Revista Brasileira de Física, Vol. 3, N.° 3, 1973

Conformal Symmetry in Lagrangian Field Theory

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Recebido em 2 de Junho de 1973

Conformal symmetry in Lagrangian field theory is discussed for Lagrangians with derivatives up to first order. Conditions for invariance and covariance of the Lagrangian and for expressing the conformal currents as moments of an "improved" energy momentum tensor are discussed.

Discute-se a simetria conforme na formulação lagrangiana da teoria de campos para o caso de lagrangianas com derivadas somente até a 1.ª ordem. Discutem-se as demais condições para. invariância e covariância da Lagrangiana, como também condições que permitam expressar as correntes conformes, na forma de momentos do tensor de energia-momento "melhorado".

1. Introduction

The idea of approximate symmetry with respect to dilatation and to the special conformal transformation group of hadronic interactions has drawn renewed interest in recent yeais. This development arose out of the experimentally observed *scalling* at high energies, which suggests the possibility of a dynamical limit where dimensíonal quantities become unimportant. The other important motivation has been the possibility of explaining, at least in part, the masses of the stable particles as arising from spontaneous breakdown of dilatation invariance.

We discuss here symmetry of a Lagrangian field theory with respect to scale and special conformal transformation. The Lagrangian is assumed to contain derivatives not higher than the first. Distinction is made between the cases in which the infinitesimal quantity $[\delta\mathscr{L}]$ defined in Eqn. (2.16) vanishes ***(invariance)*** and the case in which it is only a divergence ***(covariance).***

It is shown that in both cases the "weak" conserved currents derived from Noether's theorem can be cast as moments of the 'improved' energy mo-

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mentum tensor. We find also necessary and sufficient conditions for *invariance* condition to hold and that a Poincare invariant theory is invariant (covariant) simultaneouly, with respect to both scale and special conformal transformations if the conformal deficienty vector V^{λ} vanishes.

In Section 2, we review the Lagrangian field theory and Noether's theorem. In Section 3, we discuss the variation of the field corresponding to infinitesimal conformal transformations. In Section 4, conformal currents are constructed and the conditions of invariance and covariance of Lagrangians under infinitesimal transformations as well as the condition for expressing currents as moments of improved energy momentum tensor are discussed. Dilatation symmetry is discussed in some detail. In Section 5, applications are made to spin 0,1/2 and 1 field theories and a short Section 6 is devoted to the presence of fields with anomalous scale transformations.

2. Review of Lagrangian Field Theory and Nother's Theorem[1]

a) Notation

We will consider a classical field theory in four-dimensional space-time. The dynamical system is described by N field components $\phi_A(x)$, $A = 1, 2, \ldots N$ – the dependent variables – which are functions of the independent variables $x = (x^0, x^1, x^2, x^3)$. We assume that a Lagrangian density function $\mathscr{L}$ can be defined as a function of x^μ, $\phi_A(x)$ and derivatives of $\phi_A(x)$ only up to first order.

The action integral is given by

$$J[\phi_1, \ldots, \phi_N] = \int_a^b dx^0 \int_R d^3x\, \mathscr{L}(x, \phi, \partial\phi)$$
$$= \int_\Omega d^4x\, \mathscr{L}(x, \phi, \partial\phi), \qquad (1)$$

where R is a three dimensional region and Ω is a cylindrical space-time region[2]. We use here the metric $g^{\mu\mu} = g_{\mu\mu} = (1, -1, -1, -1)$, $g^{\mu\nu} = g_{\mu\nu} = 0$ $(\mu \neq \nu)$.

The dynamical equations are then obtained from Hamilton's principle by requiring that the functional $J[\phi_1, \ldots \phi_N]$ be an extremum for all admissible variations $\delta\phi_A$, with region Ω kept fixed (e.g. $\delta x^\mu = 0$). By considering the particular case of $\delta\phi_A$ which vanish on the boundary of Ω, we obtain Euler-Lagrange differential equations

$$[\mathscr{L}]_A \equiv -\frac{\partial\mathscr{L}}{\partial\phi_A} + \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu\phi_A}\right) = 0. \tag{2}$$

Here $\partial_\mu F = \frac{a}{\partial x^\mu} F$ are the usual partial derivatives where coordinates other than x^μ are kept constant. We will use $\partial_\mu F\,|$ to indicate partial derivatives w.r.t. x^μ, which regards coordinates other than x^p, ϕ_A and all $\partial^\lambda\phi_A$ as constants[3]. For convenience of notation, we introduce the vector $\phi = (\phi_1, \phi_2, \ldots \phi_N)$ and tensor $\nabla\phi$ with components $\partial_\mu\phi_A$, so that

$$J[\phi] = \int_\Omega \mathscr{L}(x, \phi, \nabla\phi)\,dx \tag{3}$$

and

$$-\frac{\partial\mathscr{L}}{\partial\phi} + \partial_\mu\pi^\mu = 0, \tag{4}$$

where

$$\pi^\mu_A = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_A)}, \qquad \pi^\mu = (\pi^\mu_1, \ldots \pi^\mu_N). \tag{5}$$

We assume throughout that partial derivativa of $\mathscr{L}$ exist up to seconcl order w.r.t. all its arguments and are continuous.

b) Noether's Theoren

We now consider arbitrary infinitesimal transformations

$$x'^\mu = x^\mu + \delta x^\mu, \qquad \phi'_A(x) = \phi_A(x) + \bar{\delta}\phi_A + \ldots, \tag{6}$$

or

$$\bar{\Delta}\phi_A(x) \equiv \phi'_A(x) - \phi_A(x) \simeq \bar{\delta}\phi_A + \ldots, \tag{7}$$

where

$$\delta x^\mu = \sum_{k=1}^{r} \varepsilon_k\, C^\mu_{(k)}(x, \phi, \nabla\phi)$$

and

$$\bar{\delta}\phi_A = \sum_{k=1}^{r} \varepsilon_k \bar{B}_A^{(k)}(x, \phi, \nabla\phi) \tag{8}$$

are arbitrary functions of **x,** ϕ, $\nabla\phi$ and E,, **(k** = 1, 2, . . . , r), are the r essential parameters of the transformation. We also introduce

$$\Delta\phi_A(x) \equiv \phi'_A(x') - \phi_A(x) = \delta\phi_A + \ldots \tag{9}$$

with

$$\delta\phi_A = \sum_{k=1}^{r} \varepsilon_k B_A^{(k)}(x, \phi, \nabla\phi). \tag{10}$$

It is easily shown that

$$\bar{\delta}(\partial_\mu\phi) = \partial_\mu(\bar{\delta}\phi), \qquad \delta\phi = \bar{\delta}\phi + (\partial_\mu\phi)\,\delta x^\mu,$$
$$\delta(\partial_\mu\phi) = \bar{\delta}(\partial_\mu\phi) + (\partial_\nu\partial_\mu\phi)\,\delta x^\nu = \partial_\mu(\delta\phi) - (\partial_\nu\phi)\,\partial_\mu(\delta x^\nu). \tag{11}$$

These relations lead to a relation between the functions *B*, B and C. The transformation carries $J[\phi]$ to

$$J[\phi'] = \int_{\Omega'} \mathscr{L}[x', \phi'(x'), \nabla'\phi'(x')]\,dx'$$
$$= \int_\Omega \mathscr{L}(x', \phi', \nabla', \phi')\left|\frac{\partial x'}{\partial x}\right| dx = \int_\Omega \mathscr{L}'(x, \phi, \nabla\phi, \nabla\nabla\phi)\,dx, \tag{12}$$

where Ω is mapped into a new region Ω' and $\mathscr{L}'(x, \phi, \nabla\phi, \nabla\nabla\phi) = \mathscr{L}(x', \phi', V'\ \phi')\,|\partial x'/\partial x|$ may contain second order derivatives. The variation of the action functional is thus

$$\Delta J = J[\phi'] - J[\phi] = \int_\Omega [\Delta\mathscr{L}]\,dx, \tag{13}$$

where.

