Revista Brasileira de Física, Vol. 3, N.° 3, 1973

Schwarzschild Coordinates Lead in a Natural Way to Eddington Coordinates in the Kerr-Nut Metric

CARLOS ARAGONE

Departamento de Fisica', Universidad Simón Bolívar, Caracas

Recebido em 21 de Máio de 1973

Schwarzschild (S-) coordinates constitute the easiest coordinates for any kind of local problems concerning Kerr-like universes because they are "more" diagonal. On the other hand, as they contain pseudosingularities, some properties concerning their global structure have to be dealt with in other coordinate systems, which can be classified essentially into Kruskal or Eddington (E-) frames (the former ones being by far the less practical). However, there is no natural way to find the E-frames. In each case, it has been given explicitly what is the transformation law linking S- and E- coordinates and consequently the new shape of the metric tensor. In the present paper, we show a natural and systematic method which allows one to find different systems of E-coordinates, each system corresponding to a specific choice of two analytic functions. From the whole set of E-coordinates for the Kerr-Nut metric we exhibit the Eddington null system which is simpler than the usual temporal E-coordinates. We also clarify the dynamical reasons for the existence of coupled systems of E-frames.

As coordenadas de Schwarzshild (S) são as mais simples para qualquer tipo de problemas locais envolvendo universos do tipo Kerr por serem "mais" diagonais. Por outro lado, pelo fato de conterem pseudo singularidades, algumas propriedades de sua estrutura global têm que ser tratadas em outros sistemas de coordenadas. Esses podem ser classificados em sistemas de Kruskal ou de Eddington, sendo os primeiros muito menos práticos. Não há, todavia, nenhuma maneira natural de se achar sistemas de Eddington (E). Em cada caso, a lei de transformação ligando coordenadas S com coordenadas E (e, consequentemente, a nova forma do tensor métrico) é dada explicitamente. Apresentamos, neste artigo, um método sistemático e natural para determinar sistemas. E cada um desses sistemas corresponde a uma escolha específica de duas funções analíticas. No conjunto de todos os sistemas e para a métrica de Kerr - Nut, exibimos o sistema nulo de Eddington, que é mais simples que as coordenadas temporais de Eddington usuais. Também esclarecemos as razões dinâmicas para a existência de sistemas de Eddington acoplados.

1. Introduction

Since the paper of Finkelstein[1], where Eddington (E-) coordinates attached to a Schwarzschild universe were rediscovered, each time problems concerning maximal analytic extensions are dealt with, either E- frames or Kruskal[2] ones have been found and exhibited.

'Postal address: Apartado Postal 5354, Caracas, Venezuela.

However, there is no physical argument about the existence of pairs of E-coordinates nor suggestions about what to do in order to find analogous systems in forthcoming and more complicated solutions or, in other similar metrics, solutions to other-than-Einsteinian gravitational theories, like the Brans-Dicke theory where massive point particles still move along geodesics.

In the next section, we are going to analyze locally the behaviour of the geodesics of the Kerr-Taub-Nut metric, carefully studied recently from a global point of view by Miller[3], pointing out the following crucial result: there exist well defined limits of certain quantities defined on each geodesic (where the particle tends to the horizons) which do not depend on the initially given values determining them.

In Sec. 3, the properties of the family of E-coordinates are discussed and it is shown how could be found the simplest system of the whole family. It so happens that such an E-system contains a null coordinate.

In the final section, an application to a recently given exact solution (the static quadrupole) is given and the conclusions are emphasized.

2. Geodesics in the Schwarzschild Coordinates

We shall start from the Kerr-Taub-Nut[3-4] metric given in Scharzschild (S-) coordinates:

$$a^{-2}ds_S^2(m, a, e, l) \equiv \rho^2\left[\frac{dr^2}{\Delta(r)} + d\theta^2\right] + \frac{\sin^2\theta}{\rho^2}\left[dt - \sigma^2\, d\varphi\right]^2$$

$$- \frac{\Delta(r)}{\rho^2}\left[dt - A d\varphi\right]^2, \tag{1-a}$$

where (r, t, θ, φ) are dimensionless quantities, a is the Kerr rotational parameter, m the mass of the source, e the charge and l the so called Nut parameter.

