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Degeneracy of the Baryon Spectrum in the Harmonic Symmetric Quark Model

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The counting problem for the number of linearly independent homogeneous polynomials in six variables with definite permutational symmetry under S_3 is solved. The solution of the analogous problem for homogeneous and harmonic polynomials is also given. The results are applied to the calculation of the degeneracies of the baryonic levels in quark models, specially in the harmonic symmetric quark model.

Resolve-se o problema da contagem do número de polinômios homogêneos em seis variáveis, linearmente independentes, com simetrias permutacionais sob S_3 definidas. Dá-se também a solução do problema análogo para polinômios homogêneos e harmônicos. Os resultados são aplicados ao cálculo das degenerescências dos níveis bariõnicos nos modelos de quarks, especialmente no modelo harmônico e simétrico.

1. Introduction

In a previous issue of this journal[1], a class of quantum-mechanical, non-relativistic, three-body problems have been treated, namely, those in which the interaction potential is a function only of the hyperdistance r, $V(r) = V(\sqrt{x^2 + y^2})$, where x and y are the relative Jacobi coordinates of the problem. As shown in Ref. 1 (hereafter referred to as I), those cases exhibit R, symmetry, a fact which allows a complete group theoretical treatment of the problem. In particular, the states of the system corresponding to a given orbital angular momentum L, projection M and given permutational symmetry were explicitly constructed.

Of special interest for physical applications in the context of quark models is the particular case in which the particles interact pairwise through elastic forces. As is well known, such a case presents the so called accidental degeneracy reflected by the existence of a larger symmetry group of the internal motion, namely, U_6.

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As an extension and application of the work done in I, we studied in this note the degeneracy of the baryonic states in the harmonic symmetric quark model[2] (HSQM), which is phenomenologically the most successful for a description of the baryons as composite states of three quarks. In that model, the spin-unitary spin properties of the baryons are described by the SU_6 group and the overall baryonic states are assumed to be completely symmetric under permutation of the constituents.

The involved counting problem was solved in an exact way, based on the results of I. It consists in counting the number of linearly independent *homogeneous polynomials* of degree λ, $P_\sigma^\lambda(\mathbf{x},\mathbf{y})$, in the six variables x_i and $y_i (i = 1, 2, 3)$, possessing given permutational symmetry σ with respect to S_3, the permutation group of three objects. Here $\sigma = S, A$ or M, according to the three different types of irreducible representations of S_3 (Symmetric S, Antisymmetric A and Mixed M, the first two being one-dimensional and the last one two-dimensional). The numbers of those $P_\sigma^\lambda(\mathbf{x},\mathbf{y})$ are denoted by α, β and γ for $\sigma = S$, A and M, respectively. Known α, β and γ, the counting of the overall states of the system, with permutational symmetry (in particular the symmetric ones required by the HSQM) is a simple matter. It is sufficient to use simple properties of the spin-unitary spin part of the state, together with the Clebsch-Gordan series of the S_3 group. Nevertheless, to get α, β and γ is a long, though completely straightforward calculation.

This paper is organized as follows. In Section 2, the main aspects of the HSQM are recalled, together with some relevant permutational symmetry considerations. In Section 3, we briefly sketch the details of the counting problem and the numbers α, β and γ are given in Tables 1 and 2. In Section 4, we considered the counting problem for models which are non-harmonic but central (i.e., described by an interaction potential of the $V(r)$ form). In this case, the corresponding numbers α', β' and γ' count the number of *homogeneous and harmonic* polynomials in six variables, with S_3 symmetry of type S, A and M, respectively. Finally, in Section 5, we briefly state the conclusions and comparisons with other models.

