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Renormalization Groups, Inversibility Postulates and the Finiteness of Z_3

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We first recal the classical techniques and properties of renormalization for the example of the photon propagator. After comparing the different methods used for formulating the renormalization group, we stress that the essential property from which that group results is that the Green's functions are inversible functions with respect to the charge, which implies criteria which are also reviewed. The renormalization group representations of the propagators being a means to express the fundamental property of inversibility, we review its successful applications in perturbation theory and discuss critically what properties these successes really proved. We review also the various implications of the renormalization group on the conjectured equations which might determine the observed or the bare charge. After recalling briefly how the construction of the group representations from inversibility properties allows us to formulate more general groups involving any type of mass or charge renormalization, we show that an inversibility property with respect to the electron mass, implied by unitarity, allows us to give another representation of the photon propagator. The comparison of the two representations of the photon propagator and of their known properties leads us to conjecture that the divergences of Quantum Electrodynamics one encounters in perturbation theory might only be due to a drawback of that method of attack.

Recordamos inicialmente as propriedades e técnicas clássicas de renormalização, tomando como exemplo o propagador do foton. Após compararmos os diferentes métodos empregados na formulação do grupo de renormalização, frisamos que o fato, de serem as funções de Green inversiveis com relação a carga, é a propriedade essencial da qual resulta o grupo de renormalização. Essa propriedade implica certos critérios que também revisamos. Sendo, as representações do grupo de renormalização para os propagadores, uma maneira de exprimir a propriedade fundamental de inversibilidade, revisamos suas aplicações bem sucedidas na teoria de perturbações e discutimos criticamente quais propriedades foram realmente provadas por esses sucessos. Revisamos também as várias implicações do grupo de renormalização sobre as equações que supostamente poderiam determinar a carga observada ou a carga nua. Após recordarmos brevemente como a construção das representações do grupo, a partir das propriedades de inversibilidade, nos permite formular grupos mais gerais envolvendo qualquer tipo de renormalização de massa ou carga, mostramos que uma propriedade de inversibilidade com relação a massa do electron, decorrente da unitariedade, nos permite dar outra representação do propagador do foton. A comparação, entre as duas represen-

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tações do propagador do foton e entre as suas propriedades conhecidas, leva-nos a conjecturar que as divergências encontradas no tratamento perturbativo da Eletrodinâmica Quântica possam ser devidas apenas a uma deficiência do método de ataque perturbativo.

Introduction

Like renorrnalization theory itself, the so-called renormalization group came into the world of theoretical physics through its connection with the divergence problems one faced in the early days of relativistic quantum field theory[1].

It took a few years to discover[2] its practical, and perhaps fundamental, importance in the understanding of high energy behaviours which, so it seems, touch the still hidden and ambiguous looking core of quantum field theory.

Even more time was needed both for realizing that the renormalization group results from an invariance of Lagrangian field theories which is independent of any cut-off or divergence considerations (as is well exemplified in the Lee Model with a cutt-off) and also for obtaining, by the use of a clear-cut mathematical argument, the first exact representations of that group[3].

Basing itself on renormalized Green's functions instead of Lagrangians and field operators, a new formulation of the renormalization group emerged[4] more recently, cutting the umbilical cord connecting it to its doubtful origins. In this way, it could be clearly established which ingredients are needed to construct the renormalization group representations, thereby allowing us to extend substantially their domain of applicability.

In particular, it was realized that a fundamental postulate of inversibility of Green's functions with respect to the coupling constant, which has in itself important consequences independent of perturbation theory[5,6], is also the only thing responsible for the existence of the standard renormalization group whose success in perturbation theory is well known.

Furthermore, the existence of another type of inversibility property which results from unitarity permits us to construct new representations of the propagators[4] which shed new light on the structure of the $Z_3(\alpha)$ function as well as on the possible origin of the infinities ocurring in its perturbative evaluation.

The following chapters are an attempt to present, starting from the most elementary and conventional points of view upon renormalization, the set of results we have just mentioned. Many of these have only appeared without the necessary accent put on their possible importance and mixed up with more technical developments in different papers[4,5] and unpublished reports[6].

In the first chapter, we recall, by taking the example of the photon propagator, the different points of view on the ambiguities and divergences encountered in perturbation theory which historically motivated renormalization theory. We present a brief review of Dyson's method and of Matthews and Salam's method of renormalization, making precise the various concepts and properties of renormalizability which come up in that typical example of renormalizable theory.

In Chapter 2, we introduce the property of normalization invariance which that theory possesses and show that the equations which derive from that invariance can also be obtained from a single manipulation made only upon the renormalized Green's function, thus controlling explicitly the mathematical property involved. These equations being the basis of the renormalization group, one thus makes very clear what are the postulates that the renormalizable Green's functions should obey for the existence of that group.

In Chapter 3, we discuss in detail the fundamental postulate which is tbe *inversibilitv with respect to the charge* α of the renormalized propagator. We also indicate various criteria equivalent to that postulate and establish their connection with the notions of renormalizability exhibited in Chapter 1. After concluding that chapter by discussing the difficulty of proving the exactness of the postulate, we review in Chapter 4 the usual applications in perturbation theory of the renormalization groups, showing that we cannot really deduce from their sucess that the aforementioned postulate is checked with certitude. We also review the implications of the renormalization group equations upon the equations which were conjectured to determine the bare and the physical charges.

Chapter 5 is devoted to a brief recall of the extensions made to the renormalization group for the various types of charge and mass renormalizations.

In Chapter 6, we show that another representation of the photon propagator results from its inversibility with respect to k^2, implied by unitarity. The confrontation and the apparent contradictions of the results of

Chapter 3 and of those which follow from the representation introduced in Chapter 6, leads us in Chapter 7 to propose a mechanism of reconciliation which, if it is true in actual quantum electrodynamics, would imply that the divergences one encounters are only a drawback of perturbative methods.

1. Different Notions and Properties of Renormalizability

a. Dyson's Renormalization Procedure in Perturbation Theory: Regularization Independence and Invariance; Cancellation of Infinities

Let us consider the Lagrangian of quantum electrodynamics (QED),

$$\mathscr{L}_0 = \mathscr{L}_{\text{Dirac}}(\psi^0) + \mathscr{L}_{\text{Maxwell}}(A^0) + ie_0 A_\mu^0 J_\mu^0, \tag{1-1}$$

[which involves the bare (or unrenormalized) photon (A_μ^0) and electron (ψ^0) fields, as well as the bare charge e_0], from which the formal Feynman – Dyson S-matrix series can be deduced.

In the problem of the photon field renormalization, we shall restrict ourselves to the consideration of the simplest term of the vacuum polarization tensor $\Pi_{\mu\nu}^{(1)}$, of first order in $\alpha_0 = e_0^2$, defined through the graph

$$= \alpha_0\, \Pi_{\mu\nu}^{(1)}(x) \sim \alpha_0 \operatorname{Tr}\left[\gamma_\mu S_F(x)\gamma_\nu S_F(-x)\right]. \tag{1-2}$$

This expression in x-space is a ***product of distributions*** $S_F(x)$, which therefore has no clear mathematical meaning. In momentum space, it is given by

$$\mathscr{F}\left[\alpha_0\, \Pi_{\mu\nu}^{(1)}(x)\right] = \alpha_0\, \Pi_{\mu\nu}^{(1)}(k) = (k^2\delta_{\mu\nu} - k_\mu k_\nu)\, \Pi^{(1)}(k^2), \tag{1-3}$$

(plus eventually a constant term which we shall not consider here), where $\Pi^{(1)}(k^2)$ is a ***divergent*** Feynman integral. One encounters here a difficulty which one can characterize as coming from the ambiguity of a product of distributions or else as the failure of an integral to converge. Disregarding the ***origin*** of such kind of difficulty (which may be ***mathematical,*** say, the failure of the perturbation expansion, or *physical*, e.g., the necessity of a fundamental cut-off provided by some mechanism foreign to QED) and accepting the usual S-matrix expansion, two different philosophies where developed to handle such ambiguous, or divergent, expressions. Both of them were of importance in the development of the formulation of the renormalization group.

a. **A** first point of view, which we call the ***regulariztion*** philosophy, was taken by Stueckelberg and Petermann'. They showed that $x^2\, \Pi_{\mu\nu}^{(1)}(x)$

is a regular function $Q^{(1)}_{\mu\nu}(x)$; therefore, taking into account Lorentz invariance, the vacuum polarization tensor can be written as

$$\Pi^{(1)}_{\mu\nu}(x) = \frac{\mathscr{P}}{x^2} Q^{(1)}_{\mu\nu}(x) + (\delta_{\mu\nu} A^{(1)} + \delta_{\mu\nu} C^{(1)} \Box + D^{(1)} \partial_\mu \partial_\nu) \delta^4(x), \tag{1-4}$$

$\mathscr{P}$ denoting the principal value, while $A^{(1)}$, $C^{(1)}$ and $D^{(1)}$ are ***arbitrarv*** "division constants" which are, in general, ***finite.*** Assuming or imposing gauge invariance, the constant $A^{(1)}$ drops out; $D^{(1)}$ is not observable, so that we shall be concerned only with $C^{(1)}$. A similar result was obtained by Caianiello[7], using Hadamard's finite part of an integral.

In the regularization philosophy we can also include Bogoliubov's formulation of S-matrix theory: there, an arbitrariness is already present in the expression for $\Pi^{(1)}_{\mu\nu}(k)$, since it is defined up to an arbitrary polynomial. The more recent developments of Zimmermann[8] are also in the same line of thought.

β) Another point of view is taken in the ***divergence philosophv,*** where one accepts to deal with the divergent Feynman integral $\Pi^{(1)}(k^2)$ and, at the same time, one introduces some limiting regularization scheme in order to give a controlable meaning to the "infinity". This can be done via many different methods which essentially consist either in simply cutting the domain of integration, or modifying the free propagators (a cut-off form factor or analytic regularization, $1/x \to 1/x^\lambda |_{\lambda\to 1}$) or modifying the space volume of integration (the continuous dimension method). In all cases, one recovers the divergence at a certain limit. For simplicity of expression, we shall speak of a *cut-off mass* Λ^2, whose infinite values make the integrals diverge.

The function $\Pi^{(1)}(k^2)$ being given by a Feynman integral, it is always possible to make a separation of the type

$$\Pi^{(1)}(k^2) = B^{(1)}(k^2) + C^{(1)}, \tag{1-5}$$

where $B^{(1)}(k^2)$ is such that $B^{(1)}(k^2, \Lambda^2)|_{\Lambda^2\to\infty} \to$ finite value, while $C^{(1)}(\Lambda^2)|_{\Lambda^2\to\infty} \to$ infinity. In this way, one splits $\Pi^{(1)}$ into a finite part $B^{(1)}$ and an infinite one $C^{(1)}$, both determined up to an arbitrary and finite additive constant.

Let us now resume briefly Dyson's renormalization procedure, first restricting ourselves to the consideration of the vacuum polarization tensor $\alpha_0 \Pi_{\mu\nu}(k^2, \alpha_0)$ in lowest order in α_0. This example is sufficient to illustrate the mechanisms of the various renormalization properties. It should be noticed too that the Lee Model furnishes an exact field theory where the propagator has exactly the same structure as that of the approximate photon propagator we will consider now.

Let us define the *unrenormalized interaction kernel* between two electrons as

$$\alpha_0 \mathbf{\Delta}'_0 (k^2, \alpha_0) = \frac{\alpha_0}{k^2 [1 - \alpha_0 (B^{(1)}(k^2) + C^{(1)})]} = \frac{\alpha_0}{k^2 d_0 (k^2, \alpha_0)}; \tag{1-6}$$

$\mathbf{\Delta}'_0 (k^2, a,)$ is the *unrenormalized photon propagator* and $d_0 (k^2, \alpha_0)$ the *unrenormalized clothing function.*

Defining the *observable charge a* (actually, a is the square of the observable charge) as the residue of $\alpha_0 \mathbf{\Delta}'_0$ at the pole $k^2 = 0$, namely,

$$a = \frac{\alpha_0}{1 - a, [B^{(1)}(0) + C^{(1)}]}, \tag{1-7}$$

one may Introduce the *charge renormalization constant* Z_3 by the relation

$$\alpha_0 = \frac{\alpha}{Z_3}, \tag{1-8}$$

from which results

$$Z_3 = [1 - a, (B^{(1)}(0) + C''')]^{-1} = 1 + \alpha(B^{(1)}(0) + C'')). \tag{1-9}$$

Expressing then $a,$ in terms of $a,$ from (1-7), i.e., $a, = \mathrm{f}(a, B^{(1)}(0) + C''')$, one obtains that *(1-6)* can be written as

$$\alpha_0 \mathbf{\Delta}'_0 (k^2, \alpha_0) = \frac{\alpha}{k^2 [1 - \alpha(B^{(1)}(k^2) - B^{(1)}(0))]} = \frac{\alpha}{k^2 [1 - \alpha B_{\mathrm{R}}^{(1)}(k^2)]} \equiv$$

$$= \frac{a}{k^2 d_{\mathrm{R}}(k^2, a)} \equiv \alpha \mathbf{\Delta}'_{\mathrm{R}} (k^2, a). \tag{1-10}$$

In this equation, the function $B_{\mathrm{R}}^{(1)}(k^2)$ is the *renormalized* closed loop. The renormalized propagator $\mathbf{\Delta}'_{\mathrm{R}} (k^2, a)$ and the renormalized clothing function

$$d_{\mathrm{R}} (k^2, \alpha) = [k^2 \mathbf{\Delta}'_{\mathrm{R}} (k^2, \alpha)]^{-1}$$

are thus related to the unrenormalized ones by proportionality factors, respectively, Z_3^{-1} and Z_3; $d_{\mathrm{R}} (k^2, a)$, which by construction is such that $d_{\mathrm{R}} (0, \alpha) = 1$, is finally obtained by

$$d_{\mathrm{R}}(k^2, \alpha) = [d_0 (k^2, \alpha_0) \cdot Z_3 (\alpha_0)]_{\alpha_0 = f(\alpha, B^{(1)}(0) + C^{(1)})} \tag{1-11}$$

Here one can make the following remarks:

(i) In the regularization philosophy, the indeterminacy or the variation of the constant $C^{(1)}$, for a given a,, implies only a corresponding variation of the physical charge a and of Z_3 as well, the form of the renormalized propagator remaining the same as function of a. Conversely, if a is given, the bare charge a, and Z_3 as well become indeterminate and vary with $C^{(1)}$.

