int_0^v K_{2,\lambda}(v,u)\,\psi_2(u)\,du + \int_v^\infty K_{2,\lambda}(u,v)\,\psi_2(u)\,du, \tag{6}$$

where

$$K_{2,\lambda}(v,u) = \frac{K_2(v,u)}{[V(v)-\lambda]^{1/2}\,[V(u)-\lambda]^{1/2}}\,(vu)\exp\left[-\frac{1}{2}(v^2+u^2)\right], \tag{7}$$

and $C^\gamma(\lambda)$ is a new constant corresponding to the eigenvalue of equation (6). λ being treated as parameter.

As our original problem corresponds to $C^\gamma = 1$, the relaxation constants λ_n which we are searching for, correspond to the roots of $C_n^\gamma(\lambda) = 1$.

As long as $\lambda < \lambda^* = 2\pi^{-1/2}$, it can be verified directly or inferred from the results of Grad' and Dorfman[7] that $K_{2,\lambda}(v,u)$ is a compact (completely continuous) self-adjoint linear operator in the Hilbert space $L_2(0,\infty)$. Therefore, for any $\lambda < \lambda^*$, there exists a nonempty set of real discrete eigenvalues $C_n^\gamma(\lambda)$, not accumulating at any nonzero value[8]. Moreover, it is continuous, monotonic and cannot change sign due to the fact that

$$\frac{1}{C_n}\frac{dC_n}{d\lambda} = \int_0^\infty \frac{\psi_{2,n}^2(v)\,dv}{V(v)-\lambda} \tag{8}$$

holds, as can be easily verified from *(6)*. However. as $\lambda = \lambda^*$. $K_{2,\lambda*}(v, u)$ is no longer compact, because of the singularity at $u \to 0$, $u \to 0$ created by the factor

$$[V(v)-\lambda^*]^{-1/2}\,[V(v)-\lambda^*]^{-1/2} \approx \frac{3}{2}\,\frac{\pi^{1/2}}{uv}$$

This limiting kernel can be represented as a sum

$$K_{2,\lambda*}(v, u) = H_{2,\lambda*}(v, u) + \Delta_2(v, u),$$

where $H_{2,\lambda*}(v, u)$ is a simplified model kemel which takes away that singularity so that the remaining term *A,* is compact.

By expanding the expression (7) in powers of v and **u.** and by imposing an arbitrary cut off speed v_1. we are led to the following choice:

$$\begin{aligned} H_{2,\lambda*}(v, u) &= 2^\gamma\,\frac{6}{5}\,\frac{u^2}{v^3}, && u < v < v_1, \\ &= 2^\gamma\,\frac{6}{5}\,\frac{v^2}{u^3}, && v < u < v_1, \\ &= 0, && v, u > v_1. \end{aligned} \tag{9}$$

Substituting $H_{2,\lambda*}(v, u)$ for $K_{2,\lambda*}(v, u)$, Eq. *(6)* becomes

$$C^\gamma(\lambda)\,\psi_2(v) = 2^\gamma\,\frac{6}{5}\left\{v^{-3}\int_0^v u^2\,\psi_2(u)\,du + v^2\int_v^{v_1} u^{-3}\,\psi_2(u)\,du\right\} \tag{10}$$

Applying a theorem due to Vidav[9] (s. Appendix) to Eq. 10. we readily see that the spectrum of $H_{2,\lambda*}(v, u)$ consists cf all the points of the interval $[0, C_2^*]$, where $C_2^* = 2^\gamma\, 24/25$. Hence, the spectrum of $K_{2,\lambda*}(v, u)$ also consists of the interval [0. C]. It has been shown[2] that $K_\lambda(v, u)$ converges strongly toward $K_{\lambda*}(v, u)$ and that the eigenvalues C_i. consequently. must fill the interval [O, C*] ever more densily as $\lambda \to \lambda^*$. Remembering that C is continuous and monotonic. we are led to the conclusion that an infinite number of eigenvalues $C_n(\lambda)$ must cross the line C = 1 as $\lambda \to \lambda^*$ for the single gas case. as $C_2^* = 2(24/25) > 1$ for that case. The crossing points mark an infinite set of discrete relaxation constant R_i. On the other hand, as $C_2^* = (24/25) < 1$ for the foreign gas case, no eigenvalue $C_n(\lambda)$ can reach the value 1. Therefore, no discrete relaxation constant below the continuum exists for the foreign gas problem. However, the question of possible discrete eigenvalues inside the continuum remains open. The situation below the continuum, for the $1 = 2$ case, can be illustrated by Figs. 1 and 2.