$$[\Delta\mathscr{L}] = \mathscr{L}[x', \phi', (x'), \nabla'\phi'(x')]\left|\frac{\partial x'}{\partial x}\right| - \mathscr{L}[x, \phi(x), \nabla\phi(x)]$$

$$= \mathscr{L}'(x, \phi, \nabla\phi, \nabla\nabla\phi) - \mathscr{L}(x, \phi, \nabla\phi)$$
$$= [\delta\mathscr{L}] + \ldots, \tag{14}$$

$$\Delta J = \delta J + \ldots. \tag{15}$$

Here δJ, $[\delta\mathscr{L}]$, indicate the terms up to first order in the infinitesimal parameters. Clearly,

$$\delta J = \int_{E_\Omega} [\delta\mathscr{L}]\, dx, \qquad (\ 5)$$

where

$$[\delta\mathscr{L}] = [\mathscr{L}(x', h', \nabla'\phi') - \mathscr{L}(x, h, \nabla\phi)] + \mathscr{L}(x, h, Vh)\, \partial_\mu(\delta x^\mu), \qquad 17$$

on using

$$\left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu(\delta x^\mu). \qquad (18)$$

On making a Taylor expansion, we have

$$[\delta\mathscr{L}] = \frac{\partial\mathscr{L}}{Cx''}\bigg|\, \delta x^\mu + \frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \pi^\lambda\delta(\partial_\lambda\phi) + \mathscr{L}\partial_\mu(\delta x^\mu)$$

$$= \frac{\partial\mathscr{L}}{\partial x^\mu}\bigg|\, \delta x^\mu + \frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \pi^\lambda\{\partial_\lambda(\delta\phi) - (\partial_\nu\phi)\,\partial_\lambda(\delta x^\nu)\} + \mathscr{L}\partial_\mu(\delta x^\mu). \qquad (19)$$

This can be recast as[4]

$$[\delta\mathscr{L}] = -[V],\ \bar{\delta}\phi_A\ C\ \partial_\mu(\pi^\mu_A\,\bar{\delta}\phi + \mathscr{L}\delta x^\mu)$$
$$= -[\mathscr{L}]_A\,\bar{\delta}\phi_A + \partial_\mu(\pi^\mu_A\delta\phi_A - \tau^{\mu\nu}\delta x_\nu). \qquad (20)$$

Here summation over components $A = 1 \ldots N$ is understood and $\tau^{\mu\nu}$ is the canonical energy momentum tensor:

$$\tau^{\mu\nu} = \pi^\mu_A\,\partial^\nu\phi_A - g^{\mu\nu}\mathscr{L},$$
$$\tau^\mu{}_\mu = \pi^\mu\partial_\mu\phi - 4\mathscr{L}. \qquad (21)$$

It may be remarked that, due the to arbitrariness in the region Ω, $\delta J = 0$ implies $[\delta\mathscr{L}] = 0$ and vice versa.

If the action is invariant under the infinitesimal transformations under consideration, we find

$$E\,\partial_\mu Z^\mu = [\mathscr{L}]_A\,\bar{\delta}\phi_A, \qquad (22)$$

where $\varepsilon Z^\mu = \pi^\mu_A\,\delta\phi_A - \tau^{\mu\nu}\delta x_\nu$. For constant parameter transformations, this leads to the "weak continuity" equation[4]

$$E\,\partial_\mu Z^\mu \overset{\Omega}{=} 0, \qquad (23)$$

where $\stackrel{\Omega}{=}$ indicates the equality when the fields satisfy Euler's equations of motion. For invariance under coordinate dependent parameter transformations, like gauge transformations, we obtain identities. We will be concerned in this paper with the constant parameter transformations. The linear independence of the *r* parameters lead to *r* weak continuity equations.

It is clear also that weak continuity equations can be defined even in the case the actions are not invariant.

For the case[5]

$$[\delta\mathscr{L}] = \varepsilon\partial_\mu\Lambda^\mu, \tag{24}$$

we clearly have

$$\varepsilon Z^\mu = \pi^\mu_A\delta\phi_A - \tau^{\mu\nu}\delta x_\nu - \varepsilon\Lambda^\mu \tag{25}$$

and'

$$\varepsilon\partial_\mu Z^\mu = [\mathscr{L}]_A\bar{\delta}\phi_A \stackrel{\Omega}{=} 0. \tag{26}$$

This case is important since Euler's equations corresponding to $[\delta\mathscr{L}]$ are then satisfied identically[6]. This would then assure that Euler's equations calculated from the transformed action $J[\phi']$ are the same as those derived from $J[\phi]$. In other words, the equations of motion are form invariant w.r.t. the infinitesimal transformations like in the case with $[\delta\mathscr{L}] = 0$, even though the invariance of action may be lost. For the case under discussion, we call the theory *cocariant,* while the former case will be called an *invariant* theory ($[\delta\mathscr{L}] = 0$).

There is a still more general case[7], viz., $[\delta\mathscr{L}] = \varepsilon\partial_\mu\Lambda^\mu - f$, with $f \stackrel{\Omega}{=} 0$ and $f \neq [\mathscr{L}]_A\bar{\delta}\phi_A$, where we can write a weak continuity equation with Z^μ given in Eqn. (2.25); the form invariance of the equations of motion may however, also be Iost.

3. Conformal Group. Transformation of Fields

a) Conformal Group[8]

The connected conformal group containing the identity (called. for simplicity, conformal group) may be defined as the group of the following transformations on the real space-time coordinates x^μ of a vector x in the four-dimensional Minskowski space:

1. Translations

$$x'^{\mu} = x^{\mu} + a^{\mu};$$

2. Restricted Lorentz group of transformations

$$(\Lambda x)^{\mu} \equiv \Lambda^{\mu}{}_{\nu} x^{\nu}\ ;\ g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\lambda} = g_{\rho\lambda},\ \Lambda^{0}_{0} > 1,\ \det \Lambda = 1;$$

3. Scale or dilatation transformations: $(g_D x)^{\mu} \equiv x'^{\mu} = e^{-\rho} x^{\mu}$ ρ real;

4. Special conformal transformations:

$$(g_C x) \equiv x'^{\mu} = (x^{\mu} - c^{\mu} x^2)/[1 - 2c.x + c^2 x^2].$$

These transformations constitute a 15 parameter group and the special transformations are non-linear. Each of these sets of transformations constitute a sub-group which is abelian except for the case of Lorentz transformations. Note that translations do not constitute an invariant subgroup.

