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Models of Flat Regge Trajectories

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Dispersion relations are exploited to obtain models of Regge trajectories. For trajectories that tend to a constant, at large s, a systematic procedure is developed and several examples are presented.

Utilizam-se relações de dispersão para obter modelos de trajetórias de Regge. Para trajetórias que tendem a uma constante, para grande s, desenvolve-se um tratamento sistemático e apresentam-se vários exemplos.

1. Introduction

The study of the consequences of analyticity and unitarity on the form of the trajectory of a Regge pole has been considerably developed in the last few years[1]. Though many general properties are now well understood, few detailed models are available. It is of particular interest to learn about trajectories that escape the treatment given, for instance, in Ref. 1. In this paper, several models of this kind are discussed. They are based upon a once subtracted dispersion relation which is transformed into an integrodifferential equation through the assumption of a specific form for the width function. The method is not new and has been used by the first-named author recently[2]. The trajectories studied here are, nevertheless, of an entirely different type, and many new features both of the equations and of the solutions are made clear. The problem to be discussed has, amusingly enough, another kind of interest: the formalism is exactly that of the theory of aircraft wings of finite span. An application of our results to this theory is contemplated for the near future.

In Section 2, we briefly review the idea of the model, formulating the integrodifferential equation and transforming it into a Fredholm integral equation. General results are then stated. Section 3 covers trajectories with an asymptotically vanishing imaginary part. Several solutions are presented and we concentrate in those that look more interesting.

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2. The Model

We assume, as usually, that the trajectory is an analytic function of s except for a cut along the positive axis starting at the lowest threshold, and that its asymptotic behavior for large s is compatible with a once subtracted dispersion relation. Detailed discussions about these assumptions may be found elsewhere[2,3].

Denoting by $\alpha(s)$ the trajectory, we write

$$\alpha(s) = \alpha(s_0) + \frac{s - s_0}{\pi} \int_{s_0}^{\infty} ds' \frac{\operatorname{Im} \alpha(s')}{(s' - s_0)(s' - s)} \tag{1}$$

The essential input will be the width function

$$G(s) = \sqrt{s}\, \Gamma(s) = \frac{\operatorname{Im} \alpha(s)}{\operatorname{Re} \alpha'(s)}, \tag{2}$$

where $\Gamma(s)$ is the usual (Breit-Wigner) width. It is now easy to show that the following integro-differential equation obtains:

$$\operatorname{Im} \alpha(s) = \frac{g(s)}{\pi} \int_{s_0}^{\infty} \frac{ds'}{s' - s} \frac{d}{ds'} [\operatorname{Im} \alpha(s')], \tag{3}$$

where the integral is, now, a principal value. The problem is that of, given $g(s)$, determining $\operatorname{Im} \alpha(s)$. Use of (1) then determines the real part of the trajectory. There is, of course, no complete theory of integro differential equations of an arbitrary type, and discussion of the existence and uniqueness of solutions is not simple. In the case of trajectories with an asymptotically vanishing imaginary part, (3) may be transformed into a Fredholm integral equation, for which general theorems exist. As a matter of fact, the result is more general: subtracting the dispersion relation once more than it is strictly necessary, one gets Fredholm equations for all cases. They are, however, slightly more complicated to treat. Let us see now how is it possible to do that.

First, change variables in equation (3) so as to avoid infinite integration limits. A convenient variable is

$$y = \frac{s' - s_0 - 1}{s' - s_0 + 1}, \tag{4}$$

or

$$s' - s_0 = \frac{1 \ t \ y}{1 - y} \tag{5}$$

In the new language, **(3)** reads

$$\frac{2 \operatorname{Im} \alpha(x)}{g(x)(1-x)^2} = \frac{1}{\pi} \int_{-1}^{+1} \frac{\operatorname{Im} \alpha'(y)}{y - x} dy, \tag{6}$$

where

$$s - s_0 = \frac{1 + x}{1 - x} \tag{7}$$

and the prime denotes differentiation with respect to y. Use was made, in a partial integration, of the fact that the imaginary part of the trajectory vanishes asymptotically. Equation **(6)** is of the form

$$\frac{f(x)}{B(x)} = \frac{1}{\pi} \int_{-1}^{+1} \frac{f'(y)}{y - x} dy, \tag{8}$$