The effect of the variation of the physical constant a, and of the renormalization constant Z_3, with an arbitrary regularization constant such as $C^{(1)}$, is – when extended to all classes of graphs – the basis of the Stueckelberg – Petermann formulation of the renormalization group.

(ii) If, as in the *divergence philosophy*, $C^{(1)}$ is infinite, one can then absorb the divergence in the definition of a,, making $\alpha\Delta'_R$ finite. Of course, if a is the finite physical charge, the unrenormalized charge

$$a, = \alpha[1 + \alpha(B^{(1)}(0) + C^{(1)})]^{-1}$$

will vanish.

When higher orders of a, $\Pi_{\mu\nu}(k^2, a)$ are computed, one obtains the unrenormalized clothing function, with the following structure:

$$\begin{aligned} d_0(k^2, \alpha_0) &= 1 - \alpha_0\, \Pi_0(k^2, \alpha_0, (C)), \\ a,\ \Pi_0(k^2, \alpha_0, \{C)) &= \alpha_0 [B^{(1)}(k^2) + C^{(1)}] \\ &\quad + \alpha_0^2 [B^{(2)}(k^2) + C^{(2)}] \\ &\quad + \alpha_0^3 [B^{(3)}(k^2, C^{(1)}) + C^{(3)}] + \\ &\quad + \alpha_0^n [B^{(n)}(k^2, \{C'')...C^{(n-2)}\}) + C^{(n)}] + \cdots \end{aligned} \qquad (1\text{-}12)$$

The set {C) of constants $C^{(n)}$ is arbitrary in the regularization philosophy and infinite in the divergence philosophy. The dependence of $B^{(3)}(k^2, C^{(1)})$ on $C^{(1)}$ comes from the graphs

and

which depend on the function $B^{(1)}(k^2) + C^{(1)}$ representing the loop o:n the internal photon line. The n-th order loop has also an expression of the form

$$B^{(n)}(k^2, \{C^{(1)} \dots C^{(n-2)}\}) + C^{(n)},$$

since it contains at most one internal loop of order $(n-2)$ and, eventually, many other internal loops of lower orders.

Defining the observable charge a by

$$\alpha = \alpha_0 [d_0(0, \alpha_0, \{C\})]^{-1}, \tag{1-13}$$

one can express a, $= f(\alpha, (C))$ as a function of the set (C) and of the given a. One can then demonstrate, to all orders of perturbation theory, the *essential* relation,

$$\alpha_0 [k^2 d_0(k^2, \alpha_0, \{C\})]^{-1}_{\alpha_0 = f(\alpha, \{C\})} = \alpha[k^2 d_R(k^2, \alpha)]^{-1}, \tag{1-14}$$

which is the *basis* of renormalization theory.

By construction [from (1-13) and (1-14)], $d_R(k^2,$a) is such that $d_R(0, \alpha) = 1$. But also *the function* $d_R(k^2,$a) *has-the fundamental propertv of depending onlv on* k^2, a *being independent of the set* $\{C\}$; *all dependence on the set* (C) *appears in the equation relating* a *and* a,, *only.*

This results in the following *regularization independence*: the structure of the observable functions (of variables k^2 and a, e.g., $\alpha/[k^2 d_R(k^2, \alpha)]$) is *independent* of the regularization procedure.

In the regularization philosophy, one can furthermore establish the existente of classes of sets $\{C\}$, i.e., classes of regularization schemes, such that, a, being fixed, the physical charge a $= \alpha(\alpha_0, \{C\})$ is the same (that is, two sets $\{\tilde{C}^{(1)}, C^{(2)}\}$ and $\{\tilde{C}^{(1)}, \tilde{C}^{(2)}\}$ for which $\alpha_0 C^{(1)} + \alpha_0^2 C^{(2)} = \alpha_0 \tilde{C}^{(1)} + \alpha_0^2 \tilde{C}^{(2)}$). We may call this property *regularization invariance.*

In the divergence philosophy, it follows from the regularization independente that *all the infinities can be absorbed in the relation between* α *and* α_0, namely, *in the charge renormalization constant Z,.* We call this property *infinities cancellation in the observables.* It is usually (and historically) considered as the *renormalization criterion.*

b) The **Matthews-Salam** Counter-Term Renormalization **Method**

We shall now take the standpoint of the divergence philosophy. Then, instead of starting from a given Lagrangian which, when a is finite, contains infinities in the unknown unrenormalized charge a, (this quantity being determined at the end of the calculation by a manipulation of infinities), we shall proceed differently.

First, we fix the physical charge a, known and small; then we construct simultaneously the renormalized propagator and the Lagrangian by adding to it, at each order of perturbation theory, the counter-terms needed to preserve the normalization of the d_R function. This has also the effect of cancelling the infinities involved. In practice, one starts from a Lagrangian of first order in e which has the same form as given by Eq. (1-1), namely,

$$\mathscr{L}^{(1)} = \mathscr{L}^{(0)}_{\text{Dirac}}(\psi) + \mathscr{L}^{(0)}_{\text{Maxwell}}(A) + ieA_\mu J_\mu, \tag{1-15}$$

where e denotes the observable charge. Then, to first order in a, the interaction kernel $\alpha\mathbf{\Delta}' = \alpha/k^2$ has the correct residue. In the next order in a, one gets the term

$$= \frac{\alpha}{k^2}\left[B^{(1)}(k^2) + C^{(1)}\right], \tag{1-16}$$

which however modifies the observed physical charge as

$$\text{a} \to \text{a}\{1 + \text{a}[B^{(1)}(\text{O}) + \text{C}'']).$$

This can be remedied by adding to the Lagrangian $\mathscr{L}^{(1)}$ a counter-terrn $\delta\mathscr{L}$ which turns out to have the form

$$\delta\mathscr{L} = (Z_3 - 1)\mathscr{L}_{\text{Maxwell}}(A), \tag{1-17}$$

which, to this order, is simply

$$\delta\mathscr{L}^{(2)} = \text{a}[B^{(1)}(0) + \text{C}''] \, \mathscr{L}^{(0)}_{\text{Maxwell}}(\text{A}).$$

In this way, a new cancelling graph,

$$= -\frac{\alpha^2}{k^2}\left[B^{(1)}(0) + C^{(1)}\right], \tag{1-18}$$

is introduced which, added to the preceding one, gives a contribution

$$\frac{\alpha^2}{k^2}\left[B^{(1)}(k^2) - B^{(1)}(\text{O})\right] = \frac{\alpha^2}{k^2} B^{(1)}_{\text{R}}(\text{k}^2). \tag{1-19}$$

More generally, one can determine to all orders in a the function involved in Eq. (1-17),

$$Z_3(\alpha) = 1 + \sum_{n=1}^{\infty} \alpha^n \eta_n, \tag{1-20}$$

the η_n being such that the clothing function d_{R} remains normalized at $k^2 = 0$. In this way, one can construct directly the renormalized function $d_R(k^2, \mathrm{a})$. Of course, *such a construction can be made independentlv of the divergence* problem (i.e., the v], could be finite) and it is most fortunate that in QED (a typical renormalizable theory) all infinities coming from graphs with closed loops are cancelled in the observable matrix elements by the effect of the v], counter-terms. This property is again a *renormalizability criterion* based on the infinities cancellation in the observables, like in a).

c) Relation between the Methods of Dyson and Matthews-Salam

The particular form of 6.9 exhibited in Eq. (1-17) allows us to establish the connection between the counter-term method and that of Dyson as well as between the two aspects of infinities cancellation we have exemplified. From Eqs. (1-15) and (1-17), we can write

$$\begin{aligned}\mathscr{L} - \mathscr{L}_{\mathrm{Dirac}} &= \mathscr{L}_{\mathrm{Maxwell}}(A) + ieA_\mu J_\mu + (Z_3 - 1)\mathscr{L}_{\mathrm{Maxwell}}(A) \\ &= Z_3 \mathscr{L}_{\mathrm{Maxwell}}(A) + ieA_\mu J_\mu, \end{aligned} \tag{1-21}$$

where $\mathscr{L}_{\mathrm{Maxwell}}(A)$ is quadratic in A,. Defining the *unrenormalized field*

$$A_\mu^0 = (Z_3)^{1/2} A_\mu, \tag{1-22}$$

Eq. (1-17) can be written just like Eq.' (1), i.e.,

$$\mathscr{L} - \mathscr{L}_{\mathrm{Dirac}} = \mathscr{L}_{\mathrm{Maxwell}}(A^0) + ie_0 A_\mu^0 J_\mu, \tag{1-23}$$

where the *unrenormalized charge*

$$e_0 = e/(Z_3)^{1/2}, \tag{1-24}$$

or

$$\alpha_0 = \alpha/Z_3, \tag{1-24'}$$

is present.

Let us stress the fact that only because 6.9 (which is needed not only for normalizing $d_{\mathrm{R}}(\mathrm{k}^2, \mathrm{a})$ but also for cancelling the divergences) has a close resemblance to $\mathscr{L}$ (or parts of 9) one is allowed to pass from the renormalized to the unrenormalized version of the Lagrangian by a simultaneous rescaling (Eqs. (1-21)-(1-24))of the field and of the coupling constant: that property of 6 2 is therefore connected with the *renormalizabilitv criterion* we defined in the preceding paragraphs.

2. Normalization Invariance and the Formulation of the $G(Z_3)$-Renormalization Group

a) Introduction

In the preceding chapter we have by means of an example recalled very briefly the renormalization procedures and indicated the essential properties which the renormalizable theories possess. We mean that the ambiguities (arbitrariness or infinite values), which one encounters when one makes use of perturbation theory in calculating Green's functions, can be eliminated from the observable quantities by a "rescaling" procedure of the fields and coupling constants. In this way, all the part of the theory which is obscure (arbitrary or infinite) is concentrated in the renormalization constants (Z_3 in the particular case studied here) which appear only in the Lagrangian. Contrariwise, the observable Green's functions (at least in perturbation theory) are well defined and do not exhibit the ambiguities of the equations from which they result. However, one is then dealing with properties which are well known and of the type one could call "experimental" in field theory. The question we shall try to answer in the next two chapters is the following one: which peculiar property must both the unrenormalized and renormalized Green's functions possess in order to exhibit the "miraculous" properties of renormalizability? In order to do that we shall start by noticing the existence of a kind of invariance which is apparently different from those discussed in Ch. 1. The fact that such an invariance could *also* be expressed directly on the renormalized Green's functions, without any appeal to Lagrangians, will give us the clue to find the answer to the question raised above.

b) Normalization Invariance

The method of constructing the renormalized solutions of a field theory allows one to introduce a new kind of arbitrariness which may be called *normalization invariance.* Instead of fixing the given observable charge **a** (defined as the residue at the pole of the interaction kernel), it is also possible to choose a value θ^2 of k^2 at which the interaction kernel has a fixed value, i.e., one gets a O-charge defined by

$$\alpha_\theta = [k^2\,\alpha\Delta'_R]_{k^2=\theta^2}\,; \tag{2-1}$$

note that one has to choose $\theta^2 \geq 0$ in order to have α_θ real.

Such a situation occurs most naturally in conserved-current vector meson theories, where it may be very convenient (because of the Ward cancellation of vertex corrections) to define the observable charge at $k^2 = 0$ rather than at the pole of the vector meson.

