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Non-Abelian Compton Effect on Spin-3/2 Targets

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The scattering of isovector photons on spin-3/2 targets is studied in detail up to second order in the frequency of the incident photon. Using Singh's lemma, new second order low-energy theorems related to isospin-symmetric amplitudes are obtained.

O espalhamento de fotons isovetoriais em alvos de spin-3/2 é estudado em detalhe até segunda ordem na frequência do foton incidente. Usando o lema de Singh, são obtidos novos teoremas de segunda ordem relacionados com amplitudes isospin-simétricas.

1. Introduction

Exact low-energy results for the Compton scattering on hadrons have been obtained by various authors[1-7] using the technique invented by Low[1]. If the relevant amplitudes satisfy unsubtracted dispersion relations, these theorems give rise to sum rules[S] which upon saturation with low-lying bound states and resonances can be of help in understanding dynamical symmetry properties[9] and also they can give useful relations between coupling constants.

In this paper, we study the Compton scattering of isovector photons on spin-3/2 targets in detail up to second order in the frequency of the incident photon. The method of derivation goes back to the pioneer work of Low[1] on physical photon scattering on spin-1/2 systems. Bég[2] considered the case in which the photons also carry a "charge" label and thus are associated with isovector currents of an octet satisfying current commutation relations (non-Abelian). He showed, in particular, that the well known Cabibbo-Radicati sum rule[8] follows from the obtained low-energy theorems. Further work on theorems of order ω^2 in the frequency of the incident photon was done by Singh[3] who proposed a lemma giving the excited-

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states contribution to the scattering amplitude. He derived in this way several new second order low-energy theorems for spin-0 and spin-1/2 targets both for physical and charged photons.

The scattering on spin-1 targets was studied to second order by Pais[4] for physical photons and by Kumar[5] for isovector photons. Leal Ferreira and Ragusa[6] considered the scattering of physical photons on spin-3/2 targets and several low-energy theorems were derived up to third order. In two previous notes[7], we have considered the scattering of isovector photons on spin-3/2 targets and new theorems related to isospin-antisymmetric amplitudes were established. In particular, a generalized form of the Cabibbo-Radicati theorem and of the magnetic moment radius theorem were obtained and conjectured to be valid for arbitrary spin targets.

In this paper, we consider in more detail the scattering of isovector photons on spin-3/2 targets and we obtain new low-energy theorems related to isospin-symmetric amplitudes, to second order in the incident photon frequency.

In Sec. 2, we give a general discussion of the low-energy theorems and Sec. 3 is devoted to their explicit expression. In Sec. 4, we have the concluding remarks.

2. The Low-Energy Theorems

We consider the tensor $T_{\mu\nu}^{\alpha\beta}$ given by

$$(2\pi)^4 \delta(p' + k' - p - k) \left[\frac{V^2 m^2}{E_{p'} E_p}\right]^{1/2} T_{\mu\nu}^{\alpha\beta} = i \int d^4x d^4y \exp(-ik'\cdot x + ik\cdot y) \times$$
$$\times \langle \mathbf{p}' | [T\{J_\mu^\alpha(x), \quad J_\nu^\beta(y)\} - i\rho_{\mu\nu}^{\alpha\beta}(x)\delta^4(x-y)] | \mathbf{p} \rangle, \qquad (1)$$

which is related to the amplitude for the scattering of isovector photon on a spin-3/2 target,

$$T^{\alpha\beta} = \varepsilon'_i(k') T_{ij}^{\alpha\beta} \varepsilon_j(k). \qquad (2)$$

Here α and β are isotopic spin indices, k' and k (p' and p) are outgoing and incident "photon" (target) momenta. Our metric is defined by $k_\mu = (\mathbf{k}, k_4) = (\mathbf{k}, ik_0) = (\mathbf{k}, i\omega)$. The covariance of $T_{\mu\nu}^{\alpha\beta}$ is ensured by the presence of $\rho_{\mu\nu}^{\alpha\beta}$ that counter balances the noncovariant nature of the T product[10]. J_μ^α is the conserved isospin current, $\partial_\mu J_\mu^\alpha(x) = 0$. Eq. (2) is the scattering amplitude in the transverse gauge $\mathbf{k}\cdot\varepsilon = \mathbf{k}'\cdot\varepsilon' = 0$.

The basic equal-time commutations relations of the current operators $J_\rho{}^a = (J_i^\alpha,\ iJ_0{}^a)$ are:

$$[J_0{}^\alpha(x),\ J_0{}^\beta(y)]\,\delta(x_0 - y_0) = i\varepsilon^{\alpha\beta\gamma} J_0{}^\gamma(x)\,\delta^4(x - y), \tag{3}$$

$$[J_0{}^\alpha(x),\ J_i^\beta(y)]\,\delta(x_0 - y_0) = i\varepsilon^{\alpha\beta\gamma} J_i^\gamma(x)\delta^4(x - y) + ia_i\,[\rho_{mi}{}^{\alpha\beta}(x)\delta^4(x - y)]. \tag{4}$$