Besides these physical parameters, it is convenient to introduce the pure numbers

$$\lambda \equiv la^{-1}, \qquad \mu \equiv ma^{-1}, \qquad r \equiv ea^{-1}, \tag{1-b}$$

and to recall the definition of the functions A, A, p, o, we write

$$A(8) \equiv \sin^2 0 - 2\lambda \cos 8, \quad (1\text{-c})$$
$$\Delta(r) \equiv r^2 - 2pr + 1 + \varepsilon^2 - ,I^2, \quad (1\text{-d})$$
$$\rho^2 \equiv r^2 + (A + \cos\theta)^2 \quad (1\text{-e})$$
$$o^2 \equiv \rho^2 + A = r^2 + \lambda^2 + 1. \quad (1\text{-f})$$

The metric (1-a) has only one non-vanishing off-diagonal element $g_{t\varphi}$. In spite of this apparent simplicity, the Schwarzschild form of the metric presents too many singularities. In fact, the components of the metric tensor $g_{\mu\nu}$ become infinite at the points

$$p^2 = r^2 + (A + \cos\theta)^2 = 0 \quad (2\text{-a})$$

and

$$\Delta(r) = 0, \quad (2\text{-b})$$

while $g \equiv \det\{g_{\mu\nu}\} = -p^4 \sin^2 8$ vanishes at either $\rho^2 = 0$ or $0 = (0, \pi)$. For $A \neq 0$, the singularities at $0 = (O, \pi)$ have to be treated in a specific way because they are not the usual degeneracies of spherical coordinates on the 2- sphere, as has been done in Miller's paper[3]. For $|\lambda| > 1$, $p^2 > 0$ throughout the variety and just in the case $|\lambda| \leq 1$ there is a ring of essential singularities given by $r = 0$, $\sin\theta = \sqrt{1 - \lambda^2}$, contained in the equatorial plane.

The other problem of the S-metric is the vanishing of the second order polynomial $\Delta(r)$, which has two different real roots, a double real root or no real root, according to whether

$$\mu^2 + \lambda^2 \gtreqless \varepsilon^2 + 1 \rightleftarrows m^2 + l^2 \gtreqless a^2 + e^2. \quad (3)$$

If A does not vanish, S- coordinates do not contain pseudosingularities and they constitute a reasonably good system of coordinates to deal with.

In the case where A has two or one real roots, S- coordinates have to be circumvented because the presence of those pseudosingularities, denoted as $r_\pm$, with

$$r, \ \text{E}\ \mu \pm (,u^2 + \lambda^2 - \varepsilon^2 - 1)^{1/2}. \quad (4)$$

The aim of this section is to analyze the local behaviour of the physical geodesics in the vicinity of the 2 horizons r, and therefore to achieve some insight in order to get rid of such kind of apparent troubles in a systematic way.

A complete set of differential equations governing the evolution of a massive or massless test particle along a geodesic is (we use the notation p, r $g_{\mu\alpha} x^a$, $D(\quad)/Ds \equiv (\quad)^{\cdot}$):

$$a^2 t^{\cdot} = \rho^{-2} \sin^{-2}\theta \cdot \Delta^{-1}\{(\sin^2 8 \cdot \sigma^4 - AA^2) e + (AA - \sigma^2 \sin^2\theta) j\}, \tag{5-a}$$

$$a^2 \varphi^{\cdot} = \rho^{-2} \sin^{-2}\theta \cdot \Delta^{-1}\{(\sin^2\theta \cdot a^2 - \Delta A) e + (A - \sin^2 O)), \tag{5-b}$$

$$a^2 \rho^2 \dot{\theta} = p_\theta, \qquad a^2 \dot{r} = \rho^{-2} \Delta(r) p_r, \tag{5-c-d}$$

$$p_t = - e = \text{const.}, \qquad p, = j = \text{const.}, \tag{6-a-b}$$

$$\begin{aligned} a^{-2}\dot{p}_\theta = & - \sin\theta \cdot (A + \cos\theta)(\dot{\theta}^2 + A-' r^2) \\ & + \sin\theta \cdot \rho^{-4}\{\sigma^4\rho^2 + \sigma^4 \sin^2\theta(\lambda + \cos\theta) - \Delta A(\lambda + \cos\theta)(A + 2\rho^2\}\dot{\varphi}^2 \\ & + 2 \sin 8 . \sigma^2 \rho^{-4}(\lambda + \cos\theta)\{\varepsilon^2 - 2\mu r - 2\lambda(\lambda + \cos\theta)\} t^{\cdot} \dot{\varphi} \\ & + \sin\theta . \rho^{-4}\{\rho^2 + (A + \cos 8)(\sin^2 8 - \Delta)\}(t^{\cdot})^2, \end{aligned} \tag{6-c}$$