Repeating the same argument as for the renormalized case of b), Ch. 1, one would construct both the θ-renormalized propagator

$$\Delta'_{R_\theta} = \frac{1}{k^2\, d_{R_\theta}(k^2, m^2, a,)}, \quad d_{R_\theta}(\theta^2, m^2, \alpha_\theta) = 1, \tag{2-2}$$

and the Lagrangian $\mathscr{L}$ which, like **Eq.** (1-17), takes the form

$$\mathscr{L} - \mathscr{L}_{\text{Dirac}} = Z_3^\theta\,(a,)\, \mathscr{L}(A_\theta) + \text{ie}, A_\mu^\theta J_\mu, \tag{2-3}$$

with

$$Z_3^\theta(\alpha_\theta) = 1 + \sum_{n=1}^{\infty} \alpha_\theta^n\, \eta_n(\theta),$$

allowing one to obtain, by rescaling, the unrenormalized field and charge:

$$A_\mu^0 = (Z_3^\theta)^{1/2} A_\mu^\theta, \quad a, = \alpha_\theta / Z_3^\theta\,(a,), \tag{2-4}$$

and therefore the expression of **Eq.** (1).

Having thus, at least formally, the same **QED** as before, one should also have the same interaction kernel between the electrons, namely,

$$\frac{\alpha_0}{d_0(k^2, m^2, \alpha_0)} = \frac{a}{d_R(k^2, m^2, a)} = \frac{\alpha_\theta}{d_\theta(k^2, m^2, a,)}. \tag{2-5}$$

Of course, one should also have the same result by choosing a different normalization point $\theta'^2 \geq 0$ and the corresponding $\alpha_{\theta'}$-charge. In other words: Eq. (2-5) should be true *for* any $\theta^2 \geq 0$. This is the normalization invariance.

Such an invariance can also be regarded as resulting from the invariance of the Lagrangian under the multiplicative transformation

$$A = A_\zeta \zeta^{1/2}, \quad e = e_\zeta / \zeta^{1/2}, \tag{2-6}$$

which leads to the invariance

$$\frac{\alpha}{d_R(k^2, m^2, \alpha)} = \frac{\alpha_\zeta}{d_{\theta(\zeta)}(k^2, m^2, \alpha_\zeta)}, \tag{2-7}$$

where $\theta^2(\zeta)$ (or θ_ζ^2) is the value of k^2 for which

$$d_R(\theta_\zeta^2, m^2, \alpha) = \frac{1}{\zeta} \tag{2-8}$$

Relations (2-5), which express the normalization invariance, are the mathematical basis of the Bogoliubov-Shirkov formulation of the renormalization group, Since we have dealt only with the Z_3-type of charge renormalization, we shall call it *$G(Z_3)$-renormalization* group.

Let us stress some of its properties. First of all, in deducing Eq. (2-5), we did not make any use of the regularization invariance and, indeed, a relation like (2-5)can be obtained in any finite field theory such as the simple convergent Galilean Lee Model. The above formulation is, therefore, not necessarily equivalent to other "renormalization group" properties related to the regularization invariance. On the other hand, although for the sake of illustration, we made an appeal to perturbation theory in obtaining the relations (2-5), one may also, when other methods of constructing field theoretical solutions are known (like in the Lee model case), consider those relations in a global way, thus insuring their validity within the restricted scheme of perturbation theory, including eventually the accidents (divergences) peculiar to that method.

Since we have obtained relations (2-5) within the framework of Lagrangian field theory, making use of bare fields and coupling constants, which may imply a lack of mathematical rigor (and which involves the big machinery of canonical field theory whose axioms are not yet well clarified), let us give in what follows another deduction of these relations which, because it only uses renormalized Green's functions, allows us to clearly exhibit the mathematical and physical postulates we make use of.

c) The **Axioms** of the **$G(Z_3)$-Renormalization** Group

We assume[4], for a physical charge α arbitrarily given, that the zero-mass photon renormalized Green's function in pure QED (electron of physical mass m) can be generally written in the form

$$\Delta'_R = 1/(k^2 d_R),$$

where – for dimensional reasons (α being dimensionless) – the renormalized clothing function d_R has the dependence $d_R(k^2/m^2, \alpha)$ and is taken with the normalization $d_R(0, \alpha) = 1$. In other words, we start by assuming the *existence* of a function d_R of two variables k^2/m^2 *and* α which, for convenience, is normalized at $k^2 = 0$. This choice of normalization can always be done by dividing an arbitrary function $f(x, a)$ by $f(0, a)$. We moreover suppose that $d_R(k^2/m^2, \alpha)$ *is* real for $k^2 \geq 0$ for α varying in some domain $\alpha_{max} > \alpha > \alpha_{min}$.

Let us then *lefine* a θ-charge, α_θ, by the relation

$$\alpha_\theta \equiv \frac{\alpha}{d_{\mathrm{R}}(\theta^2/m^2, \alpha)}, \qquad \theta^2 \geq 0. \tag{2-9}$$

We assume besides that such a defining relation is ***invertible with respect to*** a for all allowed 0 values. We can therefore solve it in the form

$$a = \alpha_\theta \phi(\theta^2/m^2,\ a,). \tag{2-10}$$

Let us also *define* a *θ-normalized* ***clothing function*** d_θ, which is also a function of α_θ, by

$$d_\theta(k^2, m^2, \alpha_\theta) \equiv \left[\frac{d_{\mathrm{R}}(k^2/m^2, \alpha)}{d_{\mathrm{R}}(\theta^2/m^2, \alpha)}\right]_{\alpha = \alpha_\theta \phi(\theta^2/m^2, \alpha\ )}, \tag{2-11}$$

with $d_\theta(\theta^2, \mathrm{m}^2, \mathrm{a},) = 1$. One then has

$$\frac{\alpha}{d_{\mathrm{R}}(k^2/m^2, \alpha)} = \left[\frac{\alpha}{d_{\mathrm{R}}(\theta^2/m^2, \alpha)}\right] \cdot \left[\frac{d_{\mathrm{R}}(\theta^2/m^2, \alpha)}{d_{\mathrm{R}}(k^2/m^2, \alpha)}\right]\Bigg|_{\alpha = \alpha_\theta \phi(\theta^2/m^2, \alpha_\theta)}, \tag{2-12}$$

as in the last equality of Eq. (1-25). For dimensional reasons (d_θ being dimensionless and θ^2, k^2, m^2 having the dimension of a mass squared), d_θ can be written generally as

$$d_\theta(\mathrm{k}^2, m^2, \mathrm{a},) \equiv z_3(k^2/\theta^2, m^2/\theta^2, \alpha_\theta), \tag{2-13}$$

with $z_3(1, \mathrm{m}^2/0^2, \mathrm{a},) = 1$

We note that the renormalized clothing function d_{R} is a special case of the function z_3, namely,

$$d_{\mathrm{R}} = z_3(\mathrm{k}^2/0^2, m^2/\theta^2, \alpha_\theta)_{\theta=0}\,.$$

We then have

$$\frac{\alpha}{d_{\mathrm{R}}(k^2/m^2, \alpha)} = \frac{\alpha_\theta}{z_3(k^2/\theta^2, m^2/\theta^2, \alpha_\theta)}, \quad \forall\theta^2 \geq 0. \tag{2-14}$$

As we shall see in § d), this last relatíon will suffice to obtain the properties of the renormalization group. One might, however, be interested in also recovering the equivalent to the first of the relations (2-5), which involves the bare constants. This can be achieved as follows.

In canonical field theory, one can formally prove, if Z_3 is finite and $\neq 0$, that one has

$$k^2 \Delta'_{\mathrm{R}}(k^2)\big|_{k^2\to\infty} = \frac{1}{Z_3}, \qquad \text{i.e.,} \qquad d_{\mathrm{R}}(k^2/m^2, \alpha)\big|_{k^2\to\infty} = Z_3, \tag{2-15}$$

the canonical bare charge being $\alpha_0 = \alpha/Z_3$. This property is explicitly verified in finite field theoretical modeis. When Z_3 is divergent in perturbation theory, it also usually happens that

$$d_R(k^2/m^2, \alpha)|_{k^2\to\infty},$$

computed perturbationwise, shows the same type of divergence as the one exhibited by Z_3. Needless to say, the equality of the divergences cannot be established rigorously. Thus, in the present Green's function approach, we take as a *definition of the renormalization constant* "Z_3" the expressiori

$$\text{“}Z_3\text{”} = \lim_{k^2\to\infty} d_R(k^2/m^2, a), \tag{2-16}$$

the "bare" charge (or "asymptotic" charge) "α_0" being, as usual, defined as "a," $= \alpha/$"Z_3".

Let us now consider the function z_3 in the limit $0 \to \infty$. Since z_3 is normalizecl for $\theta^2 \to \infty$, one has from *(2-14)*

$$\alpha_\theta|_{\theta\to\infty} = \frac{\alpha}{d_R(\theta^2/m^2, \alpha)}\bigg|_{\theta\to\infty} = \text{“}\alpha_0\text{”}, \tag{2-17}$$

whereas, from Eqs. *(2-12)* and (2-13),

$$z_3(k^2/\theta^2, m^2/\theta^2, \alpha_\theta)\Big|_{\theta\to\infty} = \frac{\text{“}\alpha_0\text{”}}{\alpha} d_R(k^2/m^2, \alpha)$$

$$= \frac{d_R(k^2/m^2, a)}{\text{“}Z_3\text{”}} \tag{2-18}$$

is the "unrenormalized clothing function, which can be written as

$$z_3(k^2/\infty, m^2/\infty, \text{“}\alpha_0\text{”}) = d_0(k^2/m^2, m^2/\infty, \text{“}\alpha_0\text{”}). \tag{2-19}$$

In this way, we obtain the first of the relations *(2-5).*

With such interpretations, one may then regard the 8-charge as an *interpolating charge* which varies from the physical ($\theta = 0$) to the bare charge ($8 = \infty$), and consider the function

$$z_3(k^2/0^2, m^2/\theta^2, \alpha_\theta)$$

as an *interpolating clothing function* which gives a continuous link between the renormalized and unrenormalized clothing functions as θ^2 goes from zero to infinity.

Let us now discuss and comment on some of the postulates we used for deducing the invariant *(2-14).*

(i) One postulate which will appear to be of paramount importance and that will be discussed in detail in Ch. 3 is that of inversibility with respect to a. It means that the expression (2-9) for a, is to be regarded as a function of a variable a, i.e., a is not a fixed numerical constant. We note that this distinguishes the present formulation from that of Gell-Mann and Low, who looked for renormalized clothing functions $d_R(k^2/m^2,a)$ such that

$$\frac{d_R^{-1}(k^2/m^2,\alpha)}{d_R^{-1}(\theta^2/m^2,\alpha)} = f(k^2/\theta^2, \alpha/d_R(\theta^2/m^2,\alpha)), \tag{2-20}$$

for k^Z, $\theta^2 \gg m^2$. Here, a may be regarded as a numerical constant: only the inversibility of d_R with respect to the variable k^2/m^2 is required. It is however worth mentioning that the right hand side of (2-20)is the function

$$z_3(k^2/\theta^2, m^2/\theta^2, \text{a}, \equiv \alpha/d_R(\theta^2/m^2, \text{a})),$$

whose existence results from the inversibility condition in a exhibited in Eq. (2-10). The postulate (2-20), therefore, coincides with the formulation given above when the limit $m^2 \to 0$ does exist.

(ii) The definitions of α_θ and d_θ can be made more general in the following sense: one could equally well, once a, is defined, have chosen a different value $\theta' \neq 8$ for the point at which the $d_{\theta'}$ function is normalized. In this way, the interpolating clothing function can be expressed in terms of an unrelated interpolating charge, this allowing us to treat, e.g., the renormalized propagator in terms of the bare charge or the other way round[4].

(iii) Finally, one rnight also question why we have chosen the definition (2-9) for a, rather than, say, α/d^2. The reason is that $\alpha/(k^2 d_R)$, the interaction kernel between electrons, is truly the fundamental building block in terms of which all the other Green's functions are constructed: the Green's functions are functionals of $\alpha\Delta'_R$.

7he invariance under the renormalization group, relations (2-14), is another way *of* expressing the invariance *of* the Green's *functions* under a change *of* the normalization parameter θ. That property is of course associated with the structure of the formal Eagrangian, being one of the basic attributes of renormalizability.

However, the advantage of formulating that property in a clear mathematical way is that it allows us to determine the precise mathematical structure the Green's functions should possess and also to know which of the postulates is basic in fixing that structure.