The infinitesimal transformations are given by

Translations:

$$\delta x^{\mu} = \varepsilon^{\mu} = -i\varepsilon_{\nu} \bar{P}^{\nu} x^{\mu};$$

Lorentz transformations:

$$\delta x^{\mu} = \varepsilon^{\mu}_{\nu} x^{\mu} = \frac{i}{2} \varepsilon_{\rho\sigma} \bar{M}^{\rho\sigma} x^{\mu},$$

$$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}\ ; \qquad \left|\frac{\partial x'}{\partial x}\right| = 1;$$

Dilatations.

$$\delta x^{\mu} = -\varepsilon x^{\mu} = i\,\varepsilon\,\bar{D}\,x^{\mu},$$

$$\left|\frac{\partial x'}{\partial x}\right| = (1 - 4\varepsilon);$$

Special transformations:

$$\delta x^{\mu} = \eta_{\nu}(2x^{\nu} x^{\mu} - g^{\mu\nu} x^2) = i\eta_{\nu} \bar{K}^{\nu} x^{\mu},$$

$$\left|\frac{\partial x'}{\partial x}\right| = (1 + 8\eta \cdot x), \tag{1}$$

where $\bar{P}^\nu = iS^\nu$, $\bar{M}^{\rho\sigma} = i(x^\rho \partial^\sigma - x^\sigma \partial^\rho)$, $D = i(x \cdot \partial)$ and $\bar{K}^\nu = -i(2x^\nu x^\lambda - x^2 g^{\nu\lambda}) \partial_\lambda$ are the fifteen infinitesimal generators. The Lie algebra of these generators also determines the Lie algebra of the abstract (connected) conformal group whose generator will be indicated by P^μ, $M^{\rho\sigma}$, D and K^μ. The Lie algebra is found to be:

$$\begin{gathered}[D, P_\mu] = -i P_\mu, \qquad [D, K_\mu] = +i K_\mu, \qquad [D, M_{\mu\nu}] = 0, \\ [K_\mu, K_\nu] = 0, \qquad [P_\mu, P_\nu] = 0, \qquad [P_\sigma, M_{\mu\nu}] = i(g_{\sigma\mu} P_\nu - g_{\sigma\nu} P_\mu), \\ [K_\sigma, M_{\mu\nu}] = i(g_{\sigma\mu} K_\nu - g_{\sigma\nu} K_\mu), \qquad [K_\mu, P_\nu] = -2i(g_{\mu\nu} D + M_{\mu\nu}), \\ [M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\sigma} M_{\nu\rho} - g_{\nu\sigma} M_{\mu\rho} - g_{\mu\rho} M_{\nu\sigma} + g_{\nu\sigma} M_{\mu\sigma}). \end{gathered} \tag{2}$$

Note that the commutation relations imply

$$e^{i\rho D} P_\mu e^{-i\rho D} = e^\rho P_\mu, \qquad e^{i\rho D} K_\mu e^{-i\rho D} = e^{-\rho} K_\mu, \tag{3}$$

and that K_μ transforms as a four-vector. The exact dilatation symmetry (with an integrable generator D that takes one-particle states into one-particle states) implies that the mass spectrum is either continuous or all masses are zero.

Introducing $J_{AB}(A, B = 0, 1, 2, 3, 5, 6)$, where $J_{AB} = -J_{BA}$, by

$$\begin{aligned} &J_{\mu\nu} = M_{\mu\nu}, \qquad J_{65} = D, \qquad J_{5\mu} = \frac{1}{2}(P_\mu - K_\mu), \\ &J_{6\mu} = \frac{1}{2}(P_\mu + K_\mu), \end{aligned} \tag{4}$$

one has

$$\begin{gathered}[J_{KL}, J_{MN}] = i(g_{KN} J_{LM} + g_{LM} J_{KN} - g_{KM} J_{LN} - g_{LN} J_{KM}), \\ g_{AA} = (+ - - , - +), \qquad g_{AB} = 0 \qquad (A \neq B), \end{gathered} \tag{5}$$

which is the Lie algebra of SO(4,2). Thus, the conformal group is locally isomorphic to the non-compact group SO(4,2) whose covering group is the spinor group SU(2,2). Three Casimir operators are then easily obtained:

$$\begin{gathered} J_{AB} J^{AB} = M_{\mu\nu} M^{\mu\nu} + 2P \cdot K + 8iD - 2D^2, \\ \varepsilon_{ABCDEF} J^{AB} J^{CD} J^{EF} \qquad \text{and} \qquad J^{AB} J_{BC} J^{CD} J_{DA}. \end{gathered}$$

b) Transformation of Fields

We postulate that, for every particle, there exists an interpolating field (with a finite number N of components) which transforms according to a representation of the conformal algebra. Thus, corresponding to a transformation

$$x'^{\mu} = (gx)^{\mu}, \qquad g \in \text{Conformal group},$$

the field $\phi(x) = (\phi_1, \ldots \phi_N)$ transforms as

$$T(g)\,\phi(x') = \phi'(x') \equiv S(g, x')\,\phi(x), \tag{6}$$

where $\{T(g)\}$ is an N-dimensional representation of the conformal group.

For the infinitesimal transformation

$$T(g) \simeq I + i \sum_{k=1}^{15} \varepsilon_k I_k + \ldots, \tag{7}$$

where the essential parameters are labelled as ε_k, $k = (1, \ldots, 15)$ for convenience. We find

$$\bar{\delta}\phi(x) = i \sum_i \varepsilon_k I_k \phi(x). \tag{8}$$

The generators I_k satisfy the Lie algebra of the conformal group.

When the fields are quantized field operators acting on the state vectors $|\Psi\rangle$ in Hilbert space which carry the representation according to

$$|\Psi\rangle \to U(g)|\Psi\rangle, \tag{9}$$

$U(g)$ being a unitary operator, we obtain the supplementary constraint"

$$\phi'(x') = U(g)^{+}\,\phi(x')\,U(g). \tag{10}$$

For infinitesimal transformations,

$$U(g) \simeq \Pi + i \sum_k \varepsilon_k G_k, \tag{11}$$

it follows

$$\bar{\delta}\phi(x) = i \sum_k \varepsilon_k [\phi(x), G_k], \tag{12}$$

where the G_k satisfy the Lie algebra of the conformal group. Since it is easier to calculate the commutators in quantum field theory, where x^{μ} is simply a parameter, we will frequently calculate the variation of ϕ regarding the ϕ's as quantized operators.

Homogeneity of space with respect to translations, according to special relativity, requires for any (observable) field $0(x)$,

$$0'(x') = 0(x) = O(t^{-1} x'), \tag{13}$$

when

$$(tx)^\mu = x'^\mu = x^\mu + \varepsilon^\mu,$$

thus

$$\delta_T 0(x) = 0 \tag{14}$$

and

$$\bar{\delta}_T 0(x) = -\mathrm{E}_{,}\ \partial^\mu 0 = i\varepsilon_\mu P^\mu 0(x). \tag{15}$$

Regarding the field as an operator in Hilbert space ($U_T \simeq e^{i\varepsilon\cdot P}$), we have

$$\bar{\delta}_T 0 = \mathrm{i}\,\varepsilon_\mu[0(x), P^\mu].$$

Thus,

$$[0(x), P^\mu] = i\partial^\mu 0(x) \equiv P^\mu 0(x), \tag{16}$$

from which follows

$$0(x) = e^{ix\cdot P} 0(0)\, e^{-ix\cdot P}. \tag{17}$$

Homogeneity w.r.t. space-time rotations requires that the interpolating N-component field 6, transforms according to a (non-unitary and irreducible) representation of the homogeneous Lorentz group, viz.,

$$\phi'(x') = S(\Lambda)\,\phi(x), \tag{18}$$

$S(\Lambda)$ constituting a representation of the Lorentz group. For infinitesimal transformations, we define

$$S(\Lambda) = I - \frac{i}{2}\varepsilon_{\rho\sigma}\Sigma^{\rho\sigma}, \tag{19}$$

so that

$$\delta_L \phi = -\frac{i}{2}\varepsilon_{\rho\sigma}\Sigma^{\rho\sigma}\phi, \tag{20}$$

and

$$\bar{\delta}_L \phi = -\frac{i}{2}\varepsilon_{\rho\sigma} m^{\rho\sigma}\phi, \tag{21}$$

where

$$m^{\rho\sigma} = \Sigma^{\rho\sigma} + i(x^\rho\partial^\sigma - x^\sigma\partial^\rho). \tag{22}$$