2. The Harmonic Symmetric Quark Model

As in I, we shall confine our considerations to Hamiltonians with a purely r-dependent (i.e., central) potential $V(r)$:

$$H = -\frac{1}{2}\nabla^2 + V(r). \tag{2-1}$$

Here $\hbar = m = 1$ (the masses of the bodies are equal) and r is the hyper-distance defined in terms of the Jacobi coordinates

$$\mathbf{x} = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3), \; \mathbf{y} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2) \tag{2-2}$$

by

$$r = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}. \tag{2-3}$$

Further, V^2 is the six-dimensional Laplacian

$$V^2 = \nabla_x^2 + V_y^2.$$

In the harrnonic quark model, the forces are pairwise elastic and we have

$$V \equiv \sum_{i<j=1}^{3} \frac{1}{2}\omega^2[\mathbf{r}_i - \mathbf{r}_j]^2 = \frac{\omega^2}{6}r^2, \tag{2-4}$$

an identity which follows directly from (2-2) and (2-3). Therefore, for the harmonic case,

$$H = -\frac{1}{2}\nabla^2 + \frac{\Omega^2}{2}r^2, \tag{2-5}$$

where $R = \omega/\sqrt{3}$. Clearly (2-5) is the Hamiltonian of a six-dimensional harmonic oscillator of frequency Ω. Introducing creation and annihilation operators in the usual way, it can be shown that the energy spectrum of H is given by

$$E_\lambda = \Omega(\lambda + 3), \; \lambda = 0, 1, 2, \ldots, \tag{2-6}$$

and, besides, that the degeneracy of the level A is equal to the nurnber of linearly independent homogeneous polynomials in six variables. When the particles of the system described by H have only spatial degrees of freedom, this number is easily obtained by combinatorial analysis and is given by $\binom{\lambda+5}{5}$. For the case of an r-dimensional harmonic oscillator, this number is given by

$$\dim [A]_r = \binom{\lambda+r-1}{r-1}. \tag{2-7}$$

The HSQM, in a first approximation, assumes that the interaction potential is independent of spin and unitary spin. The quarks are described by the representations of dimension 6 of SU,. It follows that three-quark states exist in the representations of dimensions 56, 20 and 70 of that group. Such representations are symmetric *S*, antisymmetric *A* and mixed M under permutation of the spin-unitary spin degrees of freedom. That is, they carry irreducible representations of the S_3 group which are symmetric, antisymmetric and mixed, the first two being one-dimensional, the last one, two-dimensional, as is well known.

The symmetry requirement of the HSQM, based on phenomenological grounds, is that the baryon states be symmetric under permutation of the space, spin and unitary spin degrees of freedom of the constituents.

In order to construct the states of the HSQM one proceeds as follows. First functions and homogeneous polynomials of definite permutational symmetry are constructed, in the spin-unitary spin variables and in the space variables, respectively. Then, one multiplies the two sets of functions and reduce the direct product into S_3 irreducible representations, selecting the symmetric ones (as required by the HSQM).

Let a, β and γ be the number of linearly independent homogeneous polynomials of degree λ in six variables x_i and y_i of symmetric, antisymmetric and mixed S_3 symmetry, respectively. Making use of the Clebsch-Gordan series of S_3, namely,

$$\begin{aligned} &S \otimes S = S, && A \otimes S = A, && M \otimes S = M, \\ &S \otimes A = A, && A \otimes A = S, && M \otimes A = M, \\ &S \otimes M = M, && A \otimes M = M, && M \otimes M = S \oplus A \oplus M \end{aligned} \tag{2-8}$$

and the permutational properties of the 56-, 20- and 70-dimensional representations of SU,, one can easily conclude that the number of symmetric baryon states of degree λ (as required by the HSQM) is given by $56\alpha + 20\beta + 35\gamma$. Therefore, the degeneracy of the level λ in the HSQM is[3]

$$D_S^\lambda = 56\alpha + 20\beta + 35\gamma. \tag{2-9}$$

For completeness, we also give here the corresponding numbers for the antisymmetric (Fermi quark model) and mixed cases:

$$D_A^\lambda = 20\alpha + 56\beta + 35\gamma, \tag{2-10}$$

$$D_M^\lambda = 140\alpha + 140\beta + 146\gamma. \tag{2-11}$$

It remains to calculate the numbers a, β and γ. This is a long, though straightforward, counting problem which will be sketched in the next section.