On contracting *Eq.* (1) with k'_μ and k_ν, one obtains from current conservation and Eqs. *(3)* and (4),

$$k'_\mu T_{\mu\nu}{}^{\alpha\beta} = T_{\nu\lambda} k_\lambda = i\left[\frac{V^2 E_{p'} E_p}{m^2}\right]^{1/2} \varepsilon^{\alpha\beta\gamma}\langle \mathbf{p}'|J_\nu{}^\gamma(0)|\mathbf{p}\rangle. \tag{5}$$

Therefore,

$$k'_\mu T_{\mu\nu}{}^{\alpha\beta} k_\nu = \tfrac{i}{2}\left[\frac{V^2 E_{p'} E_p}{m^2}\right]^{1/2} \varepsilon^{\alpha\beta\gamma}(k'_\nu + k_\nu)\langle \mathbf{p}'|J_\nu{}^\gamma(0)|\mathbf{p}\rangle. \tag{6}$$

From the identity

$$k'_\mu T_{\mu\nu}{}^{\alpha\beta} k_\nu = k'_\mu T_{\mu 4}{}^{\alpha\beta} k_4 + k'_4 T_{4\nu}{}^{\alpha\beta} k_\nu + k'_i T_{ij}{}^{\alpha\beta} k_j - k'_4 T_{44}{}^{\alpha\beta} k_4, \tag{7}$$

and *Eq.* (6), follows the relation

$$k'_i T_{ij}{}^{\alpha\beta} k_j = \omega\omega' T_{00}{}^{\alpha\beta} + \left[\frac{V^2 E_{p'} E_p}{m^2}\right]^{1/2} \frac{i}{2}\varepsilon^{\alpha\beta\gamma} \times$$

$$\times \langle \mathbf{p}'|[(\omega + \omega')J_0{}^\gamma(0) + (k'_i + k_i)J_i^\gamma(0)]|\mathbf{p}\rangle. \tag{8}$$

Next, we divide $T_{ij}{}^{\alpha\beta}$ into two parts,

$$T_{ij}{}^{\alpha\beta} = U_{ij}{}^{\alpha\beta} + E_{ij}{}^{\alpha\beta}, \tag{9}$$

where $U_{ij}{}^{\alpha\beta}$ refers to the unexcited or one-particle (target) pole contribution,

$$\left[\frac{m}{V^4 E_{p'}}\right]^{1/2} U_{ij}{}^{\alpha\beta} = \left[\frac{\langle \mathbf{p}'|J_i^\alpha|\mathbf{k}\rangle\langle \mathbf{k}|J_j^\beta|\mathbf{0}\rangle}{E(\mathbf{k}) - m - \omega} + \frac{\langle \mathbf{p}'|J_j^\beta|-\mathbf{k}'\rangle\langle -\mathbf{k}'|J_i^\alpha|\mathbf{0}\rangle}{E(\mathbf{k}') - m + \omega'}\right], \tag{10}$$

where a summation over the intermediate spin states is implied and we have taken the target initially at rest, $\mathbf{p} = 0$. We then recall that the frequency of the outgoing photon is given by the relation

$$m(\omega' - \omega) = k\,.\,k' - \omega\omega' = \omega\omega'(\cos\theta - 1). \tag{11}$$

Using Eqs. (8) and (9) and splitting $T_{00}^{\alpha\beta}$ into its unexcited and excited parts, we have for $E_{ij}^{\alpha\beta}$ the relation

$$k'_i E_{ij}^{\alpha\beta} k_j = \omega\omega' E_{00}^{\alpha\beta} + \omega\omega' U_{00}^{\alpha\beta} - k'_i U_{ij}^{\alpha\beta} k_j + iV[E_{p'}/m]^{1/2} \times$$
$$\times \frac{1}{2} \varepsilon^{\alpha\beta\gamma} \langle p' | [(\omega + \omega') J_0^{\gamma}(0) - (k'_i + k_i) J_i^{\gamma}(0)] | 0 \rangle, \tag{12}$$

where

$$\left[\frac{m}{V^4 E_{p'}}\right]^{1/2} U_{00}^{\alpha\beta} = \left[\frac{\langle \mathbf{p}' | J_0^{\alpha} | \mathbf{k} \rangle \langle \mathbf{k} | J_0^{\beta} | \mathbf{0} \rangle}{E(\mathbf{k}) - m - \omega} + \frac{\langle \mathbf{p}' | J_0^{\beta} | -\mathbf{k}' \rangle \langle -\mathbf{k}' | J_0^{\alpha} | \mathbf{0} \rangle}{E(\mathbf{k}') - m + \omega'}\right], \tag{13}$$

and, since $\rho_{00}^{\alpha\beta} = 0$, $E_{00}^{\alpha\beta}$ is given by a similar expression containing all but the single-particle intermediate state. As is well known[1], $E_{00}^{\alpha\beta}$ is of order ω^2 and this statement has been casted in a more precise way by Singh[3] who has shown that

$$E_{00}^{\alpha\beta} = k'_i k_j \Lambda_{ij}^{\alpha\beta}(k, k'), \tag{14}$$

where $\Lambda_{ij}^{\alpha\beta}$ is free of kinematical singularities and symmetric under the interchange $\alpha \leftrightarrow \beta$, $i \leftrightarrow j$, $k \leftrightarrow k'$, that is, it obeys crossing symmetry.