A result which is both strange and physically important will be shown, illustrated and discussed, in the next Chapters: the Green's functions are not any functionals of $\alpha\Delta'_R$, but instead possess a much more stringent structure.

d) Representations of the **Propagators** Invariant under the **G(Z_3)-Renormalization** Group

From the invariance under the renormalization group, expressed by

$$\frac{\alpha_\theta}{z_3(k^2/\theta^2, m^2/0^2, a,)} = \frac{a}{d_R(k^2/m^2, a)}, \quad \forall\theta^2 \mathbf{r}\ 0, \tag{2-21}$$

and the normalization condition

$$z_3(1, m^2/\theta^2, a,) = 1, \tag{2-22}$$

one has, successively,

$$\frac{\alpha_\theta}{z_3(k^2/0^2, m^2/\theta^2, a,)} = \frac{\alpha_{\theta'}}{z_3(k^2/\theta'^2, m^2/\theta'^2, \alpha_{\theta'})}, \tag{2-23}$$

and, for $k^2 = \theta'^2$,

$$\alpha_{\theta'} = \frac{a',}{z_3(\theta'^2/\theta^2, m^2/0^2, a,)} \tag{2-24}$$

Eliminating $\alpha_{\theta'}$ in Eq. (2-23), one gets the fundamental functional equation for $G(Z_3)$ [Ref. 3]:

$$z_3(k^2/0^2, m^2/0^2, a,) = z_3(\theta'^2/\theta^2, m^2/0^2, \alpha_\theta)\cdot$$
$$\cdot z_3(k^2/W^2, m^2/\theta'^2, \alpha_\theta/z_3(\theta'^2/\theta^2, m^2/0^2, a,)). \tag{2-25}$$

Differentiating with respect to k^2/θ^2 and taking then $k^2 = \theta'^2$, one gets the associated Lie differential equation:

$$\frac{a}{\partial(k^2/\theta^2)} \log z_3(k^2/\theta^2, m^2/\theta^2, a,)$$
$$= \left\{\left[\frac{\partial}{\partial x} z_3(x, m^2/k^2, \alpha_\theta/z_3(k^2/\theta^2, m^2/\theta^2, \alpha_\theta))\right]_{x=1}\right\}(\theta^2/k^2)$$
$$= -\frac{\theta^2}{k^2}\phi_3(m^2/k^2, \alpha_\theta/z_3(k^2/\theta^2, m^2/\theta^2, \alpha_\theta)). \tag{2-26}$$

With the initial condition (2-22), one obtains the three equivalent *representations*

$$z_3(k^2/\theta^2, m^2/\theta^2, a,)$$

$$= 1 - \alpha_\theta \int_1^{k^2/\theta^2} \frac{dt}{t} F_3(m^2/\theta^2 t, a, z_3^{-1}(t, m^2/\theta^2, a,)) \qquad (2\text{-}27a)$$

$$= 1 - \alpha_\theta \int_{\theta^2/m^2}^{k^2/m^2} \frac{dt}{t} F_3(1/t, \alpha_\theta z_3^{-1}(tm^2/\theta^2, m^2/\theta^2, a,)), \qquad (2\text{-}27a')$$

with $F_3(x, y) \equiv (1/y)\phi_3(x, y)$, and

$$z_3(k^2/\theta^2, m^2/\theta^2, a,)$$

$$= \exp\left[-\int_1^{k^2/\theta^2} \frac{dt}{t} \phi_3(m^2/\theta^2 t, \alpha_\theta z_3^{-1}(t, m^2/\theta^2, \alpha_\theta))\right]; \qquad (2\text{-}27b)$$

$$\overline{z_3(k^2/\theta^2, m^2/\theta^2, a,)}$$

$$= \alpha_\theta + \int_1^{k^2/\theta^2} \frac{dt}{t}\left[\frac{\alpha_\theta}{z_3(t, m^2/\theta^2, \alpha_\theta)}\right]^2 F_3(m^2/d^2t, a, z_3^{-1}(t, m^2/0^2, a,)) \qquad (2\text{-}27c)$$

$$\int_{\theta^2/m^2}^{k^2/m^2} \frac{dt}{t}\left[\frac{\alpha_\theta}{z_3(tm^2/\theta^2, m^2/\theta^2, \alpha_\theta)}\right]^2 F_3(1/t, \alpha_\theta z_3^{-1}(tm^2/\theta^2, m^2/\theta^2, \alpha_\theta)). \qquad (2\text{-}27c')$$

In particular, for $8 \to 0$, one gets the *renormalized* d_R *function,* a function of the *physical* charge a:

$$d_R(k^2/m^2, \alpha) = 1 - \alpha \int_0^{k^2/m^2} \frac{dt}{t} F_3(1/t, \alpha d_R^{-1}(t, \alpha)) \qquad (2\text{-}28a)$$

$$= \exp\left[-\int_0^{k^2/m^2} \frac{dt}{t} \frac{a}{d_R(t, \alpha)} F_3(1/t, \alpha d_R^{-1}(t, \alpha))\right], \qquad (2\text{-}28b)$$

$$\frac{\alpha}{d_R(k^2/m^2, \alpha)} = \alpha + \int^{k^2/m^2} \frac{dt}{t}\left[\frac{\alpha}{d_R(t, \alpha)}\right]^2 F_3(1/t, \alpha d_R^{-1}(t, \alpha)), \qquad (2\text{-}28c)$$

while, for $\theta \to \infty$, the *unrenormalized* d_0 *function,* a function of the "bare" charge "a_0", is

$$d_0(k^2/m^2, co/m^2, \alpha_0) = 1 - \alpha_0 \int_{\infty/m^2}^{k^2/m^2} \frac{dt}{t} F_3(1/t, \alpha_0 d_0^{-1}(t, \infty/m^2, \alpha_0)) \qquad (2\text{-}29a)$$

$$= \exp\left[-\int_{\infty/m^2}^{k^2/m^2} \frac{dt}{t} \frac{\alpha_0}{d_0(t, \infty/m^2, \alpha_0)} F_3(1/t, \alpha_0 d_0^{-1}(t, \infty/m^2, \alpha_0))\right], \qquad (2\text{-}29b)$$

$$\frac{\alpha_0}{d_0(k^2/m^2, \infty/m^2, \alpha_0)} = \alpha_0 + \int_{\infty}^{k^2/m^2} \frac{dt}{t} \left[\frac{\alpha_0}{d_0(t, \infty/m^2, \alpha_0)}\right]^2 \cdot$$

$$\cdot F_3(1/t, \alpha_0 d_0^{-1}(t, \infty/m^2, \alpha_0)), \qquad (2\text{-}29c)$$

the function F_3 being the same in every case. Obviously, the integrals

$$\alpha \int_0^{k^2/m^2} \frac{dt}{t} F_3, \quad \alpha_0 \int_{\infty/m^2}^{k^2/m^2} \frac{dt}{t} F_3,$$

appearing in Eqs. (2-28a) and (2-29a), are respectively representations of the renormalized and unrenormalized vacuum polarization functions, expressed in terms of the renormalized or unrenormalized charges. The integral

$$\alpha_\theta \int_1^{k^2/\theta^2} \frac{dt}{t} F_3,$$

in Eq. (2-27a′), is the vacuum polarization function renormalized at the energy $k^2 = \theta^2$ and expressed in terms of a,.

The expressions (2-27), (2-28) and (2-29c) can instead be interpreted as representing the propagators written in the Lehmann spectral representation.

It should be noticed that for the integrals in (2-27)-(2-29) to exist, the function ϕ_3 should be analytic in t for $t \geq 0$, z_3 being therefore ≥ 0. It may of course happen, and this is the case in QED, that the property $z_3 \geq 0$ is spoiled by the use of perturbatíon expansions of z_3. Also, if F_3 is independent of its second variable, the representation (2-27a') can be used even when z_3 becomes negative, which is actually the case in the divergent Lee Model.

We next note, as first shown by Sekine[9,10], that a consequence of the functional equation (2-25) is that the operator $U(t = \theta'^2/\theta^2)$, defined by

$$U(t)\begin{Bmatrix} k^2/\theta^2 \\ m^2/\theta^2 \\ \alpha_\theta \end{Bmatrix} \rightarrow \begin{Bmatrix} k^2/\theta^2\, t \\ m^2/\theta^2\, t \\ {}_t = \alpha_\theta z_3^{-1}(t, m^2/\theta^2, \alpha_\theta), \end{Bmatrix} \qquad (2\text{-}30)$$

satisfies the *group properties,* namely,

$$U(t_1)\,U(t_2) = U(t_1\,t_2), \tag{2-30a}$$

and the existence of

$$U(1) \quad \text{and} \quad U(1/t), \tag{2-30b}$$

which justify the denomination of renormalization group.

Extending the techniques just described, one can establish the $G(Z_3)$-representation of the vertex function and of the two Lorentz-invariant scalars appearing in the electron propagator (Ref. *4*). Defining the renormalized electron propagator

$$S'_R = [\gamma\cdot p\ \ d_2(p^2/m^2,\alpha) + mb(p^2/m^2,\alpha)]^{-1},$$

with

$$d_2(-1,\alpha) = b(-1,\alpha) = 1,$$

one has, in particular, the representation of the *electron-mass clothing function*:

$$\frac{M^2(p^2/m^2,\alpha)}{m^2} = \left[\frac{b(p^2/m^2,\alpha)}{d_2(p^2/m^2,\alpha)}\right]^2 = \exp\left[\int_{-1}^{p^2/m^2}\frac{dt}{t}\,F_5(1/t,\alpha d_R^{-1}(t,\alpha))\right] \tag{2-31}$$

(for the def. of F_5 see Eq. *(4-12)*).

e) The Connection of the G(Z_3) Representations with the Renormalization Properties of Perturbation Theory

The calculation in perturbation theory of the function $d_R(x,a), x \equiv k^2/m^2$, resulting from the QED Lagrangian, allows us to determine, as we have seen in Ch. *1,* a series

$$d_R(x,\alpha) = 1-\alpha\,\Pi_R(x,\alpha) = 1-[\alpha\,\Pi_R^{(1)}(x) + \alpha^2\,\Pi_R^{(2)}(x) + \ldots],$$

with $\Pi_R^{(i)}(0) = 0$.

Let us see that this expression can be matched with the representation *(2-28a)*:

$$d_R(x,a) = 1-a\int_0^x \frac{dt}{t}F_3(1/t,\alpha d_R^{-1}(t,a)), \tag{2-33}$$

allowing thus to determine the kernel $F_3(1/x, \nu)$. Performing a Taylor expansion of $F_3(1/x, \nu)$,

$$F_3(1/x, \nu) = \varphi_1(1/x) + \nu\varphi_2(1/x) + \nu^2\varphi_2(1/x) + \nu^2\varphi_3(1/x) + \ldots, \qquad (2\text{-}34)$$

one gets

$$d_R(x, \alpha) = 1 - \alpha\left\{\int_0^x \frac{dt}{t}\varphi_1(1/t) + \alpha\int_0^x \frac{dt}{t}\varphi_2(1/t)\left[1 + \alpha\int_0^t \frac{dv}{v}\varphi_1(1/v) + \ldots\right]\right.$$

$$\left. + \alpha^2\int_0^x \frac{dt}{t}\varphi_3(1/t)\left[1 + 2\alpha\int_0^t \frac{dv}{v}\varphi_1(1/v) + \ldots\right]\right\}.$$

$$= 1 - \alpha\left\{\int_0^x \frac{dt}{t}\varphi_1(1/t) + \alpha^2\int_0^x \frac{dt}{t}\varphi_2(1/t) + \alpha^3\int_0^x \frac{dt}{t}\left[\varphi_3(1/t) + \right.\right.$$

$$\left.\left. + \varphi_2(1/t)\int_0^t \frac{dv}{v}\varphi_1(1/v\right]\right\} + \cdot \qquad (2\text{-}35)$$

where, in each order α^n, there occurs a new function φ_n and a combination of functions φ_i, $i < n$, of lower orders, allowing us thus to compute the $\varphi_n(1/x)$. The matching therefore determines the series **(2-34)** which, if converges for all $x \geq 0$ and for ν in some domain around the origin, sums up to $F_3(1/x, \nu)$. As we will see in Ch. 4, only the first three functions $\varphi_i(1/x)$ were so far explicitly computed (at least in the limit $x \to \infty$), giving that $\varphi_1(0)$, $\varphi_2(0)$, $\varphi_3(0)$ are finite numerical constants different from zero. For $i > 3$, one also demonstrated that the limit $\varphi_i(1/x)|_{x\to\infty}$ is regular, i.e., that the $\varphi_i(0)$ are finite numbers.

Since the same kernel F_3 occurs in the representation **(2-29a)** of the unrenormalized clothing function

$$d_0(k^2/m^2, \alpha_0) = z_3(k^2/\theta^2, m^2/\theta^2, \alpha_\theta)|_{\theta^2\to\infty}$$
$$= 1 - \alpha_0\int_{\theta^2\to\infty}^{k2/m^2} \frac{dt}{t}F_3(1/t, \alpha_0 d_0^{-1}(t, \alpha_0)), \qquad (2\text{-}36)$$

one easily sees that, in perturbation expansion, the function

$$d_0(x, \alpha_0) := 1 - \left[\alpha_0\int_{\theta^2\to\infty}^x \frac{dt}{t}\varphi_1(1/t) + \alpha_0^2\int_{\theta^2\to\infty}^x \frac{dt}{t}\varphi_2(1/t) + \cdots\right], \qquad (2\text{-}37)$$

presents divergences at each order, which are produced by the fact that there exists some $\varphi_i(0) = \text{const.} \neq 0$, these divergences being controlled by the limiting process $\theta^2 \to \infty$.