$$\begin{aligned} a^{-2}\dot{p}_r = & \Delta^{-2}\{r\Delta - p^2(r - \mu)\} \dot{r}^2 + r\dot{\theta}^2 \\ & + \rho^{-4}\{A^2[r\Delta - p^2(r - \mu)] + r \sin^2\theta \cdot a^2(2p^2 - \sigma^2)\}\dot{\varphi}^2 \\ & + 2A \rho^{-4}\{r \sin^2\theta + \rho^2(r - \mu) - r\Delta\} t^{\cdot} \dot{\varphi} \\ & - \rho^{-4}\{\rho^2(r - \mu) + r \sin^2\theta - r\Delta\} (t^{\cdot})^2, \end{aligned} \tag{6-d}$$

with the first integral

$$\begin{aligned} \rho^{-2}\Delta p_r^2 + \rho^{-2} p_\theta^2 + \rho^{-2}(\sin^{-2}\theta - \Delta^{-1}) j^2 + 2\rho^{-2}(\sigma^2\Delta^{-1} - A \sin^{-2}\theta) je & \\ + \rho^{-2}(A^2 \sin^{-2}\theta - \sigma^4\Delta^{-1}) e^2 = - \varepsilon_1 a^2. & \end{aligned} \tag{7}$$

Each geodesic is determined by the values initially assigned to e, j, θ_0, r,, and p_{r_0}; p_{θ_0} can be calculated through Eq. (7), and (φ_0, t_0), the initial values of the hidden variables associated with the Killing vectors, are immaterial. In Eq. (7), ε_1 takes the values (+ 1, 0, – 1) according we are concerned with a temporal, lightlike or spatial geodesic. So, as a matter of principle, each trajectory can be imagined to be a function of the affine parameter s and of the five initial numbers (e, j, θ_0, r,, p_{r_0}).

Let us study the behaviour of a definite geodesic which started at some point with r, $> r_+$, when the particle approaches the first horizon from the outside, that is, when $r(s) \to r_+$ with $r > r_+$. As the particle comes close to the hypersurface $r = r_+$, 8 approaches some definite value which we denote as θ_+. Then, going back to Eqs. (5a-b), we have:

$$a^2 \rho^2 \Delta t^{\cdot} \xrightarrow[r \to r_+]{} \sigma_+^2(\sigma_+^2 e - j), \tag{8-a}$$

$$a^2 \rho^2 \Delta\dot{\varphi} \xrightarrow[r \to r_+]{} \sigma_+^2 e - j, \tag{8-b}$$

where $\sigma_+^2(r_+) \equiv r_+^2 + \lambda^2 + 1$.

Computing the same limit in the first integral (7) by taking into account that $\dot\theta \to \dot\theta_+$ ($|\dot\theta_+| < \infty$) when $r \to r_+$, and introducing Eqs. (5-c) and (5-d) into Eq. (7), it turns out that

$$a^4 \rho^4 \dot r^2 \xrightarrow[r\to r_+]{} (\sigma_+^2 e - j)^2, \tag{8-c}$$

or, in an equivalent way, with e, $= (+1, -1)$ chosen according to the motion of the particle,

$$a^2 \rho^2 \dot r \xrightarrow[r\to r_+]{} \varepsilon_2(\sigma_+^2 e - j). \tag{8-d}$$

Now it is possible to imagine the geodesic parametrized by means of the radial coordinate instead of the affine parameter s and then to obtain a detailed description of the behaviour near the pseudosingularities. of tht: troubles concerning $t(r)$ and $\varphi(r)$. In fact, after Eqs. (8-a-b) and (8-d),

$$\text{A} \frac{dt}{dr} \xrightarrow[r\to r_+]{} \varepsilon_2 \sigma_+^2 = \varepsilon_2(r_+^2 + \lambda^2 + 1) = \varepsilon_2(2\mu r_+ + 2\lambda^2 - \varepsilon^2), \tag{9-a}$$

$$\Delta \frac{d\varphi}{dr} \xrightarrow[r\to r_+]{} \varepsilon_2 \,. \tag{9-b}$$

These equations show where the pseudosingularities of the Kerr-Nut field in S-coordinates stem from. They constitute the Laurent part of the complex functions ($\varphi(r)$, $t(r)$), with a complete independence of the specific geodesic one is dealing with. This last property comes from the fact that neither ε_2 nor ε, σ_+^2 depend upon the particular constants (e, j, θ_0, r_0, p_{r_0}) of the chosen geodesic; on the contrary, the limits (9-a-b) are quantities depending only on the physical parameters (m, e, a, 1).

This is the clue to direct the search for less singular varieties, as we are going to see in the next secion.