Taking the field operator point of view,

$$U_L \simeq e^{(-i/2)\varepsilon_{\rho\sigma} M^{\rho\sigma}}, \tag{23}$$

and

$$\bar{\delta}_L \phi = -\frac{i}{2} \varepsilon_{\rho\sigma}[\phi(x), M^{\rho\sigma}], \tag{24}$$

so that

$$[\phi(x), M^{\rho\sigma}] = m^{\rho\sigma} \phi(x). \tag{25}$$

Using Eqn. *(3.17)* and the identity

$$[\phi(x), M^{\rho\sigma}] = e^{ix\cdot P} [\phi(0), M^{\rho\sigma}(-x)] e^{-ix\cdot P}, \tag{26}$$

where $(M^{\rho\sigma} = M^{\rho\sigma}(0))$

$$M^{\rho\sigma}(-x) = M^{\rho\sigma} + (x^\rho P^\sigma - x^\sigma P^\rho), \tag{27}$$

we can show that

$$[\phi(0), M^{\rho\sigma}] = \Sigma^{\rho\sigma} \phi(0) \tag{28}$$

Conversely, if we take this relation as a definition of $\Sigma^{\rho\sigma}$, we can recover Eqn. *(3.20)*.

For dilatations, we define

$$[\phi(0), D] = i\, L\, \phi(0), \tag{29}$$

where L is an N x N matrix and $D \equiv D(0)$. We may now use the identity similar to Eqn. *(3.26)* to obtain $[\phi(x), D]$. In the present case,

$$D(-x) = D + x^\lambda P_\lambda, \tag{30}$$

so that

$$\begin{aligned} [\phi(x), D] &= i(L + x \,.\, i?)\, \phi(x) \\ &\equiv d\, \phi(x). \end{aligned} \tag{31}$$

Then,

$$\bar{\delta}_D \phi(x) = -i\varepsilon[\phi(x), D] = -i\, \varepsilon\, d\, \phi(x), \tag{32}$$

where $U_D \simeq e^{-i\varepsilon D}$. It follows that

$$\phi'(x) \simeq e'' \phi(e^\varepsilon x).$$

Comparing with $\phi'(x) = S(g, x)\phi(g^{-1} x)$, we see that under finite dilatations, $x'^{\mu} = e^{-\rho} x^{\mu}$,

$$\phi'(x') = e^{\rho L} \phi(x) = e^{\rho L} \phi(e^{\rho} x'), \tag{33}$$

and, correspondingly, $U_D = e^{-i\rho D}$, that is,

$$e^{i\rho D} \phi(x) e^{-i\rho D} = e\& \phi(e^{\rho} x). \tag{34}$$

Also,

$$\delta_D\phi = \varepsilon L \phi(x), \qquad \delta x^{\mu} = -\varepsilon x^{\mu}. \tag{35}$$

For special conforma1 transformations, we define that the field operator $\phi(0)$ satisfies[5]

$$[\phi(0), K_{\mu}] = \kappa_{\mu} \phi(0). \tag{36}$$

From

$$\begin{aligned} K_{\mu}(-x) &\equiv e^{-ix\cdot P} K_{\mu} e^{ix\cdot P} \\ &= K_{\mu} + 2(x_{\mu} D + x^{\nu} M_{,,}) + (2x_{\mu} x . P - x^2 P_{\mu}) \end{aligned} \tag{37}$$

and an identity analagous to the used above we find

$$[\phi(x), K^{\mu}] = K^{\mu} \phi(x), \tag{38}$$

where

$$k_{\mu} = \kappa_{\mu} + i(2x_{\mu} x \cdot \partial - x^2 \partial_{\mu}) + 2(x_{\mu} iL + x^{\nu} \Sigma_{\mu\nu}) \tag{39}$$

and

$$\bar{\delta}_C \phi(x) = i \eta_{\mu}[\phi(x), K^{\mu}] = i \eta_{\mu} k^{\mu} \phi(x) \tag{40}$$

or

$$\phi'(x) \simeq [I + i\eta^{\mu}\{2(x_{\mu} iL + x^{\nu} \Sigma_{\mu\nu}) + \kappa_{\mu}\}] \phi(g^{-1} x), \tag{41}$$

or

$$S(g_C, x) \simeq I + i\eta^{\mu}\{2(x_{\mu} iL + x^{\nu} \Sigma_{\mu\nu}) + \kappa_{\mu}\}. \tag{42}$$

Thus,

$$\delta_C\phi(x) = i\eta^{\mu}\{2(x_{\mu} iL + x^{\nu} \Sigma_{\mu\nu}) + \kappa_{\mu}\} \phi(x). \tag{43}$$

It may be noted that in 66, (or δx^{μ}) no derivatives of the iield appear. It follows that $[\delta\mathscr{L}]$ contains derivatives only up to first order. In this case[6], Λ^{μ} is a function of x and 6, alone. Note also that P^{μ}, $m^{P''}$, d, k^{μ} satisfy the commutation relations of the Lie algebra of conforma1 group and that

κ^μ makes transitions between fields with different Lorentz transformationi laws;we will assume it to vanish in the discussions to follow. Also it follows that $[L,\Sigma^{\rho\sigma}] = 0$ and, if the field ϕ constitutes an irreducible representation of homogeneous Lorentz group, L is a multiple of identity matrix.

4. Conforma1 Currents as Moments of a Symmetric Energy-Momentum Tensor

We may now calculate $[\delta\mathscr{L}]$ from Eqn. (2.19):

$$[\delta\mathscr{L}] \equiv \varepsilon_\mu I^\mu_{\mathrm{T}} \mathscr{L} + \frac{1}{2}\varepsilon_{\rho\sigma} I^{\rho\sigma}_{\mathrm{L}} \mathscr{L} + \varepsilon I_{\mathrm{D}} \mathscr{L} + \eta_\nu I^\nu_{\mathrm{C}} \mathscr{L}, \tag{1}$$

where

$$I^\mu_{\mathrm{T}}\mathscr{L} = \frac{\delta\mathscr{L}}{\partial x^\mu}\Bigg|,$$

$$\begin{aligned}
I^{\rho\sigma}_{\mathrm{L}}\mathscr{L} &= (x^\sigma g^{\rho\mu} - x^\rho g^{\sigma\mu})\,\partial_\mu\mathscr{L}\,| - i\left(\frac{\partial\mathscr{L}}{\partial\phi}\Sigma^{\rho\sigma}\phi + \pi^\lambda\,\Sigma^{\rho\sigma}\,\partial_\lambda\,\phi\right)\\
&\quad + (\pi^\rho\,\partial^\sigma - \pi^\sigma\,\partial^\rho)\,\phi,\\
I_{\mathrm{D}}\mathscr{L} &= -x^\mu\,\partial_\mu\,\mathscr{L}\,| - 4\mathscr{L} + \frac{\partial\mathscr{L}}{\partial\phi} L\phi + \pi^\lambda\,(L + I)\,\partial_\lambda\,\phi\\
I^\nu_{\mathrm{C}}\mathscr{L} &= (2x^\mu\,x^\nu - g^{\mu\nu}\,x^2)\,\partial_\mu\,\mathscr{L}\,| - 2x^\nu\,(x^\mu\,\partial_\mu\,\mathscr{L}\,| + I_{\mathrm{D}}\mathscr{L})\\
&\quad + 2x_\mu([x^\mu\,g^{\nu\lambda} - x^\nu\,g^{\mu\lambda}]\,\partial_\lambda\mathscr{L}\,| - I^{\nu\mu}_{\mathrm{L}}\mathscr{L}) + V^\nu + i\left(\frac{\partial\mathscr{L}}{\partial\phi}\kappa^\nu\,\phi + \pi^\lambda\,\kappa^\nu\,\partial_\lambda\phi\right),
\end{aligned} \tag{2}$$

where

$$V^\nu = 2i\,\pi_\lambda(iLg^{\nu\lambda} + \Sigma^{\nu\lambda})\,\phi \tag{3}$$

is the conforma1 deficiency vector (note that V^ν does not depend on $\partial\mathscr{L}/\partial\phi$. Also, we will assume $\kappa^\nu = 0$.