In that way can be established the link between the normalization invariance (and the resulting $G(Z_3)$ structure) with the divergences aspect of renormalization theory.

3. The Postulate of a-Inversibility

a) Explicitness of the Inversibility Property in the Representations

It is well known that the renormalization group furnishes a powerful tool for solving many problems in field theory and that, as we shall review in the next Chapter, it very simply predicts many asymptotic properties of perturbation theory which can usually be obtained only at the price of performing (when even that can be done) very tedious graph calculations.

From all the formulations proposed to explain the origin of that group and, particularly, from the one given in Ch. 2, Section c), in which no use is made of the cloudy concept of Lagrangian quantum field theory, but one clearly knows what one has to use in order. to get the fundamental functional equation, the remarkable results we just mentioned seem exceedingly astonishing. Indeed, they result from the properties of the arbitrary clothing function, d_R, one started from and on which we only made apparently innocent manipulations!

The fact that one obtains, from the renormalization group equations, useful predictions on the asymptotic behaviours of the perturbative expressions (i.e., the Feynman graphs) and that therefore certain important assumptions on the limits $k \to \infty$ or $m = 0$ have to be made, and the analyticity in **a** admitted as well, is not enough to explain its success: ***the** renormalization group* has a non trivial content which is independent of the approximations and supplementary postulates one usually adds when applying it. This is what we will discuss and illustrate now.

Let us first consider the $G(Z_3)$ representation (2-28*a*) for $d_R(k^2/m^2, a)$. One might be tempted to consider it trivial, since it can be obtained by the following dimensional argument (Ref. 4).

As $d_R(k^2/m^2, a)$ is arbitrary, dimensionless and normalized, let us define

$$\varphi_1(x, \alpha) \equiv \frac{\alpha}{d_R(x, \alpha)} \tag{3-1}$$

From dimensional considerations, one can always write that

$$\frac{\partial}{\partial k^2} d_R(k^2/m^2, \alpha) = \frac{1}{k^2} \varphi_2(k^2/m^2, \alpha) d_R(k^2/m^2, \alpha), \tag{3-2}$$

where φ_2 is dimensionless. From (3-1), one writes a in terms of $\alpha/d_R(x,a)$ and x, namely,

$$\alpha = \varphi^{-1}_{1(x)}(\alpha/d_R(x, \alpha)). \tag{3-3}$$

Substituting this last expression on the RHS of *(3-2)* and performing an integration in k^2, with the initial condition $d_R(0, a) = 1$, one obtains the $G(Z_3)$ representation (2-28*a*) with

$$F_3(x, y) \equiv -\frac{1}{y} \varphi_2(x, \varphi^{-1}_{1(x)}(y)).$$

Let us see however that the ***postulate of inversibility*** with respect to a of **Eq.** *(3-1)* is the essential assumption which leads to Eq. (2-28*a*). Should *it* not be true for the entire range of k^2 values and some k-independent domain of a values, one would then get that

$$\frac{\partial}{\partial k^2} d_R(k^2/m^2, \alpha)$$

is given by a set of different functions of $x \equiv k^2/m^2$ and $v \equiv \alpha/d_R(x, a)$, defined for different domains of a and k^2/m^2:

$$\frac{\partial}{\partial k^2} d_R(x, \alpha) = -\alpha \frac{1}{k^2} [F_3^0(x, y) + \sum_{ij} \theta(\alpha - \psi_i(x)) \theta(\psi_j(x) - \alpha) F_3^{ij}(x, y)], \tag{3-4}$$

where $\theta(x) = 1$ for $x > 0$ and zero for $x < 0$, the $a = \psi_i(x)$ being the roots of the equation

$$d_R(x, \alpha) - \alpha \frac{\partial}{\partial \alpha} d_R(x, \alpha) = 0, \tag{3-5}$$

i.e., $(\partial/\partial k^2) d_R$ would be a function of k^2/m^2, α/d_R *and* a.

In order to avoid a possible confusion with ***another*** property of inversibility (with respect to *m)* to be introduced later on in Ch. 6, we shall call ***a-inversibility*** the one presently discussed, and denote it $G_m(Z_3)$, m being a fixed constant.

b) The Equivalent Properties of Reciprocity and Uniqueness

Let us now consider the representations (2-27*a*) and *(2-27b)* of z_3 and note that the functions z_3 of three variables are expressed in terms of a two variable function, namely,

$$F(m^2/\theta^2\, t, \alpha_\theta/z_3\,(k^2/\theta^2, m^2/\theta^2, \alpha_\theta)),$$

and that

$$\frac{\partial}{\partial k^2}\,[\alpha_\theta^{-1}\, z_3\,(k^2/\theta^2, m^2/\theta^2, \alpha_\theta)]$$

is a function of k^2/m^2 and of the invariant $\alpha_\theta z_3^{-1}$ and is, therefore, independent of θ^2. This means that the class of functions z_3 has been severely restricted by the construction of the group equations. Obviously, here too, the postulate of a-inversibility is responsible for that restriction. Let us elucidate in more detail the significance of that restriction and show that the functional equation (2-25), which results from the inversibility postulate, entails a property of reciprocity between different ways of constructing a renormalized propagator, as well as the uniqueness of the propagators constructed by attributing one value a, associated with a normalization point θ (Ref. 6).

Let a clothing function

$$z_3\,(k^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})$$

be given which, in a specific field theoretical model, might be constructed employing the Matthews-Salam method. Let us also suppose. for the moment, that $\theta_1^2 \neq 0$ and $\neq \infty$. From that function, one would then construct by the method of Ch. 2, Sec. c), a new clothing function $\bar{z}_3$ (normalized at a finite value $\theta_2^2 \neq 0$) defined by

$$\bar{z}_3\,(k^2/\theta_2^2, m^2/\theta_2^2, \theta_1^2/\theta_2^2, \bar{\alpha}) \equiv \left.\frac{z_3\,(k^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})}{z_3\,(\theta_2^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})}\right|_{\alpha_{\theta_1} = \phi\left(\alpha_{\theta_2}, \frac{\theta_2^2}{\theta_1^2}, \frac{m^2}{\theta_1^2}\right)} \tag{3-6}$$

and a function of the charge $\bar{\alpha}$ defined by

$$\bar{\alpha}_{\theta_2} \equiv \frac{\alpha_{\theta_1}}{z_3\,(\theta_2^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})} \tag{3-7}$$

As is obvious by looking at the variables involved, the function $\bar{z}_3$ generally depends on the point θ_1 one started from.

Suppose now that instead of starting from θ_1 to pass then to θ_2 (as we have just done), we started from θ_2 directly, constructing the function

$$z_3\,(k^2/\theta_2^2, m^2/\theta_2^2, \alpha_{\theta_2}),$$

where, physically, we would fix a,, to be the charge at that point, namely,

$$\alpha_{\theta_2} = \bar{\alpha}_{\theta_2}. \tag{3-8}$$

The reciprocity of the two constructions consists in their equivalence, i.e., the validity of the following condition:

$$z_3(k^2/\theta_2^2, m^2/\theta_2^2, \text{a,,}) = \bar{z}_3(k^2/\theta_2^2, m^2/\theta_2^2, \theta_1^2/\theta_2^2, \text{E,,}), \tag{3-9}$$

with

$$\alpha_{\theta_2} = \bar{\alpha}_{\theta_2} \equiv \alpha_{\theta_1}\{z_3(\theta_2^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})\}^{-1}.$$

Replacing (in the LHS) a,, by the expression for $\bar{\alpha}_{\theta_2}$ as a function of a,,, Eq. (3-7), and taking, for the second member, the definition (3-6)-withoiit performing the change of variables a,, $\to \bar{\alpha}_{\theta_2}$ – the condition (3-9) is the functional equation (2-25) for $G(Z_3)$.

Let us note too that the a-inversibility implies the inversibility with respect to a,, of the relation (2-24),

$$\alpha_{\theta_2} = \frac{\alpha_{\theta_1}}{z_3(\theta_2^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})}, \tag{3-10}$$

whose reciprocal is

$$\alpha_{\theta_1} = \frac{\alpha_{\theta_2}}{z_3(\theta_1^2/\theta_2^2, m^2/\theta_2^2, \alpha_{\theta_2})} \tag{3-11}$$

We note that condition (3-9) expresses also that $\bar{z}_3$, as defined by Eq. (3-6), does not depend on the point θ_1 we started from, i.e., *once one* a,, *is fixed in some domain, for a given $\theta_i^2 \geq 0$, the propagator is uniquely determined.* In particular, if the α-inversibility is true for any $k^2 \geq 0$ up to $k^2 \to \infty$ (θ^2 being thus allowed to vary in the domain [0, ∞]), the reciprocity says that it is equivalent to constructing directly the renormalized propagator by the Matthews-Salam method or to deduce it, by a change of variables and normalization, from Dyson's unrenormalized one, i.e., that

$$[d_R^{MS}(k^2/m^2, \alpha) \equiv z_3(k^2/\theta_2^2, m^2/\theta_2^2, \alpha_{\theta_2})|_{\theta_2 \to 0}]_{\alpha_{\theta_2} = \alpha \text{ given}}$$
$$= \left\{ d_R^D(k^2/m^2, \alpha) \equiv \left. \frac{z_3(k^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})}{z_3(\theta_2^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})} \right|_{\substack{\theta_1 \to \infty \\ \theta_2 \to 0}} \right\}, \tag{3-12}$$

where a,, is such that

$$\left. \frac{\alpha_{\theta_1}}{z_3(\theta_2^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})} \right|_{\substack{\theta_1 \to \infty \\ \theta_2 \to 0}} = \text{a (given)};$$

the superscripts M.S. and D stand for Matthews-Salam and Dyson, respectively.

The inversibility also tells us that the propagator normalized at infinity, i.e., the unrenormalized propagator, a function of the bare charge a,, can also equivalently be computed directly (it is Dyson's unrenormalized d_0 function) or constructed from the renormalized Matthews-Salam $d_{\rm R}$ function like in Eqs. (2-18)-(2-19),i.e., that

$$[d_0^{\rm D}(k^2/m^2, m^2/\infty, \alpha_0) \equiv z_3(k^2/\theta_2^2, m^2/\theta_2^2, \alpha_{\theta_2})|_{\theta_2\to\infty}]_{\alpha_{\theta_2}=\alpha_0\,(\text{given})}$$

$$= \left\{ \frac{d_{\rm R}^{\rm M.S.}(k^2/m^2, \alpha)}{d_{\rm R}^{\rm M.S.}(\infty/m^2, \alpha)} \equiv \left. \frac{z_3(k^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})}{z_3(\theta_2^2/\theta_1^2, m^2/\theta_1^2, \alpha_{\theta_1})} \right|_{\substack{\theta_1\to 0 \\ \theta_2\to\infty}} \right\}, \tag{3-13}$$

where a,, $|_{\theta_1\to 0}$ is such that

$$\frac{\alpha}{z_3(\infty/0, m^2/0, \alpha)} = \alpha_0\,(\text{given}).$$

Finally, one also should have, from Eqs. (3-10)-(3-ll), that the inversion of the relation

$$\alpha_0 = \frac{\alpha}{d_{\rm R}(x, \alpha)|_{x\to\infty}}$$

gives the relation

$$\alpha = \frac{\alpha_0}{d_0(0, \alpha_0)} \tag{3-14}$$

c) The Zero Mass Case

We have just seen and discussed how the renormalization group implies that the three variable function z_3 can be expressed in terms of a two variable function F_3. That important constraint on the class of functions z_3 is even more drastically sharpened *if one* supposes that, in the expressions (2-27), the limit $m\to 0$ exists, i.e., $F_3(0, \alpha/z_3) \neq 0$ or ∞. One then has

$$z_3(k^2/\theta^2, \alpha_\theta) = 1 - \alpha_\theta \int_1^{k^2/\theta^2} \frac{dt}{t} F_3(0, \alpha_\theta/z_3(t, \alpha_\theta)). \tag{3-15}$$

Of course, the function $d_{\rm R}$ we started from does not necessarily still exist in that limit.

d) a-Inversibility and Normalization Invariance

We have seen from the Lagrangian point of view, in Ch. 2, Sec. b), that the existence of a normalization invariance leads to the equations of the $G(Z_3)$ group. In Ch. 2, Sec c), we have seen that a-inversibility makes possible the formulation of normalization invariance and, reciprocally, the representation of the group, Eq. (2-28*a*) results from the a-inversibility.

As we shall see in the next Section, the correctness of the $G(Z_3)$ group cannot be taken as exactly proven; one can, however, assert that normalization invariance is based on the physical postulate (or on the realization) that the *definition of the phvsical coupling constants is alwavs an arbitrarv convention.* However, the equations which govern (as well as allows us to calculate) the observables are such that if two observers had chosen different conventions separately and, as a consequence, had obtained different values of some observable constants which uniquely fix the solutions of the equations (and, therefore, the physical system), they would have at the end obtained the same result, since the relation between the two parametrizations is biunique.

Such an invariance is in every way analogous to the one resulting from groups of kinematical invariance (Galilean, relativistic).

