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Production of Heat in a Finite Circular Cylinder with Radiation Boundary Conditions

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An "interior value problem" for the production of heat in a finite circular cylinder is solved by use of Laplace and finite Hankel transforms.

O problema da produção de calor por um cilindro, considerado como "problema de valor interior", e resolvido, fazendo-se uso das transformadas de Laplace e de Hankel (finita).

1. In heat engines, cylindrical solids have an important role to play and hence a study of the temperature variation of these cylindrical solids which are used in the working of compound engines, air compressors, ordinary steam engines and internal combustion engines' are of great use.

Many authors have attempted to solve problems of heat conduction in which boundary conditions are prescribed (such as temperature or heat flux) and the temperature at internal points is required. Such problems (which may be called "direct problems") can be solved by classical methods[1]. Masket and Vastano[6] have solved a problem of transient heat conduction in which one is required to find the temperature or heat flux at the surface and named such probiems "interior value problems" as here one determines boundary values from interior values. Recently, Kalla[3] and Kalla and Battig" have also considered such type of problems.

The object of the present paper is to consider a problem of transient heat conduction in a finite circular cylinder which is generating heat with a given temperature distribution on any interior plane which is normal to the axis of the cylinder, and to determine the temperature at any point on one of the flat surfaces of the cylinder when there is radiation from the cylindrical surface into a medium at a given temperature.

The solution has been obtained by an appeal to Lapiace and finite Hankel transforms[S]. Severa1 interesting particular cases are also mentioned.

2. We shall denote the classical Laplace tarisform of a function $f(r,z.t)$ as

$$\bar{f}(r,z,p) = \int_0^\infty e^{-pt} f(r,z,t)\,dt. \tag{1}$$

The finite Hankel transform of a function $f(r,z,t)$ is defined as[8]

$$f_J(\alpha_i, z, t) = \int_0^a r\, f(r,z,t)\, J_0(\alpha_i r)\, dr, \tag{2}$$

where α_i is a root of the equation

$$\alpha_i J_0'(\alpha_i a) + h\, J_0(\alpha_i a) = 0. \tag{3}$$

If f satisfies Dirichlet's conditions in the closed interval $[0, a]$ and if its finite Hankel transform is defineú as in (2) in which α_i is a root of the transcendental equation (3), then

$$f(r,z,t) = \frac{2}{a^2}\sum_i \frac{\alpha_i^2\, f_J(\alpha_i, z, t)}{h^2 + \alpha_i^2}\, \frac{J_0(r\alpha_i)}{[J_0(a\alpha_i)]^2}, \tag{4}$$

the sum being taken over all positive root:; of Eq. (3).

3. Let us consider the radial and axial heat flows in a finite circular cylinder bounded by the surfaces $z = 0$, $z = h$ and $r = a$, which is generating heat and is initially at temperature $n(r, z)$.

Thus our problem may be described mathematically as: to obtain the solution of the partial differential equation

$$\frac{\partial\theta}{\partial t} = k\left\{\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2}\right\} + A(r,z,t), \tag{5}$$

subject to the following conditions:

$$\theta(r,0,t) = m(r,t), \quad z = 0, \quad t > 0, \quad \text{to be found out} \tag{6}$$

$$\theta(r,h,t) = p(r,t), \quad z = h, \quad t > 0, \tag{7}$$

$$\left(\frac{\partial\theta}{\partial r} + h\theta\right)_{r=a} = h\, q(z,t), \quad r = a, \quad t > 0, \tag{8}$$

$\theta(r,s,t)$, $0 < s < h$, is a known function.

Taking the finite Hankel transform of both sides of Eq. 5, we get

$$\frac{\partial \theta_J}{at} = k\left\{\frac{\partial^2 \theta_J}{\partial z^2} + a\, J_0(a\alpha_i)\, hq(z,t) - \alpha_i^2\, \theta_J\right\} + A_J(\alpha_i, z, t), \tag{10}$$

with corresponding conditions

$$\begin{aligned}
&\theta_J(\alpha_i, 0, t) = m_J(\alpha_i, t), && z > 0, && t > 0,\\
&\theta_J(\alpha_i, h, t) = p_J(\alpha_i, t) && z = h, && t > 0,\\
&\theta_J(\alpha_i, z, O) = n_J(\alpha_i, z), && t = 0.
\end{aligned}$$