The currents in conformally "covariant" theory, satisfying the weak continuity equation, are also easily found. Writing (the sign in front of J_{T} being a matter of convenience)

$$\begin{aligned}
\varepsilon\,Z^\lambda &= -\varepsilon_\mu\,J^{\lambda\mu}_{\mathrm{T}} + \frac{1}{2}\varepsilon_{\rho\sigma}\,J^{\lambda\rho\sigma}_{\mathrm{L}} + \varepsilon J^\lambda + \eta_\nu\,J^{\lambda\nu}_{\mathrm{C}},\\
\varepsilon\,\Lambda^\lambda &= -\varepsilon_\mu\,\Lambda^{\lambda\mu}_{\mathrm{T}} + \frac{1}{2}\varepsilon_{\rho\sigma}\,\Lambda^{\lambda\rho\sigma}_{\mathrm{L}} + \varepsilon\,\Lambda^\lambda_{\mathrm{D}} + \eta_\nu\,\Lambda^{\lambda\nu}_{\mathrm{C}},
\end{aligned} \tag{4}$$

We have, in Poincaré invariant theory (A, = A, = O),

$$\begin{aligned}
J^{\lambda\mu}_{\mathrm{T}} &= \tau^{\lambda\mu},\\
J^{\lambda\rho\sigma}_{\mathrm{L}} &= -J^{\lambda\sigma\rho}_{\mathrm{L}} = -i\,\pi^{\lambda}\,\Sigma^{\rho\sigma}\,\phi + (x^{\rho}\tau^{\lambda\sigma} - x^{\sigma}\,\tau^{\lambda\rho}),\\
J^{\lambda}_{\mathrm{D}} &= x_{\mu}\,\tau^{\lambda\mu} + \pi^{\lambda}\,L\phi - \Lambda^{\lambda}_{\mathrm{D}},\\
J^{\lambda\nu}_{\mathrm{C}} &= -(2x^{\nu}\,x_{\mu} - g^{\nu}_{\mu}\,x^{2})\,\tau^{\lambda\mu} + 2ix_{\mu}\,\pi^{\lambda}(iLg^{\nu\mu} + \Sigma^{\nu\mu})\,\phi + i\pi^{\lambda}\,\kappa^{\nu}\phi - \Lambda^{\lambda\nu}_{\mathrm{C}}
\end{aligned} \tag{5}$$

where

$$\partial_{\lambda}\,J^{\lambda\cdots} \overset{\Omega}{=} 0. \tag{6}$$

Poincare invariance leads to restrictions:

$$\left.\frac{\partial\mathscr{L}}{\partial x^{\mu}}\right| = 0 \tag{7}$$

e.g., $\mathscr{L}$ cannot depend explicitly on the coordinates, and

$$i\left(\frac{\partial\mathscr{L}}{\partial\phi}\,\Sigma^{\rho\sigma}\,\phi + \pi^{\lambda}\,\Sigma^{\rho\sigma}\,\partial_{\lambda}\phi\right) = (\pi^{\rho}\,\partial^{\sigma} - \pi^{\sigma}\,\partial^{\rho})\,\phi, \tag{8}$$

which may be used to determine the matrices ***CP".***

Exploiting the fact that J^{λ} and $J^{\lambda} + \partial_{\mu}\,\chi^{\lambda\mu}$, where $\chi^{\lambda\mu} = -\chi^{\mu\lambda}$, have the same divergence and charge (if χ^{i0} vanishes sufficiently rapid at the surface at infinity), we can write the currents in a simpler form. In terms of Belinfante tensor[10]

$$\hat{\Theta}^{\lambda\mu} = \tau^{\lambda\mu} + \frac{1}{2}\,\partial_{\nu}\,\chi^{\nu\lambda\mu}, \tag{9}$$

where

$$\chi^{\lambda\mu\rho} = -i[\pi^{\lambda}\,\Sigma^{\mu\rho}\,\phi - \pi^{\mu}\,\Sigma^{\lambda\rho}\,\phi - \pi^{\rho}\,\Sigma^{\lambda\mu}\,\phi]. \tag{10}$$

The currents take the form ($\kappa^{\nu} = 0$)

$$\begin{aligned}
J^{\lambda\mu}_{\mathrm{T}} &= \hat{\Theta}^{\lambda\mu}, \qquad J^{\lambda\rho\sigma}_{\mathrm{L}} = (x^{\rho}\,\hat{\Theta}^{\lambda\sigma} - x^{\sigma}\,\hat{\Theta}^{\lambda\rho}),\\
J^{\lambda}_{\mathrm{D}} &= x_{\mu}\,\hat{\theta}^{\lambda\mu} - \frac{1}{2}\,V^{\lambda} - \Lambda^{\lambda}_{\mathrm{D}},\\
J^{\lambda\nu}_{\mathrm{C}} &= -(2x^{\nu}\,x_{\mu} - g^{\nu}_{\mu}\,x^{2})\,\hat{\theta}^{\lambda\mu} + x^{\nu}\,V^{\lambda} - \Lambda^{\lambda\nu}_{\mathrm{C}}
\end{aligned} \tag{11}$$

A further simplification can be achieved by introducing "improved energy-momentum tensor[11], $\Theta^{\lambda\mu}$:

$$\theta^{\mu\nu} = \hat{\theta}^{\mu\nu} + \frac{1}{2}\,\partial_{\lambda}\,\partial_{\rho}\,\chi^{\lambda\rho\mu\nu}. \tag{12}$$

where $i, ?, \chi^{\lambda\rho\mu\nu}$ is symmetric and divergenceless on the indices μ and ν and

$$\chi^{\lambda\rho\mu\nu} = g^{\lambda\rho}\,\sigma_+^{\mu\nu} + g^{\mu\nu}\,\sigma_+^{\lambda\rho} - g^{\lambda\mu}\,\sigma_+^{\nu\rho} - g^{\lambda\nu}\,\sigma_+^{\mu\rho} + \frac{1}{3}(g^{\mu\lambda}\,g^{\nu\rho} - g^{\mu\nu}\,g^{\lambda\rho})\,\sigma_{+\alpha}^{\alpha}, \quad (13)$$

$\sigma^{\mu\nu}$ being any arbitrary tensor function of the fields and $\sigma_{\mp}^{\mu\nu} = \frac{1}{2}[\sigma^{\mu\nu} \pm \sigma^{\nu\mu}]$. The currents then become

$$\begin{aligned} &J_{\mathrm{T}}^{\lambda\mu} \doteq \theta^{\lambda\mu}, \qquad J_{\mathrm{L}}^{\lambda\rho\sigma} \doteq (x^\rho\,\theta^{\lambda\sigma} - x^\sigma\,\theta^{\lambda\rho}),\\ &J_{\mathrm{D}}^{\lambda} \doteq x_\mu\,\theta^{\lambda\mu} - \frac{1}{2}(V^\lambda + 2\partial_\rho\,\sigma^{\lambda\rho}) - \Lambda_{\mathrm{D}}^{\lambda},\\ &J_{\mathrm{C}}^{\lambda\nu} \doteq -(2x^\nu\,x_\mu - g_\mu^\nu\,x^2)\,\theta^{\lambda\mu} + x^\nu(V^\lambda + 2\partial_\rho\,\sigma^{\lambda\rho}) - 2\sigma^{\nu\lambda} - \Lambda_{\mathrm{C}}^{\lambda\nu}, \end{aligned} \quad (14)$$

where the equality $=$ means that we have dropped all terms whosí: divergence w.r.t. index λ vanishes identically. /