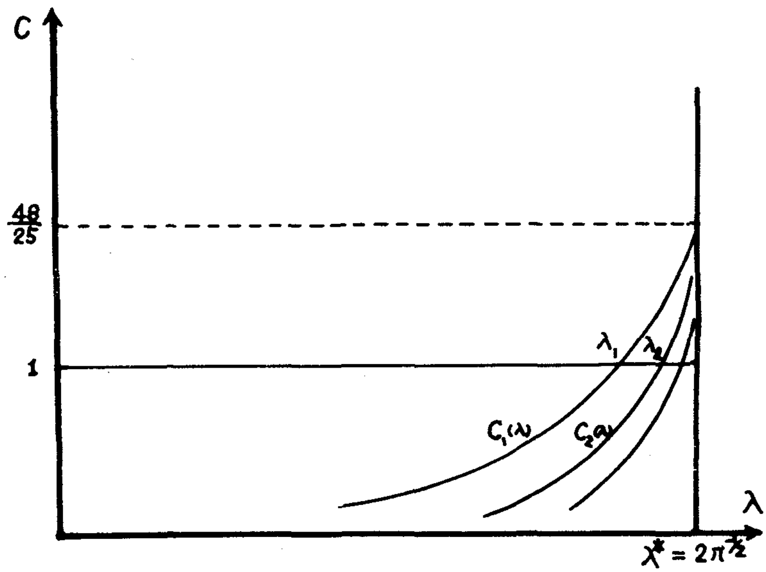

Fig. 1 - Schematic picture of eigenvalues $C_n(\lambda)$ for a single hard-sphere gas showing the crossing points λ_n as the discrete relaxation constants below the continuum. (not to scale).

(b) Case $1 = 3$

For this case, the kernel of an equation similar to (4), obtainable from (1), is given by[10]

$$\pi^{1/2} K_3(v, u) = 2^{\gamma} \frac{1}{(vu)^4} \Big\{ \left[(150 + 30v^2)\, u - (20 + v^2)\, u^3 + 5u^5\right]$$

$$+ \left[-(150 + 30v^2) + (120 + 21v^2)\, u^2 - (45 + 6v^2)\, u^4 + 10u^6\right]$$

$$\exp(u^2) \int_0^u \exp(-x^2)\, dx \Big\} + \left[1 - (-1)^{\gamma}\right] \left(\frac{2}{35} \frac{u^3}{v^2} - \frac{2}{63} \frac{u^5}{v^4}\right) \qquad (11)$$

and the limiting kernel similar to (9) is given by

$$
\begin{aligned}
H_{3,\lambda*}(v,u) &= 2^{\gamma}\frac{6}{7}\,\frac{u^3}{v^4}, && v_1 > v > u,\\
&= 2^{\gamma}\frac{6}{7}\,\frac{v^3}{u^4}, && v_1 > u > v,\\
&= 0, && \text{otherwise.}
\end{aligned} \tag{12}
$$

Therefore, the equation corresponding to (10) now reads

$$
C^{\gamma}\,\psi_3(v) = 2^{\gamma}\frac{6}{7}\left\{v^{-4}\int_0^{v} u^3\,\psi_3(u)\,du + v^3\int_v^{v_1} u^{-4}\,\psi_3(u)\,du\right\} \tag{13}
$$

and the spectrum of $K_{3,\lambda*}(v, u)$ consists of the interval $[0, C_3^*]$, with $C_3^* = 2^{\gamma}(24/49)$.