Now, by an appeal to the Laplace transform, Eq. *(10)* reduces to an ordinary differential equation

$$\frac{d^2\bar{\theta}_J}{dz^2} - \left(\alpha_i^2 + \frac{p}{k}\right)\bar{\theta}_J = -a\, J_0(\alpha_i a)\, hq_J(\alpha_i, p) -$$

$$-\frac{1}{k} n_J(\alpha_i, z) - \frac{1}{k}\bar{A}_J(\alpha_i, z, p), \tag{12}$$

whose solution is

$$\bar{\theta}_J(\alpha_i, z, p) = \bar{m}_J(\alpha_i, p)\frac{\sinh\left[(h-z)(\alpha_i^2 + p/k)^{1/2}\right]}{\sinh\left[h\left(\alpha_i^2 + \frac{p}{k}\right)^{1/2}\right]} +$$

$$+\ \bar{p}_J(\alpha_i, p)\frac{\sinh\left[z(\alpha_i^2 + p/k)^{1/2}\right]}{\sinh\left[h(\alpha_i^2 + p/k)^{1/2}\right]}$$

$$-\frac{1}{f(D)}\left[a\, J_0(\alpha_i a)\, h\bar{q}(z,p) + \frac{1}{k} n_J(\alpha_i z) + \frac{1}{k}\bar{A}_J(\alpha_i, z, p),\right. \tag{13}$$

where D is the usual differential operator and $[1/f(D)]z$ is that function of z which, when operated upon by $f(D)$, gives z . $f(D) \equiv D^2 - \left(\alpha_i^2 + \frac{p}{k}\right)$.

Now, the inverse Laplace transform of

$$\frac{\sinh\left[(h-z)\left(\alpha_i^2+\frac{p}{k}\right)^{1/2}\right]}{\sinh\left[h\left(\alpha_i^2+\frac{p}{k}\right)^{1/2}\right]}$$

is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\sinh\left[(h-z)\left(\alpha_i^2+\frac{p}{k}\right)^{1/2}\right]}{\sinh\left[h\left(\alpha_i^2+\frac{p}{k}\right)^{1/2}\right]}e^{pt}\,dp.$$

The integrand has simple poles at $p=-k(\alpha_i^2h^2+m^2\pi^2)/h^2$, $m=0,1,2,\ldots,$ with the following residues:

$$\frac{2\pi km}{h^2}\sin\left(\frac{m\pi z}{h}\right)\exp\left[-\frac{\alpha_i^2h^2+m^2\pi^2}{h^2}kt\right]. \tag{14}$$

Similarly. the inverse Laplace transform of

$$\frac{\sinh\left[z\left(\alpha_i^2+\frac{p}{k}\right)^{1/2}\right]}{\sinh\left[h\left(\alpha_i^2+\frac{p}{k}\right)^{1/2}\right]}$$

is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\sinh\left[z\left(\alpha_i^2+\frac{p}{k}\right)^{1/2}\right]}{\sinh\left[h\left(\alpha_i^2+\frac{p}{k}\right)^{1/2}\right]}e^{pt}\,dp.$$

Here, also the integrand has simple poles at

$$p=-k(\alpha_i^2h^2+m^2\pi^2)/h^2. \qquad m=0,1,2,\ldots,$$

with residues

$$(-1)^m \frac{2\pi k m}{h^2} \sin\left(\frac{m\pi z}{h}\right) \exp\left[-\frac{\alpha_i^2 h^2 + m^2\pi^2}{h^2} kt\right]. \tag{15}$$