3. A Family of Eddington Patches

The non-geodesic dependence of the limits (9) along each geodesic in the Kerr-Nut field gives a hint on how to get rid of these pseudosingularities. Let us consider the family of functions $\mathscr{F}_+ \equiv \{f(r) : f(r_+) = 1)$. Obviously, any function $g(r)$ of the same kind of smoothness as $\{f\}$ such that it takes the value σ_+^2 on r_+ has the structure $g(r) = \sigma_+^2 f(r)$.

Let us pick up a pair (f_1, f_2) of functions belonging to $\mathscr{F}_+$ and define the new system of coordinates $(\hat r, \hat\theta, \hat\varphi, \hat t)$ by

$$\hat{r} \equiv r, \qquad \hat{\theta} \equiv \theta, \tag{10-a}$$

$$d\hat{\varphi} \equiv d\varphi - \varepsilon_2 \frac{f_1(r)}{\Delta(r)} dr, \tag{10-b}$$

$$d\hat{t} \equiv dt - \varepsilon_2 \frac{\sigma_+^2 f_2(r)}{\Delta(r)} dr \tag{10-c}$$

We assert ehat the Kerr-Nut field does not present a pseudosingularity at r $= r_+$ in the $(\hat{t}, \hat{\theta}, \hat{\varphi}, \hat{r})$ representation, as we are going to see immediately. By means of Eqs. (10), the Kerr-Nut metric can be written as

$$\begin{aligned} a^{-2} ds^2_{\wedge} \equiv {} & \rho^2 \Delta^{-1} d\hat{r}^2 + \rho^2 d\hat{\theta}^2 \\ & + \rho^{-2}(\sin^2 \hat{\theta} \cdot \sigma^4 - A^2\Delta)(d\hat{\varphi} + \varepsilon_2 f_1 \Delta^{-1} d\hat{r})^2 \\ & + 2\rho^{-2}(\Delta A - \sigma^2 \sin^2 \hat{\theta})(d\hat{\varphi} + \varepsilon_2 f_1 \Delta^{-1} d\hat{r})(d\hat{t} + \varepsilon_2 \sigma_+^2 f_2 \Delta^{-1} d\hat{r}) \\ & + \rho^{-2}(\sin^2 \theta - \Delta)(d\hat{t} + \varepsilon_2 \sigma_+^4 f_2 \Delta^{-1} d\hat{r}). \end{aligned} \tag{11}$$

In this representation, we are paying a costly price for losing pseudosingularities: we have three off-diagonal non-vanishing components of the metric tensor $\{g_{\hat{r}\hat{\varphi}}, g_{\hat{\varphi}\hat{t}}, g_{\hat{r}\hat{t}}\}$ instead of the unique non-null off-diagonal component $g_{t\varphi}$ that we have in the Schwarzshild coordinates.

Computing (11), we get:

$$\begin{aligned} a^{-2} ds^2_{\wedge} = {} & \rho^{-2}\{\Delta^{-1}[\rho^4 - (Af_1 - \sigma_+^2 f_2)^2] + \sin^2 \theta \cdot \Delta^{-2}(\sigma^2 f_1 - \sigma_+^2 f_2)^2\} d\hat{r}^2 \\ & + \rho^2 d\hat{\theta}^2 + \rho^{-2}(\sin^2 \hat{\theta} - \Delta) d\hat{t}^2 \\ & + 2d\hat{\varphi}\, d\hat{r}\, \varepsilon_2 \rho^{-2}[A(\sigma_+^2 f_2 - Af_1) + \sigma^2 \sin^2 \theta \cdot \Delta^{-1}(\sigma^2 f_1 - \sigma_+^2 f_2)] \\ & + 2d\hat{\varphi}\, d\hat{t}\, \rho^{-2}(\Delta A - \sigma^2 \sin^2 \hat{\theta}) \\ & + 2d\hat{r}\, d\hat{t}\, \varepsilon_2 \rho^{-2}[(Af_1 - \sigma_+^2 f_2) - \sin^2 \theta \cdot \Delta^{-1}(\sigma^2 f_1 - \sigma_+^2 f_2)], \end{aligned} \tag{12-a}$$