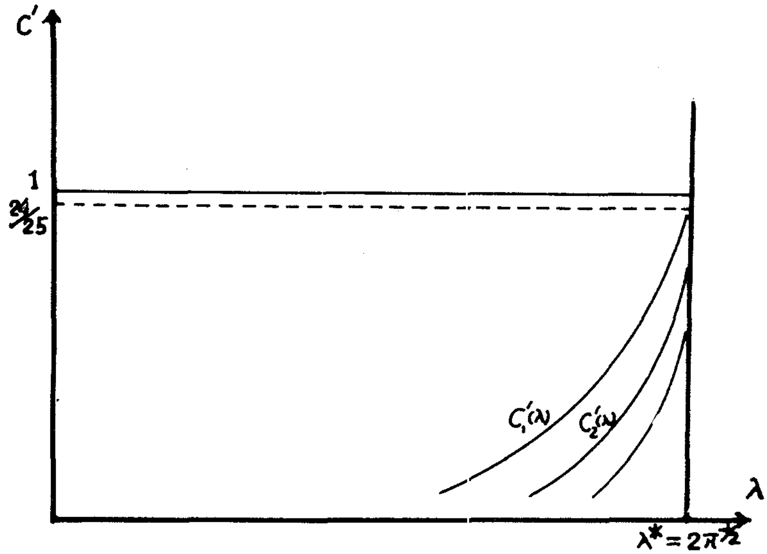

Fig. 2 - Schematic picture of eigenvalues $C_n'(\lambda)$ for a foreign hard-sphere gas showing no discrete relaxation constant below the continuum. (not to scale).

Now, for the single gas problem $C_3^* = (43/49) < 1$ and, for the foreign gas problem, $C_3^* = (24/49) < 1$, so we carinot expect $C(\lambda)$ to reach the value 1. Therefore, we conclude that in the case of $l = 3$ there is no discrete eigenvalue below the continuum for both the single and foreign gas problems. The situation is similar to that illustrated by Fig. 2.

The author wishes to thank Professor G. H. Wannier for his critical comments, Professor N. Corngold for calling his attention to the work of Kuščer and Professor I. Kuščer for offering him suggestions and referring him to the work of Vidav.

References

1. C. C. Yan, Phys. of Fluids 12, 2306 (1969).
2. I. Kuščer and M. M. R. Williams, Phys. Fluids 10, 1922 (1967).
3. D. Hilbert, Math. Ann. **72,** 562 (1912).
4. C. L. Pekeris, Proc. Nat'l Acad. Sci. U.S.A. 41, 611 (1955).
5. H. Grad, in *Rarefied Gas* Dvnamics. J. A. Laurmann, ed., Academic Press, Inc. New York, 1963, vol. I, p. 26.
6. G. W. Ford and M. Schreiber, Proc. Nat'l Acad. Sci., U.S.A., ***60,*** 802 (1968).
7. R. Dorfman, Proc. Nat'l Acad. Sci., U.S.A., 50, 804 (1963).
8. F. Riesz and B. Sz-Nagy, Functional Analvsis, Ungar Publishing Co., New York, 1955.
9. Ivan Vidav, On an Integral Operator, a preprint of May, 1967.
10. C. L. Pekeris, Z. Alterman, L. Findelstein and K. Frankowski, Phys. Fluids 5, 1608 (1962). There is misprint on page 1611. In table II, the next to the last term in the top line should read $(45 + 6p^2)p_1^4$ instead of $(45 + 5p^2)p_1^4$.
11. G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities. Cambridge University Press, 1952, pp. 175-176.

Appendix

In this appendix. Vidav's theorem is reproduced in a slightly generalized form

Let us define an integral operator T, acting in the Hilbert space $L_2(0, 1)$, by

$$T\phi(x) = \alpha_{l-1}\, x^{-l} \int_0^x y^{l-1}\, \phi(y)\, dy \tag{A-1}$$

and its adjoint operator T^* by

$$T^*\phi(x) = \alpha_{l-1}\, x^{l-1} \int_x^1 y^{-l}\, \phi(y)\, dy. \tag{A-2}$$

The operator T is not compact but is known to be bounded[11].