Combining Eqs. *(14)* and *(15)* with the inverse of $\bar{m}(\alpha_i, p)$ and $\bar{p}(\alpha_i, p)$, respectively, *by* the product theorem (Ref. *7*, p. 38), we have

$$\bar{\theta}(\alpha_i, z, t) =$$

$$\frac{2\pi k}{h^2} \sum_{m=1}^{\infty} m \sin\left(\frac{m\pi z}{h}\right) \int_0^t \bar{m}(\alpha_i, T) \exp\left\{-k\left[\alpha_i^2 + \left(\frac{m\pi}{h}\right)^2\right](t-T)\right\} dT$$

$$+ \frac{2\pi k}{h^2} \sum_{m=1}^{\infty} (-1)^m m \sin\left(\frac{m\pi z}{h}\right) \int_0^t \bar{p}(\alpha_i, T) \exp\left\{-k\left[\alpha_i^2 + \left(\frac{m\pi}{h}\right)^2\right](t-T)\right\} dT$$

$$-\mathscr{L}^{-1}\left[\frac{1}{f(D)}\left\{a J_0(\alpha_i a) h \bar{q}(z, p) + \frac{1}{k} \bar{n}_J(\alpha_i, z) + \frac{1}{k} \bar{A}_J(\alpha_i, z, p)\right\}\right], \tag{16}$$

where $\mathscr{L}^{-1}$ is the inverse Laplace operator.

Replacing z by s in Eq. (13), we get

$$\bar{m}_J(\alpha_i, p) = \bar{\theta}_J(\alpha_i, s, p) \frac{\sinh\left[h\left(\alpha_i^2 + \frac{p}{k}\right)^{1/2}\right]}{\sinh\left[(h-s)\left(\alpha_i^2 + \frac{p}{k}\right)^{1/2}\right]} -$$

$$-\bar{p}_J(\alpha_i, p) \frac{\sinh\left[s\left(\alpha_i^2 + \frac{p}{k}\right)^{1/2}\right]}{\sinh\left[(h-s)\left(\alpha_i^2 + \frac{p}{k}\right)^{1/2}\right]} +$$

$$+ \frac{\sinh\left[h\left(\alpha_i^2 + \frac{p}{k}\right)^{1/2}\right]}{\sinh\left[(h-s)\left(\alpha_i^2 + \frac{p}{k}\right)^{1/2}\right]} \chi_L(s, p), \tag{17}$$

where

$$\chi_L(s, p) = \frac{1}{f(D)}\left[a\, J_0(\alpha_i a)\, h\bar{q}(s, p) + \frac{1}{k}\,\bar{n}(\alpha_i, s) + \frac{1}{k}\,\bar{A}_J(\alpha_i, s, p)\right].$$

Now, following the same procedure as we have in Eq. (16), we obtain the inverse Laplace transform of Eq. *(17)* as

$$m_J(\alpha_i, t) = \frac{2\pi k}{(h-s)^2} \sum_{m=1}^{\infty} (-1)^{m+1}\, m \left\{ \int_0^t \exp\left[-k\left(\alpha_i^2 + \left(\frac{m\pi}{h-s}\right)^2\right)(t-T)\right]\right.$$

$$\left[\sin\frac{m\pi h}{h-s}\,\theta_J(\alpha_i, s, T) + \sin\frac{m\pi s}{h-s}\, p_J(\alpha_i, T)\right.$$

$$\left. + \sin\frac{m\pi h}{h-s}\,\chi(s, T)\right]\cdot dT,$$

where

$$\chi(s, T) = \mathscr{L}^{-1}[\chi_{\mathscr{L}}(s, p)],$$

which reduces to the following form by use of the Hankel inversion theorem[4]

$$m(r, t) = \frac{4\pi k}{a^2(h-s)^2} \sum_i \frac{J_0(r\alpha_i)}{[J_0(a\alpha_i)]^2}\, \frac{\alpha_i^2}{h^2 + \alpha_i^2} \sum_{m=1}^{\infty} (-1)^{m+1}\, m\, \cdot$$

$$\int_0^t \exp\left[-k\left(\alpha_i^2 + \left(\frac{m\pi}{h-s}\right)^2\right)(t-T)\right]\left[\sin\frac{m\pi h}{h-s}\,\theta_J(\alpha_i, s, T) +\right.$$

$$\left. + \sin\frac{m\pi s}{h-s}\, p_J(\alpha_i, T) + \sin\frac{m\pi h}{h-s}\,\chi(s, T)\right] dT, \tag{19}$$

where the summation is taken over all the positive roots of Eq. **(3).**

If $m(r, t)$ is known and the value of $\theta(r, z, t)$ is to be determined, i.e., the temperature at any interval point is required, then the solution can be obtained by applying the Hankel inversion theorem to *(16)* to obtain the following expression:

$$\theta(r, z, t) = \frac{4\pi k}{a^2 h^2} \sum_i \frac{J_0(r\alpha_i)}{[J_0(a\alpha_i)]^2}\, \frac{\alpha_i^2}{h^2 + \alpha_i^2}\cdot$$

$$\left\{ \sum_{m=1}^{\infty} (-1)^m m \sin\frac{m\pi z}{h} \int_0^t \exp\left[-k\left(\alpha_i^2 + \left(\frac{m\pi}{h}\right)^2\right)\right](t-T)\, \bar{m}(\alpha_i, T)\,dT \right.$$

$$\left. + \sum_{m=1}^{\infty} m \sin\frac{m\pi z}{h} \int_0^t \exp\left[-k\left(\alpha_i^2 + \left(\frac{m\pi}{h}\right)^2\right)(t-T)\right]\bar{p}(\alpha_i, T)\,dT \right\}$$

$$-\frac{2}{a^2}\sum_i \frac{J_0(\alpha_i r)}{[J_0(\alpha_i a)]^2}\,\frac{\alpha_i^2}{h^2+\alpha_i^2}\,\mathscr{L}^{-1}\,.$$

$$\cdot\left[\frac{1}{f(D)}\left\{a\,J_0(\alpha_i a)\,h\bar{q}(z,p) + \frac{1}{k}\bar{n}_J(\alpha_i z) + \frac{1}{k}\bar{A}_J(\alpha_i, z, p)\right\}\right]\cdot \tag{20}$$

4. Let us consider some particular cases of the general result (19).

If we set that there are no sources of heat present in the cylinder, the surface $z = h$, $t > 0$ is maintained at zero temperature, that there is radiation into a medium at zero temperature and initially the cylinder is at a temperature $\varepsilon s(0 < s < h)$, then the expression for the temperature distribution at the surface $z = 0$, at any instant, can be given in a simplified form (by using a result given in Ref. 2, p. 229) which reads

$$m(r,t) = \frac{4\pi k}{a^2(h-s)^2}\sum_i \frac{\alpha_i^2}{h^2+\alpha_i^2}\,\frac{J_0(\alpha_i r)}{[J_0(\alpha_i a)]^2}\,\frac{s}{\alpha_i}\,J_1(\alpha_i a)\,.$$

$$\sum_{m=1}^{\infty}(-1)^{m+1} m \sin\frac{m\pi h}{h-s}\,\cdot$$

$$\cdot\left\{\mu a\,\frac{1-\exp(-\gamma_i^2 t)}{\gamma_i^2} - \varepsilon a[1-(-1)^m]\,\frac{\exp(-\alpha_i^2 kt)-\exp(-\gamma_i^2 t)}{\gamma_i^2-\alpha_i^2 k}\right\} \tag{21}$$

where $\gamma_i^2 = k\left[\alpha_i^2 + \left(\frac{m\pi}{h-s}\right)^2\right]$, $\theta(r, s, t) = \mu s$, μ being a constant and the first summation is taken over all positive roots of Eq. (3).

Similarly, if we set $p(r,t) = q(z,t) = n(r,t) = 0$, $A(r,s,t) = \delta \cdot s$ and $\theta(r,s,t) = \mu s$, then, using Ref. 2 (p. 232), Eq. (19) reduces to the following form:

$$m(r,t) = \frac{4\pi k}{a^2(h-s)^2} \sum_i \frac{\alpha_i^2}{h^2+\alpha_i^2} \frac{J_0(\alpha_i r)}{[J_0(\alpha_i a)]^2} \frac{as}{\alpha_i} J_1(a\alpha_i) \cdot$$

$$\cdot \sum_{m=1}^{\infty} (-1)^{m+1} m \sin\frac{m\pi h}{h-s} \left\{ \left[\mu - \frac{\delta}{\alpha_i^2}(1-(-1)^m) \right] \frac{1-\exp(-\gamma_i^2 t)}{\gamma_i^2} \right.$$

$$\left. + \frac{\delta}{\alpha_i^2}(1-(-1)^m) \frac{\exp(-\alpha_i^2 kt) - \exp(-\gamma_i^2 t)}{\gamma_i^2 - \alpha_i^2 k} \right\}. \tag{22}$$

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