which tends to

$$\begin{aligned} a^{-2} ds^2_{\wedge, r \to r_+} \to {} & (r_+ - \mu)^{-1}[2r_+ + Af'_{1+} - \sigma_+^2 f'_{2+} + \tfrac{1}{2} \sin^2 \hat{\theta} \cdot \rho_+^{-2} \cdot \\ & \cdot (2r_+ + \sigma_+^2 f'_{1+} - \sigma_+^2 f'_{2+})] d\hat{r}^2 \\ & + \rho_+^2 d\hat{\theta}^2 + \rho_+^{-2} \sin^2 \hat{\theta} \cdot \sigma_+^4 d\hat{\varphi}^2 + \rho_+^{-2} \sin^2 \hat{\theta} \cdot d\hat{t}^2 \\ & + 2d\hat{\varphi}\, d\hat{r}\, \varepsilon_2[A + \tfrac{1}{2}\sigma_+^2 \sin^2 \hat{\theta} \cdot \rho_+^{-2}(2r_+ + \sigma_+^2 f'_1 - \sigma_+^2 f'_{2+}) \cdot (r_+ - \mu)^{-1}] \\ & - 2\rho_+^{-2} \sigma_+^2 \sin^2 \hat{\theta} \cdot d\hat{\varphi}\, d\hat{t} \\ & - 2d\hat{r}\, d\hat{t}\, \varepsilon_2\{1 + \sin^2 \hat{\theta} \cdot \rho_+^{-2}[r_+ + \tfrac{1}{2} \sigma_+^2(f'_{1+} - f'_{2+})] \cdot \\ & \cdot (r_+ - \mu)^{-1}\}. \end{aligned} \tag{12-b}$$

Since $(r_+ - \mu) \neq 0$, the disappearance of the pseudosingularity located at r_+ in the Eddington-like representation given in Eq. (12-a) is evident.

The r_- pseudosingularitymay be treated in the same·form as we did with r_+ with the replacement of $\mathscr{F}_+$ by $\mathscr{F}_- \equiv \{f(r): f(r-) = 1\}$ whenever necessary in the arguments.

Special interest lies in the subset $(\sigma_+^2/\sigma_-^2)\mathscr{F}_+$ n $\mathscr{F}_- \equiv \mathscr{F}_{+-}$. Had we chosen f_1, f_2 E $\mathscr{F}_{+-}$, we would have obtained a new metric where both pseudosingularities had been smoothed out.

Each time we are considering an Eddington system originated from a pair $(f_1, f_2) \in \mathscr{F}_{+-}$, we are allowed to speak of double E-coordinates.

The standard form of the Kerr-Nut field in E-coordinates is obtained when it is chosen $f_1(r) \equiv 1$ and $\sigma_+^2 f_2(r) \equiv$ 2pr + 2d^2 –c^2. In such cases, the metric (12-a) becomes:

$$
\begin{aligned}
a^{-2}\, ds^2_{\text{`}} = \rho^{-2}&(\sigma^2 + 2\mu\hat{r} + 4\lambda\cos\hat{\theta} - \sin^2\hat{\theta} + 2\lambda^2 - \varepsilon^2)\, d\hat{r}^2 \\
&+ \text{p}^2\, d\hat{\theta}^2 + \rho^{-2}(\sin^2\hat{\theta}\cdot\sigma^4 - \text{A}^2\text{A})\, d\hat{\varphi}^2 + \rho^{-2}(\sin^2\hat{\theta} - \text{A})\, d\hat{t}^2 \\
&+ 2d\hat{\varphi}\, d\hat{r}\, \varepsilon_2\, \rho^{-2}[\sigma^2\sin^2\hat{\theta} + A(2\mu\bar{\hat{r}} + 2\text{d}^2 - \varepsilon^2 - \text{A})] \\
&+ 2d\hat{\varphi}\, d\hat{t}(\Delta A - \sigma^2\sin^2\hat{\theta})\, \rho^{-2} \\
&- 2d\hat{r}\, d\hat{t}\, \varepsilon_2\, \rho^{-2}(2\mu\dot{\hat{r}} + 2\tilde{\text{A}}^2 - \text{c}^2 - 2\lambda\cos\hat{\theta}).
\end{aligned} \tag{13}
$$

But the most interesting system of E-coordinates, it is our feeling, is the system constituted by the choice $f_1 \equiv 1$, $\sigma_+^2 f_2 \equiv 0^2(\text{r}) = r^2 + \lambda^2 + 1$. When we select this pair of key functions to determine the new double E-system, **we get for the Kerr-Nut gravitational field the representation**

$$
\begin{aligned}
a^{-2}\, ds^2_{\text{NE}} = \rho^2\, d\hat{\theta}^2 &+ \rho^{-2}(\sin^2\hat{\theta}\cdot \text{a}^4 - A^2\Delta)\, d\hat{\varphi}^2 \\
&- \rho^{-2}(\Delta - \sin^2\hat{\theta})\, d\hat{t}^2 + 2d\hat{\varphi}\, d\hat{r}\, \varepsilon_2\, \text{A} \\
&- 2\varepsilon_2\, d\hat{r}\, d\hat{t} + 2\rho^{-2}(\Delta A - \sigma^2\sin^2\hat{\theta})\, d\hat{\varphi}\, \text{d?},
\end{aligned} \tag{14}
$$

and each choice of $\varepsilon_2 = (+1, -1)$ leads to a system of coordinates with the same properties, i.e., $|g_{\hat{r}\hat{t}}| = 1$ and $g_{\hat{r}\hat{r}} = 0$. This means that the lines of variable r become lightrays instead of space trajectories of the field as happens in the Schwarzschild representation.