Foliowing Pais[4], we write now the "complete minimal basis" for $E_{ij}^{\alpha\beta}$:

$$E_{ij}^{\alpha\beta} = \sum_{n=1}^{42} B_n^{\alpha\beta}(\omega, \omega') E_{ij}^{(n)}$$
$$= \sum_{n=1}^{42} (\{I^{\alpha}, I^{\beta}\} S_n(\omega, \omega') + [I^{\alpha}, I^{\beta}] A_n(\omega, \omega')) E_{ij}^{(n)}, \tag{15}$$

where the $E_{ij}^{(n)}$ are the basis element for spin[6] $J = 3/2$. We have decomposed the amplitudes in its isospin symmetric and antisymmetric parts, I^{a} being the appropriate target isospin matrix. With some convenient modifications, the basis elements $E_{ij}^{(n)}$ are[1'],

$n = 1$: δ_{ij},

$n = 2$: $\varepsilon_{ijm} J_m$,

$n = 3$: $\{J_i, J_j\} - \frac{5}{2}\delta_{ij}$,

$n = 4$: $k'_i k'_j + k_i k_j$,

$n = 5$: $k_i k'_j - \mathbf{k} \cdot \mathbf{k}' \delta_{ij}$,

$$n = 6: \quad k'_i k_j$$

$$n = 7: \quad \delta_{ij} \mathbf{J} \cdot (\mathbf{k}' \times \mathbf{k}) - \mathbf{k} \cdot \mathbf{k}' \varepsilon_{ijm} J_m,$$

$$n = 8: \quad \varepsilon_{ijm}(k'_m \mathbf{J} \cdot \mathbf{k}' + k_m \mathbf{J} \cdot \mathbf{k}),$$

$$n = 9: \quad \varepsilon_{ijm}(k'_m \mathbf{J} \cdot \mathbf{k} + k_m \mathbf{J} \cdot \mathbf{k}'),$$

$$n = 10: \; k'_i(\mathbf{J} \times \mathbf{k}')_j - k_i(\mathbf{J} \times \mathbf{k})_j + (i \leftrightarrow j),$$

$$n = 11: \; k_i(\mathbf{J} \times \mathbf{k}')_j - k'_i(\mathbf{J} \times \mathbf{k})_j + (i \leftrightarrow j) - 2\mathbf{k} \cdot \mathbf{k}' \varepsilon_{ijm} J_m,$$

$$n = 12: \; \delta_{ij}[(\mathbf{J} \cdot \mathbf{k}')^2 + (\mathbf{J} \cdot \mathbf{k})^2],$$

$$n = 13: \; \delta_{ij}\{\mathbf{J} \cdot \mathbf{k}', \mathbf{J} \cdot \mathbf{k}\} - \mathbf{k} \cdot \mathbf{k}' \{J_i, J_j\},$$

$$n = 14: \; k_i\{J_j, \mathbf{J} \cdot \mathbf{k}\} + k'_j\{J_i, \mathbf{J} \cdot \mathbf{k}'\},$$

$$n = 15: \; k'_i\{J_j, \mathbf{J} \cdot \mathbf{k}'\} + k_j\{J_i, \mathbf{J} \cdot \mathbf{k}\},$$

$$n = 16: \; k_i\{J_j, \mathbf{J} \cdot \mathbf{k}'\} + k'_j\{J_i, \mathbf{J} \cdot \mathbf{k}\} - 2\mathbf{k} \cdot \mathbf{k}' \{J_i, J_j\},$$

$$n = 17: \; k'_i\{J_j, \mathbf{J} \cdot \mathbf{k}\} + k_j\{J_i, \mathbf{J} \cdot \mathbf{k}'\},$$

$$n = 18 \text{ to } n = 25: \; 0(\omega^4),$$

$$n = 26: \; (\langle J_i, J_j, J_r \rangle + \langle J_j, J_i, J_r \rangle)(\mathbf{k}' \times \mathbf{k})_r - \frac{41}{10} \mathbf{J} \cdot (\mathbf{k}' \times \mathbf{k}) \delta_{ij},$$

$$n = 27: \; \varepsilon_{ijr}(k'_m k'_n + k_m k_n)\langle J_m, J_,, J_r \rangle,$$

$$n = 28: \; \varepsilon_{ijr}(k'_m k_n + k_m k'_n)\langle J_m, J_n, J_r \rangle - \frac{41}{10} \mathbf{k} \cdot \mathbf{k}' \varepsilon_{ijr} J_r$$

$$n = 29 \text{ to } n = 42: \; 0(\omega^4), \tag{16}$$

. where $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle = \mathbf{ABC} + \mathbf{CAB} + \mathbf{BCA}$.

Note in particular that $E_{ij}^{(3)}$ is an irreducible second order tensor. In this way, the generalized form of the Cabibbo-Radicati theorem will involve only the spin independent amplitude $\mathbf{A}$,.