The arbitrariness in the choice of $\sigma^{\mu\nu}$ may allow is to write

$$J_{\mathrm{D}}^{\lambda} \doteq x_\mu\,\theta^{\lambda\mu}, \qquad J_{\mathrm{C}}^{\lambda\nu} = -(2x^\nu\,x_\mu - g_\mu^\nu\,x^2)\,\theta^{\lambda\mu}. \quad (15)$$

Since, in a Poincare invariant theory, $\theta^{\lambda\mu}$ and $\hat{\theta}^{\lambda\mu}$ can be shown to be symmetric tensors, it is then easily shown that

$$\partial_\lambda\,J_{\mathrm{D}}^{\lambda} \overset{\Omega}{=} \theta_\mu^\mu, \qquad \partial_\lambda\,J_{\mathrm{C}}^{\lambda\mu} \overset{\Omega}{=} -2x^\nu\,\partial_\lambda\,J_{\mathrm{D}}^{\lambda} \overset{\Omega}{=} -2x^\nu\,\theta_\mu^\mu, \quad (16)$$

or

$$\begin{aligned} &\frac{d}{dt}\int J_{\mathrm{D}}^{0}\,d^3x = \frac{d}{dt}\int x_\mu\,\theta^{0\mu}\,d^3x \overset{\Omega}{=} \int \theta_\mu^\mu\,d^3x,\\ &-\frac{d}{dt}\int J_{\mathrm{C}}^{0\nu}\,d^3x = 2\int x^\nu\,\theta_\mu^\mu\,d^3x. \end{aligned} \quad (17)$$

In such a theory, the trace θ_μ^μ determines whether the dilatation and conformal charges are conserved or not. It may be remarked that θ_μ^μ is much 'softer' than the trace of the canonical energy-momentum tensor in the sense that it involves less derivatives of the field.

The necessary conditions in Poincaré invariant theory, to obtain Eqns. (4.16) and (4.17) are

$$\begin{aligned} &\partial_\lambda\left[\frac{1}{2}(V^\lambda + 2\partial_\rho\,o^{\bullet\bullet}) + \Lambda_{\mathrm{D}}^{\lambda}\right] = 0,\\ &\partial_\lambda[x^\nu(V^\lambda + 2\partial_\rho\,\partial^{\lambda\rho}) - 2\sigma^{\nu\lambda} - \Lambda_{\mathrm{C}}^{\lambda\nu}] = 0, \end{aligned} \quad (18)$$

while the conditions that theory be conformal 'covariant' are, from Eqns. (4.2) and (4.4), $(I_T\mathscr{L} = I_L\mathscr{L}) = 0$:

$$I_D\mathscr{L} = \partial_\lambda \Lambda_D^\lambda, \qquad I_C^\nu\mathscr{L} = -2x^\nu I_D\mathscr{L} + V^\nu = \partial_\lambda \Lambda_C^{\lambda\nu}, \tag{19}$$

where we have assumed $\kappa^\nu = 0$(Note that κ^ν makes transitions between fields with different L.T. laws). It is clar from Eqns. (4.19) that[11] a scale invariant theory is also invariant w.r.t. the special conformal transformations if and only if

$$V^\nu = 0. \tag{20}$$

In this case, we may choose $\sigma^{\mu\nu} = 0$ to satisfy Eqns. (4.18). In case Eqn. *(4.20)*is not satisfied, the scale invariance leads only to special conformal 'covariance' (c-covariance) and $V^\nu = \partial_\lambda \Lambda_C^{\lambda\nu}$. Eqns. (4.18) can then be satisfied by the choice

$$\sigma^{\lambda\rho} = -\frac{1}{2}\Lambda_C^{\rho\nu}. \tag{21}$$

This is the case, for example, with the massless scalar ϕ^4 theory and the improved tensor $\theta^{\lambda\mu}$ involves a contribution from scalar fields but not for example from a massless spin 1/2 field for which $V' \neq 0$

If the theory is c-invariant we have $V' = 2x^\nu I_D\mathscr{L}$ so that c-invariance implies a scale invariant theory if and only if $V^\nu = 0$ If this is not the case only 'covariance' w.r.t. scale transformations is obtained. In this case, we can satisfy Eqns. (4.18) by choosing.

$$\sigma^{\lambda\rho} = -x^\lambda \Lambda_D^\rho. \tag{22}$$

For a theory with only 'covariance' w.r.t. scale and c-transformations, we have

$$V^\lambda = \partial_\rho \Lambda_C^{\rho\lambda} + 2x^\lambda \partial_\rho \Lambda_D^\rho \tag{23}$$

and the choice for $\sigma^{\lambda\rho}$ is

$$\sigma^{\lambda\rho} = -\left(\frac{1}{2}\Lambda_C^{\rho\lambda} + x^\lambda \Lambda_D^\rho\right) \tag{24}$$

Thus, if the theory has symmetry w.r.t. conformal transformations and **is Poincaré invariant, it is always possible to write the currents in the** form of Eqn. (4.15) and the conservation of dilatation and special conformal currents implies then

$$\theta_\mu^\mu \stackrel{\Omega}{=} 0 \tag{25}$$

We also note that a Poincaré invariant theory has symmetry w.r.t. conformal group only if we may write the conformal deficienty vector V' of the Lagrangian in the form given by Eqn. (4.23), from which A, and A, can be identified and the improved traceless tensor $\theta^{\mu\nu}$ then defined with a choice of $\sigma^{\mu\nu}$ given by Eqn. (4.24). For the case of conformal invariance, the tensor $\theta^{\lambda\mu}$ may be identified with the Belinfante tensor $\hat{\theta}^{\lambda\mu}$ whose trace must vanish. The lack of vanishing of $\hat{\theta}^{\lambda\mu}$ thus provides a measure of lack of (exact) conformal invariance in a Poincare invariant theory but if does not exclude conformal 'covariance', for which θ^{μ}_{μ} is required to vanish. Eqn. (4.19) shows that if V' = 0 the theory with conformal symmetry is either invariant or 'covariant' w.r.t. both scale and special conformal transformations.

A remark on the scale invariance condition mey be interesting. Working with natural units $\hbar = c = 1$ all quantities in the Lagrangian have dimensions of a length L. Let us denote them by

$$[m] = L^{-1}, \quad [\phi_A] = L^{l_A}, \quad [\partial_\mu \phi_A] = L^{l_A - 1}, \quad [f] = L^{l_f}, \qquad (26)$$

where f are the coupling constants appearing in the Lagrangian. Since in Poincaré invariant theories $[\mathscr{L}] = L^{-4}$ we obtain on applying Euler's theorem for homogeneous functions

$$-4\mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi} l \phi + \pi^\lambda (l - I)\,\partial_\lambda \phi - \Sigma m \frac{\partial \mathscr{L}}{\partial m} + \Sigma l_f \frac{\partial \mathscr{L}}{\partial f}, \qquad (27)$$

where $1 \equiv (l_A I_{,,})$ is a diagonal matrix. Then,

$$I_D \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi}(L + l)\phi + \pi^\lambda (L + l)\,\partial_\lambda \phi - \Sigma m \frac{\partial \mathscr{L}}{\partial m} + \Sigma l_f f \frac{\partial \mathscr{L}}{\partial f}. \qquad (28)$$