The lines of variable $\hat{t}$ keep the same character they had in the S-representation (1-a). In fact, as

$$
a^{-2}\, ds^2_{\text{S}} = -\rho^{-2}(\Delta - \sin^2\hat{\theta})\, d\hat{t}^2 = a^{-2}\, ds^2_{\text{NE}}, \tag{15}
$$

the lines of variable 2 can be temporal trajectories, lightlike or spatial paths whether the fixed quantities ($\hat{\text{H}}$,, *i.*, , 4,) make positive, null or ne-, gative the term $[\Delta(r_0) - \sin^2\hat{\theta}_0]$.

From the differential form (10) of the transformations, it is immediate to write down the finite corresponding transformation law. It turns out to be

$$\hat{\varphi}_{\mathrm{NE}} \equiv \varphi - \varepsilon_2 (r_+ - r_-)^{-1} \log\{|r - r_+|/|r - r_-|\} + \varphi_0 , \tag{16-a}$$

$$\hat{t}_{\mathrm{NE}} \equiv t - \varepsilon_2 r - \varepsilon_2 (r_+ - r_-)^{-1} \log\{|r - r_+|^{\sigma_+^2} \cdot |r - r_-|^{-\sigma_-^2}\} + t_0 , \tag{16-b}$$

which are one to one in each interval of r where A and $\sigma^2 \Delta^{-1}$ increase or decrease monotonically. It is also worthwhile to work out the starting metric (1-a) in terms of the null Eddington systems (16). With $dt - \sigma^2 d\varphi = d\hat{t}_{\mathrm{NE}} - \sigma^2 d\hat{\varphi}_{\mathrm{NE}}$, one easily gets

$$\begin{aligned} a^{-2} d\hat{s}^2_{\mathrm{NE}} &= \rho^2 d\hat{\theta}^2 + \sin^2 \hat{\theta} \cdot \rho^{-2} (d\hat{t} - \sigma^2 d\hat{\varphi})^2 \\ &\quad - 2(d\hat{t} - A d\hat{\varphi})[\varepsilon_2 d\hat{r} + \tfrac{1}{2} \rho^{-2} \Delta (d\hat{t} - A d\hat{\varphi})], \end{aligned} \tag{17}$$

where the null anholonomous structure of the metric is emphasized. This kind of coordinates, when introduced into the standard Reissner-Nordstrom solution in Schwarzschild coordinates, lead to the wellknown null-E-representation[5]:

$$ds^2_{\mathrm{NE}} = \hat{r}\, d\hat{\theta}^2 + \hat{r}^2 \sin^2 \hat{\theta} \cdot d\hat{\varphi}^2 - (1 - 2mr^{-1} + e^2 r^{-2})\, d\hat{t}^2 - 2\varepsilon_2\, d\hat{t}\, d\hat{r}, \tag{18-a}$$

or what is the same

$$ds^2_{\mathrm{NE}} = \hat{r}^2 d\hat{\Omega} - 2d\hat{t}[\varepsilon_2 d\hat{r} + \tfrac{1}{2}(1 - 2mr^{-1} + e^2 r^{-2})\, d\hat{t}], \tag{18-a'}$$

while the starting diagonal metric is given by

$$ds^2_{\mathrm{S}} = r^2 d\Omega + (1 - 2mr^{-1} + e^2 r^{-2})^{-1} dr^2 - (1 - 2mr^{-1} + e^2 r^{-2})\, dt^2. \tag{18-b}$$

The differential transformation law linking (18-a) with (18-b) looks like

$$\hat{r} = r, \quad \hat{\theta} = \theta, \quad \hat{\varphi} = \varphi, \quad d\hat{t} = dt - \varepsilon_2 (1 - 2mr^{-1} + e^2 r^{-2})\, dr, \tag{18-c}$$

where the two possíbilities for null E-systems stem again from the signature of ε_2.

4. Application

It is worthwhile to apply the previous method to a less treated field, presenting singularities in the components of the gravitational field, which it seems could be annihilated through a coordinate transformation.