As we shall be interested in the part of $E_{ij}^{\alpha\beta}$ which is of order ω^2, we have not written tliose basis elements which are already of $0(m^4)$. Upon contraction with $k'_i k_j$, we obtain

$$k'_i E_{ij}^{\alpha\beta} k_j = k\,.\mathbf{k}' B_1^{\alpha\beta} - \frac{5}{2} k\,.\mathrm{k}' B_3^{\alpha\beta} + (\omega^2 + \omega'^2)\mathbf{k}\cdot k' B_4^{\alpha\beta}$$

$$+ \omega^2 \omega'^2 B_6^{\alpha\beta} + \mathbf{J}\cdot(\mathbf{k}' \times \mathbf{k})[B_2^{\alpha\beta} + (\omega^2 + \omega'^2) B_{10}^{\alpha\beta} - \frac{41}{10}\mathbf{k}'\cdot\mathbf{k}$$

$$\times (B_{26}^{\alpha\beta} + B_{28}^{\alpha\beta})] + \{\mathrm{J}\,.k',\ \mathbf{J}\cdot k)[B_3^{\alpha\beta} + (\omega^2 + \omega'^2) B_{15}^{\alpha\beta}]$$

$$+ [(\mathrm{J}\,.\mathbf{k}')^2 + (J\,.\mathbf{k})^2][\mathrm{k}\,.\mathbf{k}'(B_{12}^{\alpha\beta} + 2B_{14}^{\alpha\beta}) + 2\omega^2 B_{17}^{\alpha\beta}]$$

$$+ (k'_i k_j + k_i k'_j)(\mathbf{k}' \times \mathbf{k})_r \langle J_i, J_j, J_r \rangle (B_{26}^{\alpha\beta} + B_{28}^{\alpha\beta})$$

$$+ (k'_i k'_j + k_i k_j)(\mathbf{k}' \times \mathbf{k})_r \langle J_i, J_j, J_r) B_{27}^{\alpha\beta} + 0(\omega^5). \tag{17}$$

The unknown term $\Lambda_{ij}^{\alpha\beta}$ of Eq. (14) can be expanded in the same basis as $E_{ij}^{\alpha\beta}$, that is, we can write

$$\Lambda_{ij}^{\alpha\beta} = \sum_{n=1}^{42} b_n^{\alpha\beta} E_{ij}^{(n)}$$

$$= \sum_{n=1}^{42} (\{I^\alpha, I^\beta\} s_n + [I^a, I^P] a_n) E_{ij}^{(n)}, \tag{18}$$

where the $b_n's$ are unknown coefficients which we have decomposed in its isospin symmetric and antisymmetric parts.

Therefore, 'from Eqs. (14) and (16), we can write

$$\omega\omega' E_{00}^{\alpha\beta} = \omega\omega'\mathbf{k}\,.k' \left(b_1^{\alpha\beta} - \frac{5}{2} b_3^{\alpha\beta}\right) + J\ (\mathrm{k}' \times k)\omega\omega' b_2^{\alpha\beta}$$

$$+ \{\mathrm{J}\cdot k', J\,.k)\omega\omega' b_3^{\alpha\beta} + 0(\omega^6). \tag{19}$$

We state now that it is possible to know which of the amplitudes $B_n^{\alpha\beta}$ present in Eq. (17) can be determined to lowest order. Going back to Eq. (12), we notice that the three last terms on its right-hand side can be calculated exactly. Moreover, by Eq. (19) $\omega\omega' E_{00}^{\alpha\beta}$ starts two powers of ω ahead of $k'_i E_{ij}^{\alpha\beta} k_j$ and therefore it cannot compete for the determination of $B^{\alpha\beta}_{1,2,3,6,17,27}$, $B_{12}^{\alpha\beta} + 2B_{14}^{\alpha\beta}$ and $B_{26}^{\alpha\beta} + B_{28}^{\alpha\beta}$ to $0(\omega)$, giving corresponding low-energy theorems.

To get more information, we expand both amplitudes $B_n^{\alpha\beta}(\omega, \omega')$ and $b_n^{\alpha\beta}(\omega, \omega')$ in powers of $\omega\omega'$.

The general expansion of these amplitudes will contain terms in $1, \omega \pm \omega'$, $o o'$, $\omega^2 \pm \omega'^2$, etc. Recalling Eq. (11), it is easy to see that to order ω^2 the expansions can be taken to contain 1, $o + o', k \cdot k'$ and wo'. Since $T_{ii}{}^{\alpha\beta}$ and $U_{ij}{}^{\alpha\beta}$ are crossing symmetric, so it is $E_{ij}{}^{\alpha\beta}$,

$$E_{ij}{}^{\alpha\beta}(k, k') = E_{ji}{}^{\beta\alpha}(-k', -k). \qquad (20)$$

From Eqs. *(15)* and (16), it then follows that for r = 1, 3 to *6*, *12* to *17*,

$$B_r{}^{\alpha\beta}(o, o') = B_r{}^{\beta\alpha}(-o', -o), \qquad (21a)$$

and for $s = 2, 7$ to *11, 26, 27, 28*

$$B_s{}^{\alpha\beta}(\omega, \omega') = -B_s{}^{\beta\alpha}(-\omega', -\omega). \qquad (21b)$$