We may write $\mathscr{L} = \Sigma g_\gamma \mathscr{L}_\gamma$, where g_γ are coupling constants constructed from the masses and couplings f. The last two terms can then be written as $\Sigma g_\gamma \alpha_\gamma \mathscr{L}_\gamma$, where the dimension of g_γ is L^{α_γ}. Then,

$$I_D \mathscr{L} = \Sigma \left\{ \frac{\partial \mathscr{L}_\gamma}{\partial \phi}(L + l)\phi + \pi^\lambda_\gamma (L + l)\,\partial_\lambda \phi + \alpha_\gamma \mathscr{L}_\gamma \right\} g_\gamma . \qquad (29)$$

The scale invariance condition then implies that for each dimensionless coupling we must have

$$\frac{\partial \mathscr{L}_\gamma}{\partial \phi}(L + l)\phi + \pi^\lambda_\gamma (L + l)\,\partial_\lambda \phi = 0 \qquad (30)$$

and, for each dimensional coupling, the expression inside the curly bracket { } must vanish. If we assume (see remarks at the end of Sec. 3) $L = -1 = -(l_A \delta_{AB})$ no dimensional couplings may be present if scale invariance holds. For interacting field theory, it is clear that not all the masses need to vanish in the scale invariant limit.

5. Illustrations for some Field Theories

a) Scalar Field Theory

To illustrate our discussion, we study the following Lagrangian for a scalar field ϕ:

$$\mathscr{L} = \frac{1}{2}[(\partial^\mu \phi)(\partial_\mu \phi) - m^2 \phi^2] + \frac{g}{3}\phi^3 + \frac{\lambda}{4}\phi^4, \tag{1}$$

$$\pi^\mu = \partial^\mu \phi, \qquad \frac{\partial \mathscr{L}}{\partial \phi} = -m^2\phi + g\phi^2 + \lambda\phi^3. \tag{2}$$

Euler's eqns. are ($\Box = \partial^\mu \partial_\mu$):

$$(\Box + m^2)\phi = g\phi^2 + \lambda\phi^3. \tag{3}$$

The Lorentz invariant condition is verified to be satisfied with $\Sigma^{\rho\sigma} = 0$. The energy-momentum tensor is

$$\tau^{\mu\nu} = \hat{\theta}^{\mu\nu} = (\partial^\mu \phi)(\partial^\nu \phi) - g^{\mu\nu}\mathscr{L}, \tag{4}$$

$$\hat{\theta}^\mu_\mu = -(\partial^\mu \phi)(\partial_\mu \phi) + 2m^2 \phi^2 - \frac{4}{3} g\phi^3 - \lambda\phi^4 \neq 0. \tag{5}$$

The theory, therefore, can at best be conformal covariant. This may also be seen from the conformal deficiency vector

$$V^\lambda = -2(\partial^\lambda \phi) L\phi = -L\,\partial^\lambda\phi^2 = -L\,\partial_\rho(g^{\rho\lambda}\phi^2) \tag{6}$$

which does not vanish due to the kinetic energy term[12]. It also shows that w.r.t. special conformal transformations we may at best obtain 'co-variance', while scale invariance is not excluded. Since ϕ and g have length dimension (-1) the scale invariance condition is

$$(L-1)(\partial^\mu\phi)\,\partial_\mu\phi) + m^2(2-L)\,\phi^2 + g\left(L-\frac{4}{3}\right)\phi^3 + \lambda(L-1)\,\phi^4 = 0. \tag{7}$$

(It is interesting to note that if we apply a scale invariance transformation

$\phi'(x') = \rho^{-1}\phi(x)$, $x' = \rho x$, to Euler's equation, one can see that it is left invariant also with the choice L = 2, $m = 0$, $\lambda = 0$).

For the kinetic energy term, $(L-1)(\partial^\mu\phi)(\partial_\mu\phi)$, to vanish identically, one must have L = 1; it then follows m = 0 and g = 0.

A massless scalar theory with

$$9 = -\frac{1}{2}\ \partial_\mu(\phi)(\partial^\mu\phi) + \frac{\lambda}{4}\phi^4 \tag{8}$$

is thus scale invariant. We find

$$I^\lambda_C \mathcal{L} = V\ \text{[illegible]}\ - \partial_\rho(g^{\rho\lambda}\phi^2), \tag{9}$$

so that $\Lambda^{\lambda\rho}_C = -g^{\rho\lambda}\phi^2$ and $\sigma^{\lambda\rho} = \frac{1}{2}g^{\rho\lambda}\phi^2$.

The improved energy-momentum tensor of Eqn. (4.12) can be easily calculated:

$$\theta^{\lambda\rho} = \hat{\theta}^{\lambda\rho} - \frac{1}{6}(\partial^\lambda\partial^\rho - g^{\lambda\rho}\Box)\phi^2 \tag{10}$$

and

$$\theta^\mu_\mu = \phi[-\lambda\phi^3 + \Box\phi] + 2m^2\phi^2 - \frac{4}{3}g\phi^3 \tag{11}$$

$$\overset{\Omega}{=} m^2\phi^2 - \frac{1}{3}g\phi^3.$$

Thus m and g are responsible for breaking scale invariance. All the currents can be written as moments of the tensor $\theta^{\lambda\mu}$ according to Eqn. (4.15).

b) Dirac Field Theory

$$\mathcal{L} = \frac{1}{2}\{\overline{\Psi}(i\gamma\cdot\partial - m)\Psi + \overline{\Psi}(-i\gamma\cdot\overleftarrow{\partial} - m)\Psi\}, \tag{12}$$

$$\overline{\Psi} = \Psi^\dagger\gamma^0, \quad \pi^\mu = \frac{i}{2}\overline{\Psi}\gamma^\mu, \quad \frac{\partial\mathcal{L}}{\partial\Psi} = \frac{i}{2}\overline{\Psi}\gamma\cdot\overleftarrow{\partial} - m\overline{\Psi},$$

$$\pi^{*\mu} = -\frac{i}{2}(\gamma^0\gamma^\mu\Psi), \quad \frac{\partial\mathcal{L}}{\partial\Psi^*} = \gamma^0\left(\frac{i}{2}\gamma\cdot\partial\Psi - m\Psi\right)\cdot \tag{13}$$

Euler's eqns. are

$$(-i\gamma \cdot \partial + m)\Psi = 0, \qquad \overline{\Psi}(i\gamma \cdot \overleftarrow{\partial} + m) = 0. \tag{14}$$

The Lorentz invariance condition is identically satisfied for $\Sigma^{\rho\sigma} = (i/4)[\gamma^{\rho}, \gamma^{\sigma}]$; note that

$$\delta_L \Psi = -\frac{i}{2}\varepsilon_{\rho\sigma} C^{\rho\sigma}, \qquad \delta_L \Psi^* = \frac{i}{2}\varepsilon_{\rho\sigma}\Sigma^{*\rho\sigma}\Psi^*$$

and

$$\delta_D \Psi = \varepsilon L\Psi, \qquad \delta_D \Psi^* = \varepsilon L^*\Psi.$$

We also use $\gamma_0 L^{\dagger} \gamma_0 = L$. For a free field with canonical dimension $1 = -(3/2)I$, the Lagrangian $\mathscr{L}$ has length dimension -4. The scale invariance is obtained for the massless theory with $L = 3/2 1$. The conformal deficiency vector vanishes identically even for the massive case so that scale invariance also implies special conformal invariance, and $\theta^{\mu\nu}$ has no contribution from a massless spin 1/2 field.