Let us consider the static monopole-quadrupole field found by Winicour, Janis and Newman[6] and independently by Young and Alton Coulter[7] on the equatorial plane ($8 = \pi/2$) in spherical coordinates:

$$ds_S^2 = \frac{\lambda^2}{\lambda^2 - 1} a(\lambda)\, b^{-1}(\lambda)\, dr^2 - b(\lambda)\, dt^2 \equiv$$

$$\equiv g_{rr}\, dr^2 + |g_{tt}|\, dt^2, \tag{19-a}$$

where $r \equiv m(\lambda + 1)$; $a(\lambda)$ and $b(\lambda)$ are given by

$$a(\lambda) = \exp\left[\frac{9}{32} q^2(\lambda^2 - 1)^2 \ln^2 (\lambda - 1/\lambda + 1) - 6q + \frac{3}{8} q^2(3\lambda^2 - 4)\right] \cdot$$

$$\cdot\, \lambda^{-2(1+q)^2} \cdot (\lambda + 1)^{1 + q(2+3\lambda) + q^2[1 + \frac{3}{8}(5\lambda - 3\lambda^3)]} \cdot$$

$$\cdot\, (\lambda - 1)^{1 + q(2-3\lambda) + q^2[1 + \frac{3}{8}(3\lambda^3 - 5\lambda)]} \equiv \alpha(\lambda)\, (\lambda - 1)^{p_a(\lambda)}, \tag{19-b}$$

$$b(\lambda) = \exp(-\tfrac{3}{2} q\lambda) \cdot (\lambda + 1)^{-1 + \frac{1}{4}q(3\lambda^2 - 1)} \cdot (\lambda - 1)^{1 - \frac{1}{4}q(3\lambda^2 - 1)} \equiv$$

$$\equiv \beta(\lambda)\, (\lambda - 1)^{p_b(\lambda)}. \tag{19-c}$$

From the structure of the functions $a(\lambda)$ and $b(\lambda)$, we see that g_{rr} has a singularity at the Schwarzschild radius $r = 2m$ $(\lambda = 1)$ for values of q, the quadrupole moment, smaller than $q_M = 1 + \sqrt{5}$. As $|g_{tt}|$ has a nonsingular behavior for values of $q \leq 2$, we are going to assume in this section that we are considering the case $0 \leq q < 2$, more similar to the classical Schwarzschild pseudosingularity in the sense that g_{rr} becomes infinite while g_{tt} remains finite at the sphere $r = 2m$.

Let us consider a radial geodesic contained in the equatorial plane. Its dynamical evolution is determined by the differential system:

$$b(\lambda)\, \dot{t} = e, \tag{20-a}$$

$$m^2 \frac{\lambda^2}{\lambda^2 - 1} a(\lambda)\, b^{-1}(\lambda)\, \dot{\lambda}^2 - b(\lambda)\, \dot{t}^2 = -\varepsilon_1 . \tag{20-b}$$

Introducing the energy into the quadratic integral of motion *(20-b)*, we have for $\lambda(s)$ the equation

$$m^2 \frac{\lambda^2}{\lambda^2 + 1} a(\lambda)\, b^{-1}(\lambda)\, \dot{\lambda}^2 = -\varepsilon_1 + e^2\, b^{-1}(\lambda), \tag{20-c}$$

which near the point $\lambda = 1$ has the structure

$$m\lambda(\lambda + 1)^{-1/2}\, \alpha^{1/2}(\lambda)(\lambda - 1)^{\frac{1}{2}q(2-3\lambda) + \frac{1}{2}q^2[1 + \frac{3}{8}(3\lambda^3 - 5\lambda)]}\, \dot{\lambda} \to \varepsilon_2 e, \quad \text{for} \quad \lambda \to 1. \tag{21}$$

From Eqs. *(20-a)* and (21), we reach the geodesic-independent behavior near the troublesome sphere $\lambda = 1$, which induces the choice of a new coordinate $\tilde{\lambda}$ ($f(1) = 1$):

$$d\hat{t} \equiv dt - m\varepsilon_2 c(\lambda - 1)^{\frac{3}{8}q^2 - 1} f(\lambda)\, d\lambda$$
$$\equiv dt - m\varepsilon_2 c(\lambda - 1)^{\frac{1}{2}p_a(1) - p_b(1) - \frac{1}{2}} f(\lambda)\, d\lambda, \tag{22-a}$$

with

$$c = 2^{1+3q+\frac{7}{8}q^2} \exp\left[\tfrac{3}{4}\, q(q-2)\right] = \alpha^{1/2}(1)\,.\,2^{-1/2}\,.\,\beta^{-1}(1). \tag{22-b}$$