Therefore, to the order that we are interested in, we have the expansions

$$S_r(\omega, \omega') = S_r(0) + \mathbf{k}\cdot\mathbf{k}' S_{r,1} + \omega\omega' S_{r,2} + 0(\omega^3),$$
$$A_r(\omega, o') = (o t\ o')A_{,,,} + 0(0^3),$$
$$S_s(\omega, \omega') = (\omega + \omega') S_{s,1} + 0(\omega^3),$$
$$A_s(\omega, \omega') = A_s(0) t\ \mathbf{k}\cdot\mathbf{k}' A_{,,,} + \omega\omega' A_{,,} + 0(\omega^3). \qquad (22)$$

Similar expansions hold for s_n and a, of Eq. *(18)*. These expansions are now substituted in Eqs. *(17)* and (19). For the isospin symmetric part of Eq. (17), we have to $0(\omega^4)$,

$$k'_i E_{ij}{}^{\{\alpha\beta\}} k_j = \{I^\alpha, I^\beta\} \Big\{ \mathbf{k}\cdot\mathbf{k}' \left(S_1(0) - \frac{5}{2} S_3(0) \right) + (\mathbf{k}\cdot\mathbf{k}')^2 \left(S_{1,1} - \frac{5}{2} S_{3,1} \right)$$
$$+ \omega^2 \omega'^2 S_6(0) + \omega\omega' \mathbf{k}\cdot\mathbf{k}' \left(S_{1,2} - \frac{5}{2} S_{3,2} + 2S_4(0) \right)$$
$$+ \mathbf{J}\cdot(\mathbf{k}' \times \mathbf{k})(\omega + \omega') S_{2,1} + \{\mathbf{J}\cdot\mathbf{k}, \mathbf{J}\cdot\mathbf{k}'\} [S_3(0) + \mathbf{k}\cdot\mathbf{k}' S_{3,1}$$
$$+ \omega\omega'(S_{3,2} + 2S_{15}(0))] + ((\mathbf{J}\cdot\mathbf{k}')^2 + (\mathbf{J}\cdot\mathbf{k})^2)[\mathbf{k}\cdot\mathbf{k}'(S_{12}(0) + 2S_{14}(0))$$
$$+ 2\omega^2 S_{17}(0)] + 0(\omega^5) \Big\}, \qquad (23a)$$

and for the isospin antisymmetric part

$$k'_i E_{ij}{}^{[\alpha\beta]} k_j = [I^\alpha, I^\beta] \Big\{ \mathbf{k}\cdot\mathbf{k}'(\omega + \omega') \Big(A_{1,1} - \frac{5}{2} A_{3,1} + \mathbf{J}\cdot(\mathbf{k}' \times \mathbf{k})$$
$$\times \Big[A_2(0) + \mathbf{k}\cdot\mathbf{k}' A_{2,1} \pm \frac{41}{10} \mathbf{k}\cdot\mathbf{k}'(A_{26}(0) + A_{28}(0))$$

$$+\ \omega\omega'(A_{2,2} + 2A_{10}(0))\Big] + \{\mathbf{J}\cdot\mathbf{k},\ \mathbf{J}\cdot\mathbf{k}'\}(\omega + \omega')A_{3,1}$$

$$+\ (k'_i k_j + k_i k'_j)(\mathbf{k}' \times \mathbf{k})_r \langle J_i, J_j, J_r\rangle (A_{26}(0) + A_{28}(0))$$
$$+\ (k'_i k'_j + k_i k_j)(\mathbf{k}' \times \mathbf{k})_r \langle J_i, J_j, J_r\rangle A_{27}(0) + 0(\omega^5)\}. \qquad (23b)$$

Similarly, for Eq. (19), we can write

$$\omega\omega' E_{00}{}^{\alpha\beta} = \{I^\alpha, I^\beta\}\left\{\omega\omega'\mathbf{k}\cdot\mathbf{k}'\left(s_1(0) - \frac{5}{2}s_3(0)\right) + \{\mathbf{J}\cdot\mathbf{k}', \mathbf{J}\cdot\mathbf{k}\}\,\omega\omega' s_3(0)\right\}$$
$$+\ [I^\alpha, I^\beta]\,\mathbf{J}\cdot(\mathbf{k}' \times \mathbf{k})\,\omega\omega' a_2(0) + 0(\omega^5). \qquad (24)$$

Eqs. (23a), (23b) and (24) are now to be substituted into Eq. (12). It is apparent that only $S_{1,2}$, $S_{3,2}$, $S_4(0)$, $S_{15}(0)$, A„ and $A_{10}(0)$, will receive an unknown contribution from $\omega\omega' E_{00}{}^{\alpha\beta}$. All the other amplitudes in Eqs. (23 a, b) will suffer no competition from $\omega\omega' E_{00}{}^{aa}$ and will therefore be completely determined by the other known terms of Eq. (12), giving fourteen low-energy theorems.