3. The Counting Problem

The more convenient basis of homogeneous polynomials in six variables, from the viewpoint of permutational symmetry, is given by the polynomials

$$Q^\lambda_{\mu\nu jj'LM}(\xi,\eta) = (\xi^2)^\mu(\eta^2)^\nu \sum_{m,m'} \langle jmj'm' | LM \rangle \mathcal{Y}^j_m(\xi)\mathcal{Y}^{j'}_{m'}(\eta) \tag{3-1}$$

with

$$\xi = \frac{1}{\sqrt{2}}(x - iy) \quad \text{and} \quad \eta = \frac{1}{\sqrt{2}}(x + iy). \tag{3-2}$$

These polynomials are homogeneous of degree

$$\lambda = 2(\mu + \nu) + (j + j') \tag{3-3}$$

and, as it was proved in I, their permutational symmetry depends on the label

$$u = 2(\mu - \nu) + (j - j') \tag{3-4}$$

in the following way:

i) for $(\mu, \nu, j, j') \neq (\nu, \mu, j', j)$ and $u \not\equiv 0 \pmod 3$, from the pair $(Q^{\lambda}_{\mu\nu jj'}, Q^{\lambda}_{\nu\mu j'j})$ one can get a symmetric and an antisymmetric polynomial;

ii) for $(\mu, \nu, j, j') \neq (\nu, \mu, j', j)$ and $u \equiv 0 \pmod 3$, from the pair $(Q^{\lambda}_{\mu\nu jj'}, Q^{\lambda}_{\nu\mu j'j})$ one can get two polynomials which transform as the components of the mixed representation of S_3;

iii) for $(\mu, \nu, j, j') = (\nu, \mu, j', j)$ [i.e., $\mu = \nu$ and $j = j'$], the polynomial $Q^{\lambda}_{\mu\mu jj}$ is symmetric when L is even and antisymmetric when L is odd.

From i) to iii) it follows immediately that for odd λ one has the same number of symmetric and antisymmetric polynomials since iii), which treats differently symmetric and antisymmetric polynomials, occurs only for even λ.

Let us count firstly the cases $(\mu, \nu, j, j') \neq (\nu, \mu, j', j)$. Since μ, ν, j and j' are nonnegative integers satisfying Eq (3-3), it turns out to be convenient to introduce two new labels $l = \mu + \nu$ and $\rho = 2(\mu - \nu)$. For these labels one has

$$l = \mu + \nu = 0, 1, 2, \ldots, [\lambda/2], \tag{3-5}$$

$$\rho = 2(\mu - \nu) = 0, 2, 4, \ldots, 2l \tag{3-6}$$

and the sum

$$\sum_{l=0}^{[\lambda/2]} \sum_{\rho/2=0}^{l} [2j(l, \rho, u) + 1][2j'(l, \rho, u) + 1] \tag{3-7}$$

gives the contribution to α and β when restricted to $u \equiv 0 \pmod 3$ and the contribution to γ when restricted to $u \not\equiv 0 \pmod 3$.

From (3-3) and (3-4) it follows that

$$(2j+1)(2j'+1) = (\lambda - 2l + u - \rho + 1)(\lambda - 2l - u + \rho + 1).$$

The restriction $u \equiv 0 \pmod 3$ can be more easily be taken into account by breaking the sum (3-7) into 9 parts:

$$l \equiv 0 \pmod 3 \text{ and } \rho \equiv 0,1,2 \pmod 3$$
$$l \equiv 1 \pmod 3 \text{ and } \rho \equiv 0,1,2 \pmod 3$$
$$l \equiv 2 \pmod 3 \text{ and } \rho \equiv 0,1,2 \pmod 3).$$

Summing up all these 9 partial contributions one gets the total contribution to α and β for all cases $(\mu, \nu, j, j') \neq (\nu, \mu, j', j)$. The same process is applied to get the contribution to γ. In this counting process it is convenient to treat separately the cases odd λ and even λ since in the first case one automatically has $(\mu, \nu, j, j') \neq (\nu, \mu, j', j)$, while in the second case one has to be careful to take off the cases $\mu = \nu$ *and* $j = j'$.