c) Vector Field Theory

$$\mathscr{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A_\mu A_\mu ,$$

where

$$F^{\mu\nu} = -F^{\nu\mu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{15}$$

$$\frac{\partial L}{\partial A_\lambda} = -m^2 A^{\lambda}, \qquad \frac{\partial \mathscr{L}}{\partial(\partial_\rho A_\lambda)} = F^{\rho\lambda} \tag{16}$$

$$\left(\text{Note that } \frac{\partial F_{\mu\nu}}{\partial(\partial_\rho A_\lambda)} = (g^{\rho}_{\mu} g^{\lambda}_{\nu} - g^{\rho}_{\nu} g^{\lambda}_{\mu}), \qquad \frac{\partial(F^{\mu\nu}F_{\mu\nu})}{A_\lambda)} = 2F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial(\partial_\rho A_\lambda)}\right)$$

Euler's equation are

$$\partial_\rho F^{\rho\lambda} = -m^2 A^{\lambda}, \qquad \partial_\lambda A^{\lambda} = -\frac{1}{m^2}\partial_\lambda \partial_\rho F^{\rho\lambda} \tag{17}$$

For $m \neq 0$, then

$$(\Box + m^2) A^{\lambda} = 0. \tag{18}$$

Applying Lorentz invariance, $\Sigma^{\rho\sigma}$ may be easily found to be

$$(\Sigma^{\mu\nu})_{\lambda\sigma} = i(g^{\mu}_{\lambda} g^{\nu}_{\sigma} - g^{\mu}_{\sigma} g^{\nu}_{\lambda}). \tag{19}$$

The commutation relations for $\Sigma^{\mu\nu}$ can be verified to be analogous to Eqn. *(2.28).* The conformal deficiency vector is

$$V' = 2i\{iF^{\nu\lambda} L_{\lambda\sigma} A^{\sigma} + F_{\mu\lambda}(\Sigma^{\nu\mu})^{\lambda\sigma} A,)$$
$$= -2(L_{\lambda\sigma} - g_{,,})\ F^{\nu\lambda} A^{\sigma}. \tag{20}$$

It vanishes if

$$L_{\lambda\sigma} = g_{\lambda\sigma} \qquad \text{or} \qquad L^{\lambda}{}_{\sigma} = g^{\lambda}_{\sigma}. \tag{21}$$

We also note that $[A^{\lambda}] = L^{-1}$. For scale invariance, if $L^{\lambda}{}_{\sigma} = g^{\lambda}_{\sigma}$, theory must be massless which is well known and theory is then conformal invariant. There is no contribution to $\theta^{\mu\nu}$ from massless vector field. We also note that

$$\tau^{\mu\nu} = F^{\mu\nu}\partial^{\nu} A_{\lambda} - g^{\mu\nu}\mathscr{L}, \tag{22}$$

$$\frac{1}{2}\chi^{\lambda\mu\rho} = F^{\lambda\mu} A^{\rho}, \tag{23}$$

$$\hat{\theta}^{\nu\mu} = \hat{\theta}^{\mu\nu} = g^{\rho\nu} F^{\mu\lambda} F_{\rho\lambda} + (\partial_{\lambda} F^{\lambda\mu}) A^{\nu} - g^{\mu\nu}\mathscr{L}$$
$$\stackrel{\Omega}{=} g^{\rho\nu} F^{\mu\lambda} F_{\rho\lambda} - m^2 A^{\mu} A^{\nu} - g^{\mu\nu}\mathscr{L} \tag{24}$$

and

$$\theta^{\mu}_{\mu} = (C,\ F^{\lambda\mu}) A, + 2m^2 A,\ A^{\mu} \stackrel{\Omega}{=} m^2 A,\ A^{\mu}. \tag{25}$$

6. Fields with Anomalous Scale Transformations

To illustrate the consequences of a modified scale invariance condition in case some of the fields do not have the normal scale transformation, we consider a field theory with the fields *{h,)* ≡ h with normal transformation and a single scalar field $\sigma(x)$ with a scale transformation given by

$$\delta x^{\mu} = -\varepsilon x^{\mu}, \qquad \delta\sigma(x) = \varepsilon T\sigma_0, \qquad \partial_{\lambda}(\delta\sigma) = 0, \tag{1}$$

where ***o,*** is a constant field with the dimensions of a mass, i.e., *[o]* $= [\sigma_0] = L^{-1}$. It is convinient to work with a dimensionless field $p(x) = \sigma(x)/M$, $p_0 = \sigma_0/M$, where M is some mass. We have $[p(x)] = [p_0] = L^0$ but $[\partial_{\mu} p(x)] = L^{-1}$, $\delta p(x) = \varepsilon T p_0$. The invariance condition is

$$I_D\mathscr{L} = -4\mathscr{L} + \frac{\partial\mathscr{L}}{\partial\phi} L\phi + \pi^{\lambda}(L + I)\,\partial_{\lambda}\phi + \frac{\partial\mathscr{L}}{\partial p} Tp_0 + \frac{\partial\mathscr{L}}{\partial(\partial_{\lambda}p)}(\partial_{\lambda}p), \tag{2}$$

where from Euler's theorem

$$-4\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi} l\phi + \pi^{\lambda}(l - I)\,\partial_{\lambda}\phi - \frac{\partial L}{\partial(\partial_{\lambda}p)}(\partial_{\lambda}p) - \sum_m m\frac{\partial\mathscr{L}}{\partial m} + \sum_f f\, l_f \frac{\partial\mathscr{L}}{\partial f}. \tag{3}$$

Hence assuming that the fields with normal transformation have the canonical dimension, viz., (L - l) = 0, the scale invariance requires

$$-\frac{\partial \mathscr{L}}{\partial p} T p_0 = -\sum_m m \frac{\partial \mathscr{L}}{\partial m} + \sum_f fl_f \frac{\partial \mathscr{L}}{\partial f} \cdot \tag{4}$$

Writing the Lagrangian $\mathscr{L} = \sum_\gamma g_\gamma \mathscr{L}_\gamma$, where the g_γ are quantities constructed out of m and f, with dimension a,, we obtain

$$\frac{\partial \mathscr{L}_\gamma}{\partial p} T p_0 = -\alpha_\gamma \mathscr{L}_\gamma , \tag{5}$$

or

$$\mathscr{L}_\gamma = \mathscr{L}_\gamma^{(0)} \exp\left[-\frac{p(x)}{Tp_0}\alpha_\gamma\right], \tag{6}$$

where $\mathscr{L}_\gamma^{(0)}$ is independent of $\sigma(x)$ but may depend on $(\partial_\lambda \sigma)$. Thus $\sigma(x)$ apprears in the Lagrangian in a very specific form. Consider, for example, the kinetic energy term of the field; it is of the form $(\partial_\mu \sigma)(\partial^\mu \sigma) A(\sigma) = M^2(\partial^\mu p)(\partial_\mu p) A(p)$, where $A(\sigma)$ is a dimensionless function. Then,

$$\mathscr{L}_{KE} = \mathscr{L}^{(0)} \exp\left[2\frac{\sigma(x)}{T\sigma_0}\right] = \frac{1}{2}(\partial^\mu \sigma)^2 \exp\left[2\frac{\sigma(x)}{T\sigma_0}\right], \tag{7}$$

with appropriate normalization factors.

Another type of anomalous scale transformation is

$$I, o = \varepsilon T[\sigma(x) - \sigma_0]$$

or (8)

$$[\sigma'(x') - o,] \simeq (I + \varepsilon T)[\sigma(x) - \sigma_0].$$

The invariance condition may be discussed as above.

The author is grateful to the *Comissão de Aperfeiçoamento do Pessoal de Ensino Superior* and to *the Conselho Nacional de Pesquisas* for partial financial supports.

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