3. Expression of the Low-Energy Theorems

We shall now establish the explicit expression of the low-energy theorems. As we are working on $E_{ij}{}^{\alpha\beta}$ to order ω^2, we need in Eq. (12) both $U_{00}{}^{\alpha\beta}$ and $U_{ij}{}^{\alpha\beta}$ to $0(\omega^2)$ and $(\mathbf{p}\,|J_\mu{}^\alpha(0)|0)$ to $0(\omega^3)$. To compute these quantities we need the $\mathbf{J} = 3/2$ isovector current matrix element. We have[12]

$$\langle \mathbf{p}'|J_\mu{}^\alpha(0)|\mathbf{p}\rangle = i\left[\frac{m^2}{V^2 E_p E_{p'}}\right]^{1/2} \bar{u}_\rho(p')\left\{\left[F_1{}^\alpha(q^2)\delta_{\rho\sigma} + \frac{F_3{}^\alpha(q^2)}{2m^2}q_\rho q_\sigma\right]\gamma_\mu\right.$$
$$\left. +\ \frac{i}{4m}[\gamma_\mu, \gamma\cdot q]\left[F_2{}^\alpha(q^2)\delta_{\rho\sigma} + \frac{F_4{}^\alpha(q^2)}{2m^2}q_\rho q_\sigma\right]\right\}u_\sigma(p), \qquad (25)$$

where $q = p' - p$ and we have supressed polarization indices. $F_i^\alpha(q^2) = F_i^V(q^2)I^\alpha$, with i = 1, 2, 3, 4, are the isovector form factors: $F_1^V(0) = 1$, $F_1^V(0) + F_2^V(0) = \mu^V$ is the isovector magnetic moment in units of 1/2m, $F_1^V(0) + F_3^V(0) = Q^V$ is the isovector quadrupole moment in units of $1/m^2$ and $F_1^V(0) + F_2^V(0) + F_3^V(0) + F_4^V(0) = \Omega^V$ is the magnetic octupole moment in units of $1/2m^2(6)^{1/2}$. The Rarita-Schwinger wave function $u_\sigma(p)$

may be expressed as follows:

$$\mathbf{u}(p) = \left(\frac{E+m}{2m}\right)^{1/2}\left(1+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{E+m}\right)\left[\mathbf{u}+\frac{\mathbf{p}}{m(E+m)}\mathbf{p}\cdot\mathbf{u}\right],$$

$$u_0(0) = \left(\frac{E+m}{2m}\right)^{1/2}\left(1+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{E+m}\right)\frac{\mathbf{p}\cdot\mathbf{u}}{m}, \tag{26}$$

where $E^2 = \mathbf{p}^2 + m^2$ and $\boldsymbol{u}$ is the wave function for $\boldsymbol{p} = 0$. The calculation of the known part of Eq. (12) is straightforward. From Eq. (11), one has the relations

$$\frac{m+E_k}{2E_k(E_k-m-\omega)} \pm \frac{m+E_{k'}}{2E_{k'}(E_{k'}-m+\omega')}$$

$$= -\frac{\text{tos }0}{m} - \frac{\omega^2+\omega'^2}{8m^3} + 0(\omega^3), \qquad \text{for } (+) \text{ sign},$$

$$= -\frac{1}{\boldsymbol{o}} - \frac{1}{\boldsymbol{o}'} + \frac{\omega^2-\omega'^2}{8m^3} + 0(\omega^3), \qquad \text{for } (-) \text{ sign}. \tag{27}$$

Using Eq. (25) in Eqs. (10) and (13), one obtains from Eq. (12),

$$k'_i E_{ij}^{\{\alpha\beta\}} k_j - \omega\omega' E_{00}^{\{\alpha\beta\}} = \frac{1}{2}\{I^\alpha, I^\beta\} u_i^\dagger \Bigg\{\delta_{ij}\Bigg[-\frac{\mathbf{k}\cdot\mathbf{k}'}{m} - \frac{(\omega\omega')^2}{4m^3} + \omega\omega'\mathbf{k}\cdot\mathbf{k}'\left(\frac{2\mu^V+1}{4m^3} - \frac{2}{m}F_1^{V'}\right) + i\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})(\omega+\omega')\frac{2\mu^V-3}{4m^2}\Bigg] - (k'_i k'_j + k_i k_j)\mathbf{k}\cdot\mathbf{k}'\frac{Q^V}{2m^3} + 0(\omega^5)\Bigg\} u_j, \tag{28}$$

and

$$k'_i E_{ij}^{[\alpha\beta]} k_j - \omega\omega' E_{00}^{[\alpha\beta]} = \frac{1}{2}[I^\alpha, I^\beta] u_i^\dagger \Bigg\{\delta_{ij}\Bigg[\!\!\Bigg[(\omega+\omega')\mathbf{k}\cdot\mathbf{k}'\left(\frac{2\mu^V-1}{4m^2} - 2F_1^{V'}\right) + \frac{i}{m}\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})\Bigg[-\mu^V + 2\mathbf{k}\cdot\mathbf{k}'\left(F_1^{V'} + F_2^{V'} + \frac{\mu^V-3}{8m^2}\right) - 2\omega\omega' \times \left(F_1^{V\prime} + F_2^{V\prime} - \frac{\mu^V}{8m^2}\right)\Bigg]\Bigg]\!\!\Bigg] - (k'_i k_j + k_i k'_j)(\omega+\omega')\frac{Q^V}{2m^2} - \frac{i\Omega^V}{2m^3}(k_i - k'_i)(k_j - k'_j)\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k}) + 0(\omega^5)\Bigg\} u_j, \tag{29}$$

where $F_{1,2}{}^{V'} = [dF_{1,2}(t)/dt]_0$.