λ	α	β	γ
$6n$	$\frac{1}{720}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 270\lambda^2 + 624\lambda + 720)$	$\frac{\lambda}{720}(\lambda^4 + 15\lambda^3 + 85\lambda^2 + 180\lambda + 84)$	$\frac{\lambda}{180}(\lambda^4 + 15\lambda^3 + 85\lambda^2 + 225\lambda + 234)$
$6n+2$	$\frac{1}{720}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 310\lambda^2 + 744\lambda + 640)$	$\frac{1}{720}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 220\lambda^2 + 204\lambda - 80)$	$\frac{1}{180}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 205\lambda^2 + 174\lambda + 40)$
$6n+4$	$\frac{1}{720}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 230\lambda^2 + 264\lambda + 80)$	$\frac{1}{720}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 140\lambda^2 - 276\lambda - 640)$	$\frac{1}{180}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 245\lambda^2 + 414\lambda + 320)$

Table 1 - Number of linearly independent homogeneous polynomials of even degree λ in six variables of symmetric α, antisymmetric β and mixed γ, S_3 symmetry. Here n is nonnegative integer: $n = 0, 1, 2, \ldots$.

λ	$\alpha = \beta$	γ
$6n+1$	$\frac{(\lambda-1)}{720}(\lambda^4 + 16\lambda^3 + 101\lambda^2 + 286\lambda + 280)$	$\frac{1}{180}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 245\lambda^2 + 414\lambda + 320)$
$6n+3$	$\frac{1}{720}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 225\lambda^2 + 354\lambda + 360)$	$\frac{\lambda}{180}(\lambda^4 + 15\lambda^3 + 85\lambda^2 + 225\lambda + 234)$
$6n+5$	$\frac{1}{720}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 265\lambda^2 + 474\lambda + 280)$	$\frac{1}{180}(\lambda^5 + 15\lambda^4 + 85\lambda^3 + 205\lambda^2 + 174\lambda + 40)$

Table 2 - Same as Table **1,** for odd λ. Note that in this case $\alpha = \beta$

For odd λ, the counting is finished and the results are given in Table 2. For even λ, the counting of the polynomials of mixed symmetry is alsa finished, but one still has to count the contribution of the cases $\mu = \nu$ and $j = j'$ to a and β.

When $\mu = \nu$ and $j = j'$, the sum

$$\sum_{\mu=0}^{[\lambda/4]} \sum_{j=0}^{[(\lambda-4\mu)/2]} \sum_{L=0}^{2j} [2L(j,\mu) + 1] \tag{3-9}$$

gives the contribution to a and β when restricted to even or odd values of L, respectively. These contributions, when added to the contribution of the cases $(\mu, \nu, j, j') \neq (\nu, \mu, j', j)$ gives a and β. The final results for λ even are given in Table 1.

To be sure that the values of a, β and γ given in Tables 1 and 2 are correct, they were checked, by a computer program, against the corresponding values obtained by direct counting.