with

$$B(x) = (1/2) g(x)(1-x)^2 . \tag{9}$$

Equation (8) appears in the theory of aircraft wings of finite span and has been dealt with by several mathematicians. The best treatment known to us is due to L. G. Magnaradze[5] and is reported in its essentials in Ref. 4. Equation (8) is shown to be equivalent to the Fredholm equation

$$f(x) = f(0) \cos t(x) - \frac{1}{\pi} \int_{-1}^{+1} K(x, y) f(y) dy, \tag{10}$$

with

$$t(x) = \int_0^x \frac{dy}{B(y)} \tag{11}$$

and

$$K(x, y) = \int_0^x \frac{R(z, y)}{\sqrt{1 - z^2}} \cos [t(x) - t(z)] dz, \tag{12}$$

with

$$R(z, y) = \frac{1}{y-z}\left[\frac{\sqrt{1-y^2}}{B(y)} - \frac{\sqrt{1-z}}{B(z)}\right]. \tag{13}$$

The knowledge of the function $B(y)$ determines everything and the Fredholm theorems assert that if there is a solution, it is unique (for a given $B(y)$). As an example, observe that taking

$$B(y) = A\sqrt{1-y^2} \tag{14}$$

in (13), $R(z, y)$ vanishes and Eq. (10) reduces to

$$f(x) = f(0)\cos\left(\frac{1}{A}\arcsin x\right) \tag{15}$$

and, for $A = 1$,

$$f(x) = f(0)\sqrt{1-y^2} \tag{16}$$

We will come back to this solution later. It is conveniente now to state general results concerning the asymptotic behavior of Regge trajectories.

Let us assume that $\alpha(s)$ tends to powers of s as $s \to \pm\infty$ along the real axis and that it is bounded by an exponential along every direction of the upper half-plane. These assumptions are not really new, being implicit in the hypothesis of validity of the dispersion relation in Eq. (1). We can write

$$\frac{\alpha(s)}{(-s)^k} \underset{s\to-\infty}{\sim} -A, \tag{17}$$

A being a real number and, by applying the Phragmén-Lindelof theorem[6], conclude that

$$\alpha(s) \underset{s\to+\infty}{\sim} -Ae^{-i\pi k}s^k. \tag{18}$$

Computing now the width function it is easy to see that

$$g(s) \underset{s\to+\infty}{\sim} -\frac{\tan(\pi k)}{k}s, \tag{19}$$

that is, all solutions of (3) with a power behavior are connected with width functions satisfying Eq. (19). These solutions have been treated in detail in Ref. 2, so that we will not bother to discuss them again here.

3. The Solutions

Consider equation (17) when $k = 0$, that is, when the trajectory tends to a constant at infinity. We have

$$\alpha(s) \underset{s \to -\infty}{\sim} -A,$$

with a real A, and the limit is the same for $s \to +\infty$. We can, therefore, conclude that the imaginary part goes to zero and the real part to a constant, at infinity. Nothing can be said about $g(s)$, except that it is not proportional to s, and this is the main point: these trajectories are different from the usual ones mainly in the form of the width function. To determine them, we use the following technique: we choose some specific function for $g(x)$, determining, in this way, Eq. (10) completely. After solving it, we must check whether $\operatorname{Im} \alpha(s)$ vanishes both at infinity and at threshold. Let us start with a particularly simple case: the one described in equations (14) and (15). We have

$$\operatorname{Im} \alpha(x) = \operatorname{Im} \alpha(0) \cos\left[\frac{1}{A} \arcsin x\right], \tag{20}$$

corresponding to

$$g(x) = \frac{2A\sqrt{1-x^2}}{(1-x)^2}. \tag{21}$$

It is necessary that $\operatorname{Im} \alpha(1) = \operatorname{Im} \alpha(-1) = 0$. Hence,

$$\cos\left[\frac{1}{A} \arcsin 1\right] = 0,$$

that is,

$$A = \pm 1/(2n+1), \qquad n = 0, 1, 2, \ldots . \tag{22}$$

Requiring, further, that $\operatorname{Im} \alpha(x)$ have a constant sign in the domain of x, it is not difficult to see that $A = \pm 1$. Inserting, finally, (20) and (21) into (6), one concludes that A must be equal to -1. The solution is, after changing variables to s again,

$$\operatorname{Im} \alpha(s) = 2 \operatorname{Im} \alpha(s_0 + 1) \frac{\sqrt{s-s_0}}{s-s_0+1}, \tag{23}$$

corresponding to

$$g(s) = -\sqrt{s-s_0}\,(s-s_0+1). \tag{24}$$

One can use equation (2) to compute $\operatorname{Re}\alpha(s)$ for $s > s_0$, getting

$$\operatorname{Re}\alpha(s) = B + \frac{2 \operatorname{Im} \alpha(s_0+1)}{(s-s_0+1)}, \tag{25}$$

$$(s > s_0)$$

B being an arbitrary constant. For $s < s_0$, $\operatorname{Re}\alpha(s)$ must be computed from the dispersion relation, Eq. (1).