Now, we have to write Eqs. (28) and (29) in J space. For that purpose, one needs the following relations[6],

$$u_i u_i^{\dagger} = u^{\dagger} u,$$

$$u_i^{\dagger}(\sigma,\mathbf{k}' \times \mathbf{k})u_i = \frac{2}{3} u^{\dagger} \mathbf{J} \cdot (\mathbf{k}' \times \mathbf{k})u,$$

$$(k'_i k_j + k_i k'_j)u_i^{\dagger} u_j = -\frac{1}{3} u^{\dagger} \{\mathbf{J} \cdot \mathbf{k}', \mathbf{J} \cdot \mathbf{k}\} u + \frac{3}{2} \mathbf{k} \cdot \mathbf{k}' u^{\dagger} u,$$

$$(k'_i k'_j + k_i k_j)u_i^{\dagger} u_j = -\frac{1}{3} u^{\dagger}[(\mathbf{J} \cdot \mathbf{k}')^2 + (\mathbf{J} \cdot \mathbf{k})^2]u + \frac{3}{4}(\omega^2 + \omega'^2)u^{\dagger} u,$$

$$(k'_i k_j + k_i k'_j)u_i^{\dagger}(\sigma,\mathbf{k}' \times \mathbf{k})u_j = -\frac{2}{9}(k'_i k_j + k_i k'_j)(\mathbf{k}' \times \mathbf{k})_r u^{\dagger} \langle J_i, J_j, J_r \rangle u + \frac{13}{9} \mathbf{k} \cdot \mathbf{k}' u^{\dagger} \mathbf{J} \cdot (\mathbf{k}' \times \mathbf{k})u;$$

$$(k'_i k'_j + k_i k_j)u_i^{\dagger}(\sigma,\mathbf{k}' \times \mathbf{k})u_j = -\frac{2}{9}(k'_i k'_j + k_i k_j)(\mathbf{k}' \times \mathbf{k})_r u^{\dagger} \langle J_i, J_j, J_r \rangle u + \frac{13}{18}(\omega^2 + \omega'^2)u^{\dagger} \mathbf{J} \cdot (\mathbf{k}' \times \mathbf{k})u. \quad (30)$$

Taking Eq. (30) into Eq. (28) and Eq. (29), and recalling Eq. (24), one finds

$$k'_i E_{ij}^{\{\alpha\beta\}} k_j = \{I^\alpha, I^\beta\} \Big\{ \frac{\mathbf{k} \cdot \mathbf{k}'}{2m} \frac{(\omega\omega')^2}{8m^3} + \frac{\omega\omega' \mathbf{k} \cdot \mathbf{k}'}{2m} \left[\frac{1 + 2\mu^V - 3Q^V}{4m^2} - 2F_1^{V'} + 2ms_1(0) - 5ms_3(0)\right] + i\mathbf{J} \cdot (\mathbf{k}' \times \mathbf{k})(\omega + \omega') \times \frac{2\mu^V - 3}{12m^2} + \{\mathbf{J} \cdot \mathbf{k}', \mathbf{J} \cdot \mathbf{k}\} \omega\omega' s_3(0) + [(\mathbf{J} \cdot \mathbf{k}')^2 + (\mathbf{J} \cdot \mathbf{k})^2] \mathbf{k} \cdot \mathbf{k}' \frac{Q^V}{12m^3} + O(\omega^5) \Big\} \quad (31)$$

and

$$k'_i E_{ij}^{[\alpha\beta]} k_j = [I^\alpha, I^\beta] \Big\{ (\omega + \omega')\mathbf{k} \cdot \mathbf{k}' \left(-F_1^{V'} + \frac{2\mu^V - 1 - 3Q^V}{8m^2}\right) + i\mathbf{J} \cdot (\mathbf{k}' \times \mathbf{k}) \times \left[-\frac{\mu^V}{3m} + \mathbf{k} \cdot \mathbf{k}' \left(\frac{\mu^V - 3}{12m^3} + \frac{2(F_1^{V'} + F_2^{V'})}{3m} + \frac{13\Omega^V}{36m^3}\right)\right.$$

$$+\ \omega\omega'\left(\frac{\mu^V}{12m^3}-\frac{2(F_1^{V'}+F_2^{V'})}{3m}-\frac{13\Omega^V}{36m^3}\right)-i\omega\omega' a_2(0)\Bigg]$$

$$+\ \{J\cdot\mathbf{k}', J\cdot\mathbf{k}\}(\omega+\omega')\frac{Q^V}{12m^2}+\frac{\Omega^V}{18im^3}(k'_ik_j+k_ik'_j)(\mathbf{k}'\times\mathbf{k})_r\langle J_i, J_j, J_r\rangle$$

$$+\ \frac{i\Omega^V}{18m^3}(k'_ik'_j+k_ik_j)(\mathbf{k}'\times\mathbf{k})_r\langle J_i, J_j, J_r\rangle+0(\omega^5)\Bigg\}. \tag{32}$$