In the case we are studying here (the $G(Z_3)$ group), the resulting arbitrariness gives rise to a single arbitrary positive parameter ($\theta^2 \geq 0$) to which the coupling constant α_θ is associated. In fact, such an arbitrariness is really much larger and can be taken as the physical justification of the groups we are going to consider in Ch. 5.

e) What is Unknown about the a-Inversibility Postulate

Though the consequences of a-inversibility we have just discussed result equivalently from the formal field-theoretical construction of the renormalization group of Ch. 2, Sec. b), the question obviously arises of determining whether they *reallv* are satisfied by the explicit solutions of field theories. Because all practical calculations can actually only be done in perturbation theory, it seems exceedingly difficult to prove or disprove the property of a-inversibility. The anomalous dependence, in α, in the representation (3-4), for instance, can very well elude the perturbative expansion, since the $\theta(\alpha-\varphi_i(x))$functions would give a series of distributions $\delta^n(\varphi_i(x))$ which would only contribute if the $\varphi_i(x)$ have zeros at finite x.

Also, an eventual violation of the reciprocity between the relations *(3-14)* does not manifest itself when the inversions of these formulae are done in perturbative expansions, since one chooses thereby, among the various possible determinations of the inverse functions, that one for which $(\alpha_0/\alpha)\to 1$ when a $\to$ 0.

Finally, the fact that one knows how to construct *one* propagator, once θ and a, are given, is by no means a proof of its uniqueness. On the contrary, there exist specific examples of finite field theoretical models for which many different propagators result from the same bare-field equations (Ref. 11), this indicating that the postulates of reciprocity and uniqueness may be violated for $0 = \infty$.

In opposition to analiticity and unitarity, which also impose stringent restrictions on the representations of the propagators but are related to well defined physical axioms, a physical axiom – more objective than the existence of a field theory – on which a-inversibility might be based, is still unknown.

Let us close this series of open questions by mentioning a yet unexplored problem which seems only to require standard mathematics for its solution. We mean the problem of obtaining the structure of the propagators which fulfills simultaneously the postulates of unitarity, analyticity and α-inversibility, bearing in mind that the representations *(2-27a-c')* – which are valid only in the domain of reality of the functions-might possibly induce a correspondingly peculiar form on the analytical cut.

4. Some Applications of $\mathbf{G}_m(\mathbf{Z}_3)$ in Quantum Electrodynamics

A. Perturbative Treatment

Gell-Mann and Low were the first to show some interesting consequences of the renormalization group. Though their construction was different from that given in Ch. *2*, Sec. c), we have seen why they arrived at an approximate $G_m(Z_3)$ structure in which *the ratio* m^2/Q^2 *ivas neglected.*

In practice, a connection between the asymptotic behaviour of Feynman graphs representing the vacuum polarization tensor, and the $G(Z_3)$ representations, can indeed be established by imposing the condition $m^2 \ll \theta^2$, k^2 in the representation *(2-27a)* of z_3. Expanding F_3 in Taylor series,

$$F_3(m^2/\theta^2 t, v) = \varphi_1(m^2/Q^2 t) + v\varphi_2(m^2/\theta^2 t) + v^2\varphi_3(m^2/\theta^2 t) + \ldots \qquad (4\text{-}1)$$

and taking, in each term, the limit $(m^2/Q^2) \to 0$, ***which one supposes to exist,*** one obtains

$$\lim_{(m^2/\theta^2)\to 0} z_3(k^2/0^2, m^2/0^2, \alpha_\theta) = z_3(k^2/0^2, \alpha_\theta)$$

$$= 1 - \alpha_\theta \int_1^{k^2/\theta^2} \frac{dt}{t} F_3(\theta, \alpha_\theta / z_3(t, \alpha_\theta)), \tag{4-2}$$

with

$$F_3(0, y) = \varphi_1 + y\varphi_2 + y^2\varphi_3 + \dots, \tag{4-3}$$

where the

$$\varphi_i \equiv \varphi_i(m^2/\theta^2\, t)|_{(m^2/\theta^2)\to 0} \tag{4-4}$$

should be finite numbers. Inserting (4-3) into Eq. (4-2), one obtains the: form of the most divergent terms which appear in the asymptotic expression of the Feynman graphs for vacuum polarization:

$$z_3(k^2/Q^2, \alpha_\theta) = 1 - \alpha_\theta \Pi_{R_\theta}(k^2/0^2, \alpha_\theta) \equiv$$
$$\equiv 1 - \alpha_\theta(\Pi_1 + \alpha_\theta \Pi_2 + \alpha_\theta^2 \Pi_2 + \cdots)$$
$$= 1 - \alpha_\theta \left[\varphi_1 L + \alpha_\theta \varphi_2 L + \alpha_\theta^3 \frac{\varphi_1 \varphi_2}{2} L^2 + \alpha_\theta^2 \varphi_3 L + \dots \right], \tag{4-4}$$

where $L \equiv \log(k^2/0^2)$ and $(k^2/\theta^2) \gg 1$.

These results agree amazingly well with the results one would obtain by tedious calculations of graphs, namely,

$\Pi_1 \sim$  , $\Pi_2 \sim 2$  +  ,

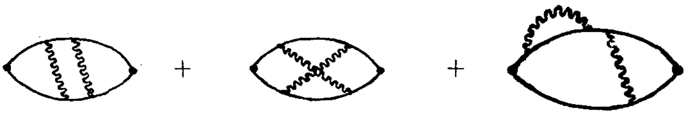
-are no more divergent then a simple log and, besides, graph calculations fix $\varphi_1 = 1/3\pi$ and $\varphi_2 = 1/4n^2$.

At the α_θ^3-order, the term in L^2, which depends on the structure of Π_1 and Π_2, is naturally ***interpreted*** as belonging to the function Π_2 in which the photon is clothed with the Π_1 loop:

 + 2  $\sim \frac{\alpha_\theta^3 \varphi_1 \varphi_2}{2} L^2 = \frac{\alpha_\theta^3}{24\pi^2} L^2$, (4-5)

this result agreeing with the explicit computation[12].

The remaining log term of Π_3 may result from the less divergent part of the above term and, from the graphs with two internal photon lines,



The sum of these graphs should therefore behave, asymptotically, at most as a simple log.

That property was verified by Rosner[13], who obtained the very simple coefficient $(-1/32\pi^3)$ for the log.

The high energy behaviour of the renormalized Feynman graphs can also be obtained from the representation (2-28a) of d_R, which also involves the series defining F_3 which, for $t \gg m$, tends to (4-3). To take into account the low energy contributions, which introduce supplementary asymptotically constant terms in the integration, it is conventional to write d_R in the following form, which is valid only for $k^2/m^2 \gg 1$:

$$d_R(k^2/m^2, a) = d_R(1, a) - a \int_1^{k^2/m^2} \frac{dt}{t} \phi(\alpha/d_R(t, \alpha)), \qquad (4\text{-}7)$$

where $d_R(1, a)$ and $\phi(\beta)$ can be computed[14] from Feynman graphs; giving

$$\phi(\beta) = \frac{1}{3\pi} + \frac{\beta}{4\pi^2} + \left[\frac{8}{3}\zeta(3) - \frac{101}{36}\right]\frac{\beta^3}{8\pi^3} + \ldots, \qquad (4\text{-}8)$$

where $\zeta(3)$ is the value of the Riemann zeta function for the value $s = 3$ of the argument.

Eq. (4-7) can also be written in the form originally given by Gell-Mann and Low, namely,

$$\log u = \int_{\alpha/d_R(1,\alpha)}^{\alpha/d_R(u,\alpha)} \frac{d\beta}{\beta^2\,\phi(\beta)}, \qquad (4\text{-}9)$$

$\psi(\beta) \equiv \beta^2\,\phi(\beta)$ being called the Gell-Mann and Low function.

When m is exactly zero, $\phi(\beta)$ reduces to $F_3(0, \beta)$, for which the first terms of its Taylor expansion are

$$F_3(0, \beta) = \frac{2}{3}\,\frac{1}{2\pi} + \frac{\beta}{(2\pi)^2} - \frac{\beta^3}{4(2\pi)^3} + \qquad (4\text{-}10)$$

As was seen in the calculation of Π_3, when the first few terms were known, the $G_m(Z_3)$ equations allows us to improve perturbation theory by taking into account, in a simple way, the effect of clothing the photon lines. That result also applies in a particularly powerful manner to the electron self--mass function, when one uses the representation (2-31), performing again a Taylor expansion on $F_5(1/t, \beta)$. Starting from the expression valid for $k^2 \gg m^2$, namely,

$$m_0(k^2) \simeq m\left(1 - \frac{3}{4\pi}\alpha \log\frac{k^2}{m^2} + \cdots\right), \tag{4-11}$$

computed from the lowest self-energy graph, which is

 ,

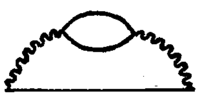
one determines that the first term $J_1(1/t)$ of the series defining $F_5(1/t, \beta)$,

$$F_5(1/t, \beta) = J_1(1/t)\beta + J_2(1/t)\beta^2 + \cdots, \tag{4-12}$$

is such that

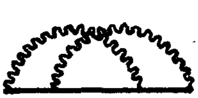
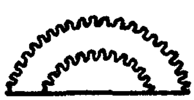
$$J_1(1/t)\big|_{t\to\infty} \to -\frac{3}{2\pi}$$

One then easily obtains that, asymptotically, one has



$$\to -\frac{\alpha^2}{8\pi^2}\log^2\frac{k^2}{m^2}, \tag{4-13}$$

and

 + 

$$\to \alpha^2 \frac{1}{8}\left(\frac{3}{2\pi}\right)^2 \log^2\frac{k^2}{m^2}, \tag{4-14}$$

in complete agreement with the delicate, lengthy, graph calculations[15]. We note that the last result does not follow from clothing the photon line but rather from the repeated clothing by photons of the electron propagator.

These very beautiful and simple results, which come by the use of the representations (2-28*a*) and (2-31), are usually considered as a success or even a proof of the renormalization group equations. In fact, though these results are consistent with $G(Z_3)$, they do not prove its exact reliability but only its validity in perturbation theory. Furthermore, the essential hypothesis made for obtaining the indicated results is that

$$\varphi_i(m^2/k^2)|_{k^2\to\infty} \to \varphi_i < \infty,$$

a property which is specific to QED, as we will see in the next Chapter, and which has nothing to do with the renormalization group. That property, which was admitted since the early days of the renormalization group[2], was only recently demonstrated in general. It is indeed a byproduct of the Callan – Symanzik equations[19], whose basic justification lies in an analysis of the convergence properties in perturbation theory of certain S-matrix elements related to the photon and electron Green's functions, that not only

$$\varphi_i(m^2/k^2)|_{k^2\to\infty} \to \text{finite value} \tag{4-15}$$

but also

$$J_i(m^2/k^2)|_{k^2\to\infty} \to \text{finite value.} \tag{4-16}$$

Without entering into a detailed comparison of the Callan-Symanzik equations and the renormalization group equations, which would not be in place here, let us only recall that the property (3-15) is the essential link between the representations (2-28) and the expression (3-9) where $\psi(\beta)$ is given by its Taylor expansion, and that precisely the relation (3-9) also follows directly from the Callan-Symanzik equations.[20]

B. Global Properties

a) A Possible Finiteness of QED

If we take the exact representation (2-28) and assume that

$$\left.\frac{\alpha}{d_R(u,\alpha)}\right|_{u\to\infty} = \alpha_0,$$

(which may be taken as a definition of a,, cf. Ch. 2, Sec. c) and also that

$$F_3(1/t, \alpha/d_R(t,\alpha))|_{t\to\infty} = F_3(0,\alpha_0)\begin{cases}\equiv 0\\ \neq \infty\end{cases}, \tag{4-17}$$

it then follows that

$$d_R(\infty, a) = Z_3(a) \simeq 1 - \alpha F_3(a,) \log(\infty) + \text{finite terms,} \tag{4-18}$$

and therefore either $Z_3 = \infty$, i.e., $\alpha_0 = 0$ or $F_3(\alpha_0) = 0$ and α_0 *is* a *fixed* number independent of α. This last result, first mentioned by Gell-Mann and Low[2], also follows from **Eq.** (4-9), α_0 being also the root of $\psi(\alpha_0) = 0$ (that root might of course lie at infinity). If α_0 is independent of α, a possibility which cannot be excluded (cf. discussion of Ch. 3, Sec. d), it would invalidate the construction of the renormalization group from a Lagrangian field theory since then two different physical propagators, unrelated by the group operations, would follow from the same Lagrangian.