Now it is instructive to write down the Eddington form of the quadrupole field. From Eqs. (22-a) and (19-a), we get

$$ds_E^2 = m^2 \left\{ \frac{\lambda^2\, \alpha(\lambda)}{(\lambda+1)\, \beta(\lambda)} - \frac{\beta(\lambda)\, \alpha(1)}{2\beta(1)^2} (\lambda - 1)^{2p_b(\lambda) - p_a(\lambda) - 2p_b(1) + p_a(1)} f_2(\lambda) \right\}$$
$$\cdot (\lambda - 1)^{p_a - p_b - 1}\, d\hat{\lambda}^2 - b(\lambda)\, d\hat{t}^2$$
$$- 2m\varepsilon_2 c\, \beta(\lambda)\, f(\lambda)(\lambda - 1)^{p_b(\lambda) + \frac{1}{2}p_a(1) - p_b(1) - \frac{1}{2}}\, d\lambda\, d\hat{t}, \tag{23-a}$$

where $f(\lambda)$ could be chosen in such a way that the bracket in $g^{\lambda\lambda}$ be of second order in (L$-$1), allowing consequently a regular behavior of the transformed $g^{\lambda\lambda}$ near the singularity $\lambda = 1$, because

$$p_a(1) - p_b(1) + 1 = \left(1 - \frac{q}{2}\right)^2 \geq 0.$$

So we have for the metric, in Eddington coordinates, near $\lambda = 1$, the structure

$$ds_E^2 \xrightarrow[\lambda \to 1]{} m^2 A(\lambda - 1)^{(1-q/2)^2}\, d\lambda^2 - \beta(1)\,(\lambda - 1)^{(1-q/2)}\, d\hat{t}^2$$
$$- 2m\varepsilon_2\, c\beta(1)\,(\lambda - 1)^{\frac{1}{8}q(q-4)}\, d\hat{t}\, d\lambda, \tag{23-b}$$

which shows that, for $q \leq 2$, the singularity at $\lambda = 1$ in g_{rr} disappears but has been shifted to a nonregular behavior of $g_{\hat{t}\hat{\lambda}}$, independent of the choice of the function $f(\lambda)$.

Such a situation corresponds to the fact that, when the 4- Riemann tensor had been calculated for the Schwarzschild metric (19-a), it should have turned out that $\lambda = 1$ was a truly singular point for it, so that there is no hope for the search for a better coordinate system where troubles at this point could be avoided. Even so, the Eddington system (22) is better suited than Schwarzschild's because, in the whole interval $0 \leq q \leq 2$,

$$\tfrac{1}{8}\, q(4 - q) \leq 1 + p_b(1) - p_a(1) = 1 + \tfrac{1}{2}\, q - \tfrac{1}{4}\, q^2, \tag{24}$$

which shows that the E-metric is "less singular" than the S-metric given at the beginning of this section.

5. Conclusions

We have shown how one could obtain Eddington systems of coordinates in a natural way, starting from S-frames of reference, by analysis of the local behavior of the geodesics in the vicinity of the horizons. This analysis exhibits in a very clear way that pseudosingularities contribute with a pole structure to each geodesic.

The pole structure for Kerr-like fields happens to be of the type $a_{\pm}(r - r_{\pm})^{-1}$ where the residues $s_{\pm}$ do not depend on the specific constraints of each geodesic but on the physical parameters (m, a, e, l) describing the field.

This gives the close connection linking Eddington systems, free of pseudo singularities, and Schwarzschild coordinates, allowing one to go, even in more complex fields, from singular quasi-diagonal systems to better systems which constitute the unit patch in order to get maximal analytic extensions of the initial local field.

It has been shown that the existence of two patches, related through the inversion of the signature of ε_2, stems from the quadratic first integral determining the parameter s. In other words, S-systems provide all the information needed in order to find better systems, in spite of seeming not so well suited because of their pseudo-singularities.

Even in fields where the singularity at the point considered corresponds to a singular structure of the Riemann tensor, E-systems provide a less singular representation of the field than S-coordinates.

Financia1 support from the *Universidad Simón Bolivar* is acknowledged.

References

1. D. Finkelstein, Phys. Rev. **110,** 965 (1958).
2. M. Kruskal, Phys. Rev. **119,** 1743 (1960).
3. J. G. Miller, Princeton preprint (1972).
4. M. Demianski and E. T. Newman, Bull. Acad. Polon. Sci. **14,** 653 (1966)
5. E. Penrose, in ***Batelle Rencontres,*** ed. by C. M. de Witt and J. A. Wheeler (1968), W. A. Benjamin Inc.
6. J. Winicour, A. Janis and E. Newman, Phys Rev. 176, 1507 (1968)
7. J. H. Young and C. Alton Coulter, Phys. Rev. **184,** 1313 (1969).