$$\operatorname{Re}\alpha(s) = 2 \operatorname{Im} \alpha(s_0+1)\left[\frac{1+\sqrt{s_0}}{s_0-1} - \frac{1+\sqrt{s_0-s}}{s_0-s-1}\right]. \tag{26}$$

$$(s < s_0)$$

The solution is, therefore, completed, with

$$B = 2 \operatorname{Im} \alpha(s_0+1) \frac{1+\sqrt{s_0}}{s_0-1}. \tag{27}$$

There is a simple way to get many new solutions of Eq. (8). Though no use is made of the Fredholm equation (10), the existence of the latter is essential, in that it ensures for a given $g(s)$, the uniqueness of the solution.

Start from Eq. (3), which we rewrite for the reader's convenience:

$$\frac{\operatorname{Im} \alpha(s)}{g(s)} = \frac{1}{\pi} \int_{s_0}^{\infty} \frac{\operatorname{Im} \alpha'(s')\,ds'}{s'-s}, \tag{3}$$

where the integral is a principal value. Changing variables to

$$u = s' - s_0 \tag{28}$$

and putting $t = s - s_0$, Eq. (3) is transformed into

$$\frac{\operatorname{Im} \alpha(t)}{g(t)} = \frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im} \alpha'(u)\,du}{u-t} \tag{29}$$

Consider now some particular form for Im $\alpha(t)$, like, for instance,

$$\operatorname{Im} \alpha(t) = t^{\nu-1}(t+1)^{1-\mu}, \tag{30}$$

with $0 < \operatorname{Re} \nu < \operatorname{Re} \mu$. The derivative of *(30)* has the same form, namely,

$$\operatorname{Im} \alpha'(t) = (\nu-1)t^{\nu-2}(t+1)^{1-\mu} + (1-\mu)t^{\nu-1}(t+1)^{-\mu}.$$

Using Ref. (8), we may compute the second member of *(29)*. Using *(30)* in the first member, we can then determine the function $g(t)$: we hava, in this way, found a solution, as we determined the Im $\alpha(t)$ corresponding to some value of $g(t)$. The existence of the Fredholm equation warrants that the solution found is the only one for that choice of $g(t)$. Let us go into details: using *(30)* in (29), we get

$$\begin{aligned}\frac{\operatorname{Im} \alpha(t)}{g(t)} &= (\nu-1)t^{\nu-2}(t+1)^{1-\mu} \cot[(\mu-\nu-1)\pi] \\ &\quad -(\nu-1)\frac{\Gamma(\mu-\nu-2)\Gamma(\nu+1)}{\pi(t+1)\Gamma(\mu-1)}\,{}_2F_1\left(2-\mu, 1;\ 3-\mu+\nu;\ \frac{1}{t+1}\right) \\ &\quad +(1-\mu)t^{\nu-1}(t+1)^{-\mu}\cot[(\mu-\nu+1)\pi] \\ &\quad -(1-\mu)\frac{\Gamma(\mu-\nu)\Gamma(\nu)}{\pi(t+1)\Gamma(\mu)}\,{}_2F_1\left(1-\mu, 1;\ 1-\mu+\nu;\ \frac{1}{t+1}\right).\end{aligned} \tag{31}$$

Using **(30)** in the first member of **(31)** one gets

$$\begin{aligned}\frac{1}{g(t)} &= \frac{(\nu-1)\cot[(\mu-\nu-1)\pi]}{t} + (1-\mu)\,\frac{\cot[(\mu-\nu+1)\pi]}{t+1} \\ &\quad -(\nu-1)\frac{\Gamma(\mu-\nu-2)\Gamma(\nu+1)}{\pi(t+1)\Gamma(\mu-1)}t^{1-\nu}(t+1)^{\mu-1}\,{}_2F_1\left(2-\mu, 1; 3-\mu+\nu; \frac{1}{t+1}\right) \\ &\quad -(1-\mu)\frac{\Gamma(\mu-\nu)\Gamma(\nu)}{\pi(t+1)\Gamma(\mu)}t^{1-\nu}(t+1)^{\mu-1}\,{}_2F_1\left(1-\mu, 1; 1-\mu+\nu; \frac{1}{t+1}\right).\end{aligned} \tag{32}$$