In the same line of thinking, if one admits that $F_5(m/t, \beta)$ is well approximated, when $t \to \infty$, by its lowest order term $F_5 \simeq -\frac{3\beta}{2\pi}$, one would then get that the bare mass of the electron is *identically* zero:

$$m_0 \equiv m_0(k^2/m^2)|_{k^2\to\infty} = m \exp\left[-\frac{3\alpha_0}{4\pi}\log(\infty)\right] = 0. \qquad (4\text{-}19)$$

Taking these two results (namely that α_0 is a fixed constant and $m_0 \equiv 0$) for granted, one may conclude[16] the finiteness of QED.

b) Conjectures Concerning the Determination of the Fine Structure Constant

b.1. $Z_3(\alpha) = 0 \to \alpha = \mathrm{Max}(e^2)$

It has been shown that the characteristic property of a composite particle is that the renormalization constant of its field is null[16]. When the corresponding Z can be calculated as a function of g^2, like in some models[17], then the condition $Z(g^2) = 0$ fixes the value of the coupling constant. Thus, in the photon case, the condition $Z_3(e^2) = 0$ could fix the value of the charge[18], and we would possibly obtain the fine structure constant. For that, it would be necessary that $Z_3(e^2)$ be a finite function of e^2 (which is not the case in perturbation theory) and one could then test the correctness of the conjecture by computing that function. Leaving the discussion of the finiteness of $Z_3(e^2)$, in non-perturbative theories, for Ch. 7, we show now that α-inversibility implies that the critical value α_c, solution of $Z_3(\alpha_c) = 0$, is also the greatest possible for the e (Ref. 5). Indeed, α-inversibility implies that

$$\alpha_k \equiv \frac{\alpha}{d_R(k^2/m^2, \alpha)}$$

is a monotonic function of a for all $k^2 \geq 0$ and, therefore,

$$\frac{\partial \alpha_k}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\frac{\text{a}}{d_{\mathbf{R}}(k^2/m^2, \alpha)}\right] \text{ is either } > 0 \text{ or } < 0, \text{ for all } k^2 \geq 0, \tag{4-20}$$

(it can eventually be zero for some isolated values of a). Since, for $k^2 = 0$, $d_{\mathbf{R}}(0, \alpha) = 1$, then from Eq. (3-18) follows that

$$\left.\frac{\partial \alpha_k}{\partial \alpha}\right|_{k \to 0} = 1$$

and thus one has

$$\frac{\partial \alpha_k}{\text{da}} > 0 \quad \text{for all } \text{k}^2 \geq 0. \tag{4-21}$$

Therefore, when $k^2 \to \infty$, a, r $\alpha/d_{\mathbf{R}}(\infty, \text{a,})$ is such that

$$\frac{d\alpha_0}{d\alpha} \equiv \frac{d}{d\alpha}\left[\frac{\alpha}{d_{\mathbf{R}}(\infty, \alpha)}\right] > 0, \tag{4-22}$$

and this implies that if a, goes to infinity at a finite value α_c of a (which is its greatest value), then α_c is a solution of $Z_3(\alpha_c) = 0$ The converse, however, is not true: one might have a maximum value of a which would not be a zero'of Z_3(a).

The property of a-inversibility being a fundamental criterion for renormalizability, such properties should be checked in all known field theoretical models in which $Z(g^2)$ can be computed. This is precisely the case for the Lee Model, where the composite V particle has the strongest possible interaction. Let us also note that since'one cannot exclude the case where a-inversibility would not hold at $k^2 \to \infty$, one would have in that case

$$\left.\frac{\partial \alpha_k}{\partial \alpha}\right|_{k \to \infty} \to 0_+ . \tag{4-23}$$

b.2. $F_3(0, a) = 0$

We have seen that, under certain conditions, the Gell-Mann and Low condition, $\psi(\alpha_0) = 0$, yields an-equation for the *bare charge* α_0, *a,* being determined independently of the physical charge. This same constant a, can also be obtained from the zero-mas OED condition $F_3(0, \text{a,}) = 0$.

It has however been argued[18], on the basis of a particular method of summing Feynman graphs, that the function $F_3(0, \beta)$ vanishes on the entire physical domain, having a zero of infinite order at the value of the physical charge, i.e., $F_3(0, \alpha) = 0$, $(\partial/\partial\alpha)^n F_3(0, \alpha) = 0$, this implying that $d_R = 1$. It was also conjectured that the property $F_3(0, \alpha) = 0$ would fix e^2, which might yield the fine structure constant.

5. The Extended Groups

Though, in pseudo-scalar meson theory, charge renormalization involves not only Z_3 but also Z_1 and Z_2 (which are different), the Z_3 normalization invariance still exists and, therefore, the $G_m(Z_3)$ renormalization group should still apply. It however fails if applied as in QED because Z_3, e.g., to the order $(G^2)^2$ diverges as $(\log)^2$, whereas the prediction of Eq. (4-4) gives a log divergence! To study rigorously such a failure, it is necessary to extend the formulation of the renormalization group in such a way as to include the Z_1 and Z_2 types of clothing effects and to deal exactly with the mass parameters as well.

It is in principle easy to generalize the method of constructing renormalization groups, starting from Green's functions as in Ch. 2, Sec. c). The only physical problem consists in choosing the most convenient combination of Green's functions in terms of which the interpolating variables are defined. The only mathematical assumption consists in admitting the possibility of performing a change of variables, i.e., a postulate of invertibility[4]. The physical justification of that postulate was explained in Ch. 3, Sec. d), but its exact validity in field theory can only be ascertained (or invalidated) a posteriori.

The number of groups that can be constructed is enormous. Indeed, instead of a single two-variable function involved in the two Z_3 groups, in zero-mass photon QED, one needs – for a complete description of the renormalization properties of a meson theory – four functions of three variables (g, m/p, μ^2/p^2) related to Z_3, Z_2, δm and $\delta\mu^2$, as well as one function of five variables related to Z_1 (the vertex is indeed defined by three external masses, two internal masses (μ, m) and g, the masses occurring as ratios), plus terms related to an eventual meson-meson coupling.

A general formulation of charge groups $G(Z_1^2 Z_2^{-2} Z_3^{-1})$ was established, but particularly interesting are its subgroups: $G(Z_1), \ldots, G(Z_1^2 Z_3^{-1})$, $G(Z_2^{-2} Z_3^{-1})$ which, under the same condition of regularity (when $m \to 0$), allow one to compute the asymptotic behaviour of almost all graphs, the results being in agreement with the expressions obtained from direct calculation of known Feynman graphs[21,22].

If θ_1^2, θ_2^2 and θ_3^2 denote respectively the normalization parameters of (i) the vertex function with zero external photon momentum, (ii) the $\gamma \cdot p$ coefficient of the inverse electron propagator and (iii) the photon clothing function, the technique we mentioned above allows us to construct the group $G(Z_3 Z_2^2 Z_1^{-2})$. When $\theta_1^2 = \theta_2^2 = -m^2$, this group reduces to $G(Z_3)$.

In this context, it has been shown[22] that one obtains the result cf Fevnman graph *computations* cf the z_3 function in QED, bv requiring that the limit $\theta_1^2 = \theta_2^2 \to -m^2 = 0$ be regular for the perturbation expansion. This requirement of regularity is also shown to be equivalent to *Ward's* identitv and is related to the mass singularity theorems[23]. If, instead, θ_1^2 and θ_2^2 are fixed and the limit $m \to 0$ performed (and assumed regular), one obtains that z_3 behaves like

$$z_3 \sim 1 - g^2 c_1 \log \frac{k^2}{\theta_3^2} - g^4 \left(c_2 \log^2 \frac{k^2}{\theta_3^2} + d_2 \log \frac{k^2}{\theta_3^2} \right) + \ldots, \qquad (5\text{-}1)$$

in'agreement with the expressions of the p.s. meson theory, thus resolving the failure of the $G(Z_3)$ group mentioned at the beginning of the chapter, the vertex function and the z_3 function being there normalized on their mass shell, i.e., the limit performed is $\theta_1^2 = \theta_2^2 = -m^2 \to 0$.

When one links together the normalization parameters of the different Green's functions, one obtains simpler groups, namely, the bountl *groups*[4]. If that link is such that the parameters reach simultaneously the values corresponding to the renormalized expressions and, besides, they tend simultaneously to infinity, a one-parameter bound group, namely, $G(Z_1^2 Z_2^{-2} Z_3^{-1})$, allows one to pass continuously from the renormalized to the unrenormalized theory, and an integral equation for the invariant of the group can be constructed, leading (when the mass is zero or can be neglected) to a Gell-Mann and Low type of equation for the group invariant. These types of groups are of practical importance in the theories of critical phenomena[10] and are even unavoidable for treating consistently the infrared divergence problems of super-renormalizable theories[24].

Groups for mass renormalization, i.e., groups relating mass renormalization to the coupling constant renormalizations, can also be constructed[4] (e.g., the $G(Z_3, m_0)$ and its bound restriction $G(\widehat{Z_3, m_0})$ for QED). With these groups, it is possible to show that if m, the electron physical mass, is different from zero, but the bare mass m does vanish, then a is fixed unless α-inversibility fails at infinity. Then the kernel of the representation for d_R, namely,

$$F_3^{(m_0=0)}(\alpha/d_R(t,\alpha)),$$

is a function of a single variable. It follows then, without any approximation, that the asymptotic behaviour of d_R, to all orders in a,, is no more divergent than a simple log., as has been shown by Feynman graph calcultions[16]. The bare charge a, is then unambiguously fixed by the condition

$$F_3^{(m_0=0)}(\alpha_0) = 0$$

which, since a is also fixed by the condition $m_0 = 0$ is consistent with the existence of a relation between a, and a. The comparison between the representation of the kernel $F_3^{(m_0=0)}$ and the one resulting from the group $G_\alpha(Z_3)$ (which we introduce in the next Chapter), will confirm this last conclusion, also showing that the perturbative expansion in α_0 would be meaningless if that situation (i.e., m $\neq 0$ $m_0 = 0$) were the actual physical situation.

6. Unitarity, k-Inversibility and the Hybrid Renormalization Group

a) It is known that when unitarity and causality are satisfied, the function d_R can be written as

$$d_R(k^2/m^2,\alpha) = 1 - \alpha k^2 \int_0^\infty \frac{\Pi_1(a)\,da}{a + k^2 - i\varepsilon}, \qquad \Pi_1(a) \geq 0, \tag{6-1}$$

and that furthermore $d_R(k^2/m^2, a) > 0$, except eventually for $k^2 = \infty$ for which one may have $d_R(\infty, a) = 0$ a case realized when $Z_3(a) = 0$. Therefore, for $k^2 \geq 0$,

$$\frac{\partial}{\partial k^2} d_R \leq 0, \quad \frac{\partial}{\partial k^2} \frac{\alpha}{d_R(k^2,\alpha)} \geq 0. \tag{6-2}$$

Thus $\alpha/d_R(k^2/m^2, a)$ is inversible with respect to the variable k^2: this is k-inversibilitv. Actually, since the photon has zero mass and k always appears in the ratio k^2/m^2, one also has an m-inversibilitv. Defining

$$\alpha_k \equiv \frac{\alpha}{d_R(k^2/m^2,\alpha)}, \tag{6-3}$$

one then can express k^2/m^2 in the form

$$\frac{k^2}{m^2} = \phi(\alpha_k, \alpha), \qquad k^2 > 0. \tag{6-4}$$

Repeating Ch. 2, Sec. d, we obtain a new renormalization group which we call hybrid[4], since it mixes the physical and the interpolating charges. We denote the group by $G_\alpha(Z_3)$ and remark that a is kept fixed in defining the group. The functional equation is similar to Eq. (2-25), the variable m^2/θ^2 being replaced by a. In this way, new $G_\alpha(Z_3)$ representations are obtained of a form similar to Eq. (2-27) where, e.g., the kemel $F_3(1/t, \alpha/z_3)$ is replaced by H, (a,α/z_3):

$$z_3(k^2/\theta^2, m^2/\theta^2, \alpha) = 1 - \alpha \int_1^{k^2/\theta^2} \frac{dt}{t} H_3(\alpha, \alpha/z_3(t, m^2/\theta^2, \alpha)), \tag{6-5}$$

$$d_{\mathrm{K}}(k^2/m^2, \alpha) = 1 - \alpha \int_0^{k^2/m^2} \frac{dt}{t} H_3(\alpha, \alpha/d_{\mathrm{R}}(t, \alpha)), \tag{6-6}$$

and similarly for the groups $G(Z_1^2 Z_2^{-2} Z_3^{-1})$ introduced in Ch. 5.

b) Though $G_\alpha(Z_3)$ is physically better founded than the $G_m(Z_3)$ group, since it results from the inversibility properties implied by unitarity, it is however the more recent and therefore the less explored of the two.