From Tables 1 and 2, one obtains

$$D_S^\lambda = 56\alpha + 20\beta + 35\gamma =$$

$$=\frac{1}{60}(18\lambda^5 + 270\lambda^4 + 1530\lambda^3 + 4185\lambda^2 + 5782\lambda + 3360), \text{ for } \lambda \equiv 0 \pmod 6,$$

$$=\frac{1}{30}(9\lambda^5 + 135\lambda^4 + 765\lambda^3 + 2015\lambda^2 + 2396\lambda + 980), \text{ for } \lambda \equiv 1 \pmod 6,$$

$$=\frac{1}{60}(18\lambda^5 + 270\lambda^4 + 1530\lambda^3 + 4205\lambda^2 + 5842\lambda + 3320), \text{ for } \lambda \equiv 2 \pmod 6,$$

$$=\frac{1}{30}(9\lambda^5 + 135\lambda^4 + 765\lambda^3 + 2025\lambda^2 + 2486\lambda + 1140), \text{ for } \lambda \equiv 3 \pmod 6,$$

$$=\frac{1}{60}(18\lambda^5 + 270\lambda^4 + 1530\lambda^3 + 4165\lambda^2 + 5602\lambda + 3040), \text{ for } \lambda \equiv 4 \pmod 6,$$

$$=\frac{1}{30}(9\lambda^5 + 135\lambda^4 + 765\lambda^3 + 2035\lambda^2 + 2516\lambda + 1120), \text{ for } \lambda \equiv 5 \pmod 6,$$

(3-10)

4. Extension to General Central Interactions

In this section, we consider possible quark models of non-harmonic type, described by central interaction potentials $V(r)$ and the corresponding counting problem. For those central potentials $V(r)$ which do not present accidental degeneracy, the counting of states of a given symmetry follows the same lines as described in section 2, with new numbers α', β' and γ' in the place of α, β and y. The a', β', γ' now count the number of linearly independent *homogeneous and harmonic* polynomials of degree λ in six variables of the type S, **A** and M, respectively. It turns out, from I, that those numbers can be obtained from α, β and y by a simple subtraction procedure, namely,

$$\alpha'(A) = \alpha(\lambda) - \alpha(\lambda - 2), \text{ etc.} \tag{4-1}$$

This follows from Eq (4-13) of I. The values of α', β' and γ', obtained in this way, are given in Tables 3 and 4. From them, one gets that the degeneracy of the level λ for a central interaction potential $V(r)$, without accidental degeneracy, is given by

$$D^{\lambda}_{S,\,centra\mathfrak{l}} = (56\alpha' + 20\beta' + 35\gamma') =$$

$$\begin{aligned}
&= \frac{1}{60}(180\lambda^4 + 1440\hat{A}^3 + 4160\lambda^2 + 5680\lambda + 3360), \text{ for } \lambda \equiv 0 \pmod 6,\\
&= \frac{1}{30}(90\hat{A}^4 + 720\hat{A}^3 + 2050\hat{A}^2 + 2440\lambda + 1000), \text{ for } \lambda \equiv 1 \pmod 6,\\
&= \frac{1}{60}(180\lambda^4 + 1440\hat{A}^3 + 4160\lambda^2 + 5640\lambda + 3280), \text{ for } \lambda \equiv 2 \pmod 6, \qquad (4\text{-}2)\\
&= \frac{1}{30}(90\lambda^4 + 720\lambda^3 + 2080\lambda^2 + 2570\lambda + 1140), \text{ for } A \equiv 3 \pmod 6,\\
&= \frac{1}{60}(180\hat{A}^4 + 1440\lambda^3 + 4100\lambda^2 + 5420\lambda + 3080), \text{ for } \lambda \equiv 4 \pmod 6,\\
&= \frac{1}{30}(90\lambda^4 + 720\lambda^3 + 2080\lambda^2 + 25503, + 1100), \text{ for } \lambda \equiv 5 \pmod 6,
\end{aligned}$$

in the case of a symmetric quark model.