Equation *(32)* provides us with a large class of solutions, including those with $g(t)$ proportional to t, which correspond to the terms proportional to the cotangent. These come out where the solution has the form of a nonvanishing power of **t**. This case has already been treated in great detai1[2] and we refrain from considering it again. Take, instead, solutions connected with the terms containing hypergeometric functions. These are new and present some interesting features. They are gotten, for instance, when ν is taken to be of the form $(2n+1)/2$, $n = 1, 2, \ldots$, and μ is an integer. Let us examine some cases.

a) $\nu = \dfrac{3}{2}$, $\mu = 2$. We have,

$$g(s) = -(s - s_0 + 1)\sqrt{s - s_0}, \qquad \operatorname{Im}\alpha(s) = \frac{\sqrt{s - s_0}}{s - s_0 + 1}, \tag{33}$$

which is exactly the solution found before (Eqs. (23) and (24));

b) $\nu = \dfrac{3}{2}$, $\mu = 3$. The solution is

$$g(s) = \frac{2(s - s_0 + 1)\sqrt{s - s_0}}{2(s - s_0)^2 + 5(s - s_0) - 1}, \qquad \operatorname{Im}\alpha(s) = \frac{\sqrt{s - s_0}}{(s - s_0 + 1)^2}. \tag{34}$$

Using now (2), one gets

$$\operatorname{Re}\alpha(s) = C + \ln(s - s_0 + 1) + \frac{1}{(s - s_0 + 1)^2}, \tag{35}$$

that is, a logarithmically increasing trajectory. Trajectories of this kind, characterized by a very slow increasing rate, can be used to describe the Pomeron, though the teachings of duality seem to exclude it from the world of Regge trajectories. To give this discussion an end, we exhibit a slightly more general solution of equation (3):

c) $\mu = 2$, $1 < \nu < 2$. The solution is

$$\operatorname{Im}\alpha(t) = \frac{t^{\nu - 1}}{t + 1} \tag{36}$$

and

$$g(t) = t(t+1)\Bigg[(\nu - 1)\cot[(1-\nu)\pi](t+1) - \cot[(3-\nu)\pi]\,t - \frac{t^{2-\nu}(t+1)}{\pi}[(\nu-1)(-\nu)\Gamma(\nu+1) - \Gamma(2-\nu)\Gamma(\nu)] - \frac{t^{2-\nu}}{\pi}\frac{\Gamma(2-\nu)\Gamma(\nu)}{\nu - 1}\Bigg]^{-1}, \tag{37}$$

with the asymptotic behavior

$$g(t) \sim t^{\nu-1}$$

From these three examples, we can observe the variety of forms that a trajectory which tends to a constant (except eventually for a logarithmic term) can have, as compared to the very restricted possibilities that are open when it behaves at infinity as a nonvanishing power of s.

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References

1. H. Fleming, T. Sawada, Lett. Nuovo Cimento *26* 1045 (1971); H. Fleming, Lett. Nuovo Cimento 3,363 (1972); H. Fleming, A. Montes Filho, ***Upper Bounds for the Asymptotic Behavior of Regge Trajectories,*** to be published in Nuovo Cimento.
2. H. Fleming, ***Dynamical Model of Regge Trajectories: Threshold Behavior and Linearity,*** Rev. Brasil. Fis. 3, 124 (1973).
3. R. Oehme, ***Complex Angular Momentum*** in R.G. Moorhouse (ed.), Strong Interactions and High Energy Physics (Oliver and Boyd, 1964).
4. N. I. Muskhelishvili, ***Singular Integral equations*** (Noordhoff, 1953), Chapter 17.
5. L. G. Magnaradze, ***On a New Integral Equation of the Theory of Aircraft Wings***, Soobshcheniya A N Gruz SSR, Vol. III, n.° *6,* 503 (1942).
6. E. Phragmen, E. Lindelof, Acta Math. 31, 381 (1908). See also Ref. 7.
7. E. C. Titchmarsh, ***The Theory of Functions,*** 2nd edition (Oxford, 1939).
8. A. Erdélyi (ed.), Bateman Manuscript Project, Tables of Integral Transforms, Vol. 2, pg. 251, n.° 34 (McGraw-Hill, 1954).