One of the great advantages of the $G_\alpha(Z_3)$ representation of propagators,

$$d_{\mathrm{R}}(x, \alpha) = 1 - \alpha \int_0^x \frac{dt}{t} H_3(\alpha, \alpha/d_{\mathrm{R}}(t, \alpha)), \tag{6-7}$$

is that it can be writ k^2*exactly* in the form

$$\log \frac{}{\theta^2} = \int_{\alpha/d_{\mathrm{R}}(\theta^2/m^2, \alpha)}^{\alpha/d_{\mathrm{R}}(k^2/m^2, \mathrm{a})} \frac{d\beta}{\beta^2 H_3(\alpha, \beta)}, \tag{6-8}$$

which generalizes the approximate formula of Gell-Mann and Low, Eq. (3-9), for $G_m(Z_3)$, which is valid only at high energies and under the conditions that the Taylor expansion (4-1) converges and that Eq. (4-15) is true. It follows at once that the bare charge a, $\equiv \alpha/d_{\mathrm{R}}(\infty, \mathrm{a})$ is a solution of the equation

$$H_3(\alpha, \alpha_0) = 0, \tag{6-9}$$

and therefore that, in general[4], a, is a *function of* α.

The difficulty with the above representation comes from the fact that H,, depends on a in its two variables, thus preventing one from establishing in an obvious way its connection, via a Taylor expansion, with Feynman graphs.

To get an idea of the shape of the two variable function $H_3(\alpha, \beta)$, one can first require the function $d_R(x, e^2)$ to possess, for the actual value a $= e^2$, all known and desirable properties. Since the representation extends itself down to the physical cut, i.e., down to the first pole at $k^2 = -(2m-\varepsilon)^2$ of the lowest positronium bound state (if we neglect the small correction of the 3-photon cut), one can require the following properties (which fix a certain dependence in β of the function $H_3(\alpha, \beta)$):

(i) Regularity at $k^2 = 0$, i.e.,

$$H_3(a, a) = 0, \quad a^2 \left.\frac{\partial H_3(\alpha, \beta)}{\partial \beta}\right|_{\beta=\alpha} = 1; \tag{6-10}$$

(ii) Unitarity, i.e.,

$$\frac{a}{\partial k^2} d_R < 0 \quad \text{for } -(2m-\varepsilon)^2 < k^2 \leq \infty \tag{6-11a}$$

(we note that k-inversibility does not fix the sign of the inequality in (6-lla)). From (6-11*a*), it follows that

$$\begin{aligned} H_3(\alpha, \beta) &\geq 0, \quad \alpha \leq \beta \leq \alpha_0, \\ H_3(\alpha, \beta) &< 0, \quad 0 < \beta \leq \alpha; \end{aligned} \tag{6-11b}$$

(iii) The known behaviour of the vacuum polarization tensor, near the lowest positronium pole, which gives

$$H_3(\alpha, \beta)|_{\beta \to 0} \to -4/\beta^2 \alpha^3. \tag{6-12}$$

In what concerns the dependence of $H_3(\alpha, \beta)$ in the a variable, one can say the following. From Ch. 4, Sec B.b) and the a-inversibility property, one knows that $[\partial \alpha_0(\alpha)/\partial \alpha] > 0$ (Eq. (4-22)) and also that there might possibly exist a maximum value a, for which $\alpha_0(\alpha_c) = \infty$.

The above requirements, Eqs. (6-10)-(6-12) and (4-22), have been used in drawing Fig. 1. On the other hand, *when* m = 0, one has

$$F_3(1/t, \beta)|_{m=0} \equiv F_3(0, \beta) = 1/3\pi + \beta/4\pi^2 + \ldots.$$

If this series does converge for $\beta \to 0$, then $F_3(0,0) \neq 0$; also, if $F_3(0, \beta) \neq 0$,

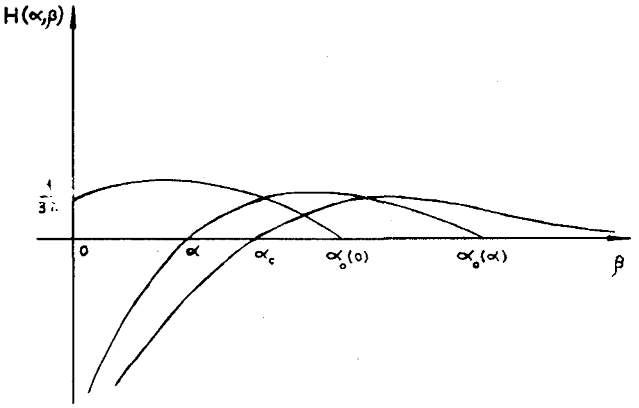


Figure 1

one has

$$z_3(k^2/\theta^2, \alpha_\theta) = 1 - \alpha_\theta \int_1^{k^2/\theta^2} \frac{dt}{t} F_3(0, \alpha_\theta/z_3(t, \alpha_\theta))$$

and the physical charge vanishes:

$$\alpha(m = 0) \equiv \frac{\alpha_\theta}{z_3(0, \alpha_\theta)} = 0. \qquad (6\text{-}13)$$

Thus, $H_3(\alpha = 0, \beta) = F_3(0, \beta)$ and the corresponding bare charge $\alpha_0[\alpha(m = O)] = a, (O)$ should be a solution of $F, (0, a,) = 0$. However, since for $a = 0$ one has also the solution $a, = \alpha = 0$, one may consider the axis $\beta = 0$ as asymptotic to the ascending branch of $H, (\alpha, \beta)$ for $r \to 0$.

Let us finally consider the case $m \neq 0$, $m_0 = 0$, discussed in Ch. 5. The function $H_3(\alpha(m_0 = O), \beta)$ is then identical to $F_3^{(m_0=0)}(\beta)$ and, if this case were an actual physical situation, it is clear from the behaviours given above that a Taylor expansion, around $\beta = O$, is meaningless.

7. Are the Divergences a Drawback of Perturbation Theory?

By comparing the two renormalization groups of QED, $G_m(Z_3)$ and $G_\alpha(Z_3)$, which result from the two different inversibility properties, we finally arrived at two pictures of the function $d_R(x, \alpha)$ which are complementar~in the better known cases but which are contradictory when we try to extrapolate our knowledge up to infinity. The situation is summarized in Table 1.

	$G_m(Z_3)$	$G_\alpha(Z_3)$
Postulate	a-Inversibility	k-Inversibility
Condition	$\frac{\partial}{\partial\alpha}(\alpha/d_R(x,\alpha))$, $d_R(x,a)$ real	$\frac{\partial}{\partial x}(\alpha/d_R(x,\alpha))$, $d_R(x,a)$ real
Physical Axiom	Lagrangian Field Theory(?)	Unitarity
Kernels	$F_3(1/t,\alpha/d_R(t,\alpha))$	$H_3(1/t,\alpha/d_R(t,\alpha))$
"Experimental Properties"	$F_3\vert_{\ } = \sum_1^\infty \varphi_n(t)(\alpha/d_R(t,a))$"-'I,,, → Perturbative asymptotic behaviour of Feynman graphs: if $\varphi_n(0) \neq \infty$, $\varphi_1 = 1/3\pi$, $\varphi_2 = 1/4n^2, \ldots$	Global low energy behaviour: $H_3(\alpha,\alpha) = 0$, $\partial_\beta H, (a,\beta)\vert_{\beta=\alpha} = 1/a^2$, $H_3(\alpha,\beta)\vert_{\beta\to 0} \to -4/\beta^2\alpha^3$.
"Natural" Expected Global Constraint at High Energy	$F_3(1/t,\alpha/d_R(t,\alpha))\vert_{t\to\infty} = F_3(0,\alpha_0) = 0$	$H_3(\alpha,\alpha_0) = 0$
Consequence	a, independent of a, i.e., $\frac{\partial}{\partial\alpha}(\alpha/d_R(\infty,\alpha)) = 0$	$\alpha_0 = f(a)$, i.e., $\frac{\partial}{\partial\alpha}(\alpha/d_R(\infty,\alpha)) \neq 0$

Table 1

Stressing the fact that both pictures admit the finite ness of *a,*, the question is therefore to obtain consistency between the two equations which are expected to fix a,. This can be done in two ways.

Either one can suppose that H, $(a,a,) = 0$ is really an identity, this meaning that the dependence in *a* disappears when

$$\alpha(k)\big|_{k\to\infty} \to \alpha_0$$

(e.g., if H, (a,β) where of the form $\sim A(\alpha)[\beta-\alpha_0]^{B(\alpha)}$; note that since nothing is known about the behaviour of $H_3(\alpha,\beta)$ in the domain $\beta \gg a$, this cannot be excluded), or one can suppose that we have in fact a non-trivial relation $H_3(\alpha,a,) = 0$ which fixes $\alpha_0(\alpha)$, i.e., the dependence of *a,* on *a.*

This situation would obviously imply new conditions on F_3 which could, eventually, be tested since F_3 is more easily accessible to "theoretical experiments" via perturbation theory; also, it is certainly more interesting to have a relation $\alpha_0(\alpha)$ than to obtain a universal value for a,, for which we have no obvious use and whose correctness seems very difficult to control.

Taking the second point of view, let us suppose that one 'chooses a simple function H, (a, β) such that the equation H, (α, a,) $= 0$ gives a relation a, $=$ f (a) (where f is not a constant). One then can easily verify[4] that the corresponding $F_3(1/t, \beta)$ function, which one can obtain from H,, is such that

$$F_3(1/t, \alpha/d_R(t, \alpha))|_{t\to\infty} \equiv 0, \tag{7-1}$$

i.e., the condition (4-17) discussed in Ch. 4, Sec. B.a), which was expected to fix a, is no longer an equation but, rather, F_3 becomes identically null at $t \to \infty$. It remains however to show that there exist functions

$$F_3(1/t, \alpha/d_R(t, \alpha))$$

which should then possess the following apparently contradictory properties:

(i) have a Taylor expansion:

[Taylor expansion of $F_3(1/t, \alpha/d_R(t, \alpha))]_{t\to\infty}$

$$= [\sum_1^\infty \varphi_n(1/t)(\alpha/d_R(t))^{n-1}]_{t\to\infty} = \sum_1^\infty \varphi_n(0)(\alpha_0)^{n-1}, \tag{7-2}$$

with $\varphi_n(0) \neq \infty$ for all n and $\neq 0$ at least for some n, and (ii) being such that

$$F_3(1/t, \alpha/d_R(t, \alpha))|_{t\to\infty} \equiv 0, \tag{7-3}$$

for all $\alpha_0 \in \alpha/d_R(\infty, a)$ in some domain $\mathscr{E}_0$.

Limiting for simplicity the discussion to the domain $t \geq 1$, one can see that a function such as

$$F_3(1/t, \beta) = \frac{F_3(0, \beta)}{1 + \beta t^{a\beta - b}}, \tag{7-4}$$

where a and b are > 0 and $F_3(0, \beta)$ has a Taylor expansion for all $\beta \geq 0$, satisfies the condition (ii) provided that β, which is an increasing function of t, $\beta = \alpha/d_R(t, a)$, reaches (at a finite value t_1 of t) the critica1 value $\beta(t_1) =$

$= \alpha/d_R(t_1, \alpha) = \beta_c \equiv b/a$. In perturbation theory, instead, the same function (7-4) gives a series

$$[\text{Taylor expan. } F_3(0,\beta)] \times [1 - \beta t^{-b}(1 + a\beta \log t + \ldots) + 0(t^{-2b})], \quad (7\text{-}5)$$

which, when $t \to \infty$ and $\beta(t) \to \alpha_0$ tends to

$$[\text{Taylor expan. } F_3(0,\beta)]$$

and, therefore, satisfies condition (i).

It is easy to verify that the difference between Eqs. (7-2) and (7-3), and the error comitted in (7-2), result from the fact that, β being a function of t, the Taylor expansion is justified for $t \lesssim t_1$ only, and that therefore one is not justified in performing on that series the limit $t \to \infty$. The function (7-4) furnishes also an illustration of the essential difference, which is not manifest in perturbation theory, between an exactly zero mass theory (represented here by the function $F_3(0, \beta)$) and a theory represented by $F_3(m^2/k^2, \beta)$, which involves a mass which might be arbitrarily small.

It is also clear that, with a function like (7-4) and for a in a domain $\mathscr{E}_\alpha$ (which is related to the domain $\mathscr{E}_0$ of a, for which $\alpha_0 > \beta_c$), Z_3 is a finite function of a. That same function, computed in perturbation theory, would give precisely the same type of divergences one usually encounters in standard quantum electrodynamics. This example, therefore, exhibits a delicate cut-off mechanism which does not involve any "deus ex machina", spoiled however by perturbation theory which is solely responsible in this case for the presence of divergences. It is of course a conjecture that the real photon propagator should be ***kind enough*** to belong to such a class of functions.

Nous désirons souligner la part importante qui a apportée Mme Irac-Astaud à la formulation que nous présentons ici et la remercier de ses nombreuses discussions critiques, qui ont contribué a éclairer ce sujet. Nous tenons a remercier par son hospitalité l'Institut de Physique Théorique de São Paulo et, particulièrement, le Professeur Jorge Leal Ferreira par i'aide éfficace qu'il nous a apportée dans la rédaction de cet article.

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24. **An** application of the bound group $G_{\dot{\mu}=0}(\widehat{Z_1 Z_3^{-2}})$ for a $\lambda\phi^4$ coupling theory in dimension $d \lesssim 4$ (i.e., for theories where the coupling constant may have a dimension) is given in B. Jouvet, Rapport P. A. M. 73.01.