λ	α'	β'	γ'
$6n$	$\frac{1}{72}(\lambda^4 + 8\lambda^3 + 27\lambda^2 + 66\lambda + 72)$	$\frac{\lambda}{72}(\lambda^3 + 8\lambda^2 + 27\lambda + 30)$	$\frac{\lambda}{18}(\lambda^3 + 8\lambda^2 + 21\lambda + 18)$
$6n+2$	$\frac{1}{72}(R^4 + 8\lambda^3 + 27\lambda^2 + 58\lambda + 56)$	$\frac{1}{72}(\lambda^4 + 8\lambda^3 + 27\lambda^2 + 22\lambda - 16)$	$\frac{1}{18}(\lambda^4 + 8\lambda^3 + 21\lambda^2 + 22\lambda + 8)$
$6n+4$	$\frac{1}{72}(\lambda^4 + 8\lambda^3 + 15\lambda^2 + 14\lambda + 16)$	$\frac{1}{72}(\lambda^4 + 8\lambda^3 + 15\lambda^2 - 22\lambda - 56)$	$\frac{1}{18}(\lambda^4 + 8\lambda^3 + 27\lambda^2 + 44\lambda + 28)$

Table 3 - Number of linearly independent *homogeneous und harmonic* polynomials of even degree λ in six variables of symmetric a', antisymmetric β' and mixed γ', S_3 symmetry.

λ	$\alpha' = \beta'$	γ'
$6n+1$	$\frac{1}{72}(\lambda^4+8\lambda^3+15\lambda^2-4\lambda-20)$	$\frac{1}{18}(\lambda^4+8\lambda^3+27\lambda^2+44\lambda+28)$
$6n+3$	$\frac{1}{72}(\lambda^4+8\lambda^3+27\lambda^2+48\lambda+36)$	$\frac{\lambda}{18}(\lambda^3+8\lambda^2+21\lambda+18)$
$6n+5$	$\frac{1}{72}(\lambda^4+8\lambda^3+27\lambda^2+40\lambda+20)$	$\frac{1}{18}(\lambda^4+8\lambda^3+21\lambda^2+22\lambda+8)$

Table 4 - Samc as Table 3, for odd *i*. Note that in this case $\alpha' = \beta'$

5. 'Conclusions

As shown in the previous sections, the degeneracies of the baryon spectrum in quark models appear in the form of a polynomial in 1. In particular, for the HSQM case and *for* large λ, one has from (3-10)

$$D_S^\lambda \simeq \frac{\lambda^3}{10}(3\lambda^2 + 45\lambda + 255). \tag{5-1}$$

From (2-7) it is also clear that the power of the leading term in D_S^λ depends on the number n of constituents as 3n–4. The form of the hadronic spectrum for large excitations is physically important in successful models such as the Hagedorn relativistic bootstrap mode1[4]. For large excitations, Hagedorn has a spectrum proportional to $(a/m^{5/2})\exp(m/T_0)$, where m is the hadron mass, a result also reproduced by the dual-resonance mode1[5]. The latter model, however, corresponds to a hadron structure with an infinite number of harmonic oscillators and the corresponding counting problem, which can be solved with the help of the celebrated theorems of Hardy and Ramanujan on partitions, gives an exponential (in contrast to polynomial) behavior, for large excitations. Therefore, one can conclude that the asymptotic form of the hadron spectrum depends, essentially, on the assumed structure of the hadrons.

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References and Notes

1. J. A. Castilho Alcarás and J. Leal Ferreira, Rev. Bras. Fis 1, 63(1971). See also G. Karl and E. Obrik, Nucl. Phys. **B8**, 609(1968).

3. For the mesons, considered as a $q\bar{q}$ structure, the corresponding number of states is immediately given by $36\binom{\lambda+2}{2}$, since after removing the CM motion, one gets a three-dimensional harmonic oscillator. The factor 36 comes from the SU_6 part, via the well known reduction $6 \otimes 6^* = 1 \oplus 35$.
4. R. Hagedorn, in *Proceedings of rhe Colloquium on High Multiplicity Hadronic Interactions,* École Polytechnique, Paris, 1970, where extensive references on the model can be found.
5. S. Fubini and G. Veneziano, Nuovo Cim. **64A**, 811(1969). S. Fubini, D. Gordon and G. Veneziano, Phys. Lett **29B**, 679(1969). K. Huang and S. Weinberg, Phys. Rev. Lett. 25, 895(1970).