It can be easily verified that, for any complex number m, with Rem $> -1/2$,

$$e_m(x) = (1 + 2\gamma)^{1/2}\, x^m, \qquad \gamma = \mathrm{Re}\, m > -1/2, \tag{A-3}$$

is an eigenfunction of T, with norm 1, corresponding to the eigenvalue $\lambda = \alpha_{l-1}(l + m)^{-1}$.

According to (A-1) and (A-2), the products TT^* and T^*T are the following operators:

$$TT^*\phi(x) = \alpha_{l-1}^2\, x^{-l} \int_0^x z^{2l-2}\, dz \int_z^1 y^{-l}\, \phi(y)\, dy, \tag{A-4a}$$

$$T^*T\phi(x) = \alpha_{l-1}^2\, x^{l-1} \int_x^1 z^{-2l}\, dz \int_0^z y^{l-1}\, \phi(y)\, dy. \tag{A-4b}$$

By changing the order of integration, we obtain

$$TT^*\phi(x) = \frac{\alpha_{l-1}}{2l-1}(T + T^*)\phi(x), \tag{A-5a}$$

$$T^*T\phi(x) = \frac{\alpha_{l-1}}{2l-1}[T + T^* - P]\phi(x),$$

where

$$P\phi(x) = \alpha_{l-1}\, x^{l-1} \int_0^1 y^{l-1}\,\phi(y)\,dy.$$

Consider now the sum $A = T + T^*$. This is a bounded self-adjoint operator defined by

$$A\phi(x) = \alpha_{l-1}\left\{x^{-l}\int_0^x y^{l-1}\,\phi(y)\,dy + x^{l-1}\int_x^1 y^{-l}\,\phi(y)\,dy\right\}. \tag{A-6}$$

Let us put

$$T^*e_m = f_m, \tag{A-7}$$

e_m being the function (A-3). Since $Te_m = \lambda e_m$ with $\lambda = \alpha_{l-1}(l + m)^{-1}$, we obtain by applying T^* on both sides of this equation and using (A-5),

$$\lambda f_m = \frac{\alpha_{l-1}}{2l-1}(\lambda e_m + f_m - Pe_m) \tag{A-8}$$

or

$$T^*e_m = f_m = \left(\frac{2l-1}{\alpha_{l-1}}\lambda - 1\right)^{-1}(\lambda e_m - Pe_m), \tag{A-9}$$

with

$$Pe_m = \lambda(1 + 2\gamma)^{1/2}\, x^{l-1}$$

Therefore,

$$\begin{aligned} Ae_m &= (T + T^*)e_m = \lambda e_m + f_m \\ &= \lambda\left[1 + \left(\frac{2l-1}{\alpha_{l-1}}\lambda - 1\right)^{-1}\right]e_m - \left(\frac{2l-1}{\alpha_{l-1}}\lambda - 1\right)^{-1} Pe_m. \end{aligned} \tag{A-10}$$

Let us write $m = r + is$. If $\gamma \to -1/2$, s being fixed, then the term

$$\left(\frac{2l-1}{\alpha_{l-1}}\lambda - 1\right)^{-1} Pe_m = \left(\frac{2l-1}{\alpha_{l-1}}\lambda - 1\right)^{-1}\lambda(1 + 2\gamma)^{1/2}\, x^{l-1}$$

tends to zero and the expression $\lambda\left[1 + \left(\frac{2l-1}{\alpha_{l-1}}\lambda - 1\right)^{-1}\right]$ converges to the value $4(2l-1)\,\alpha_{l-1}/[(2l-1)^2 + 4s^2]$. Hence, the norm of the function

$$Ae_m - \frac{4(2l-1)\,\alpha_{l-1}}{(2l-1)^2 + 4s^2}\, e_m$$

is arbitrarily small if γ is sufficiently close to $(-1/2)$. It follows that e_m is an approximate eigenfunction of A and the number $4(2l-1)\,\alpha_{l-1}/[(2l-1)^2 + 4s^2]$ belongs to the spectrum of A. Since this holds for each real s, the spectrum of A consists of all the points of the interval $\left[0, \frac{4\alpha_{l-1}}{2l-1}\right]$.