By comparing Eqs. (23*a*) and (31), we obtain the following low-energy theorems for the *symmetric case:*

$$S_1(0)=-\frac{1}{2m}, \tag{33a}$$

$$S_{2,1}=i\frac{2\mu^V-3}{12m^2}, \tag{33b}$$

$$S_3(0)=0, \tag{33c}$$

$$S_{1,1}=0, \tag{33d}$$

$$S_{3,1}=0, \tag{33e}$$

$$S_6(0)=-\frac{1}{8m^3}, \tag{33f}$$

$$S_{12}(0)+2S_{14}(0)=\frac{Q^V}{12m^3}, \tag{33g}$$

$$S_{17}(0)=0. \tag{33h}$$

From Eqs. (23*b*) and (32), one obtains for the *antisymmetric case*

$$A_{1,1}=\frac{2\mu^V-1}{8m^2}-F_1^{V'}-\frac{Q^V}{6m^2}, \tag{34a}$$

$$A_2(0)=\frac{\mu^V}{3im}, \tag{34b}$$

$$A_{2,1}=i\left[\frac{\mu^V-3}{12m^3}+\frac{2(F_1^{V'}+F_2^{V'})}{3m}+\frac{2\Omega^V}{15m^3}\right], \tag{34c}$$

$$A_{3,1}=\frac{Q^V}{12m^2}, \tag{34d}$$

$$A_{26}(0)+A_{28}(0)=\frac{\Omega^V}{18im^3}, \tag{34e}$$

$$A_{27}(0)=\frac{i\Omega^V}{18m^3}. \tag{34f}$$

The first-order theorems (33*a*), (33*b*) and (33*c*), and the second-order theorem ((33*g*) are, of course, a trivial extension of those obtained by Pais[4] for physical Compton scattering on arbitrary spin targets.

Theorems (33*d*) and (33*e*) are new second-order theorems.

Theorems (33f) and (33*h*) refer to amplitudes which, due to transversality, will not be present in the Compton amplitude in Eq. (2).

Theorem (34*b*) satisfies the general relation

$$A_{,} (O) = \frac{\mu^V}{2imJ}, \tag{34}$$

that has been conjectured before[S] for arbitrary spin.

Theorems (34*a*) and (34*c*) to (34f) have already been discussed before[7] and here we shall quote the main **results** for completeness. These theorems can be casted in the following generalized forms:

$$A_{1,1} = \frac{\langle r^2 \rangle}{6}, \tag{35a}$$

$$A_{2,1} = i\left[-\frac{\langle R^2\rangle^V}{5J} + \frac{\mu^V}{4Jm^3} - \frac{1}{4m^3}\right], \tag{35b}$$

$$A_{3,1} = \frac{Q^V}{4J(2J-1)m^2}, \tag{35c}$$

$$A_{26}(0) + A_{28}(0) = \frac{\Omega^V}{12iJ(2J-1)(J-1)m^3}, \tag{35d}$$

$$A_{27}(0) = \frac{\Omega^V}{12J(2J-1)(J-1)m^3}, \tag{35e}$$

where $\langle r^2\rangle^V$ and $\langle R^2\rangle^V$ are, respectively, the isovector charge and **magnetic**-moment mean-square-radius given by the usual **definitions**[13],

$$\langle r^2\rangle^V I^\alpha = \left\langle \mathbf{0} \left| \int J_0{}^\alpha(\mathbf{r}) r^2\, d\mathbf{r} \right| \mathbf{0} \right\rangle = i^2\, \mathrm{V} \lim_{p=0} \lim_{p'=p} \nabla_p^2 \langle \mathbf{p}' | J_0{}^\alpha | \mathbf{p} \rangle, \tag{36}$$

$$\langle R^2\rangle^V I^\alpha = \frac{1}{2}\left\langle \mathbf{0}, \lambda' \left| \int (\mathbf{r} \times \mathbf{J}^\alpha)_z\, r^2\, d\mathbf{r} \right| \mathbf{0}, \lambda \right\rangle \Big|_{\lambda' = \lambda = J}$$

$$= \frac{(-i)^3 V}{2} \varepsilon_{3ji} \lim_{p=0} \lim_{p'=p} \nabla_p^2 \frac{\partial}{\partial p_j} \langle \mathbf{p}', J | J_i{}^\alpha | \mathbf{p}, J \rangle. \tag{37}$$

As was shown[7], these generalized forms are valid for $J \leq 3/2$ (of course, for $J = 0$ we have only (35*a*), for $J = 1/2$ only (35*a*, b), for $J = 1$ only (35*a*, b, c) and for $J = 3/2$ all will be present), and they were conjectured to be valid for arbitrary spin targets. In particular, theorem (35*a*) is the generalized form of the Cabibbo-Radicati theorem and (35*b*) is the generalized form of the magnetic moment radius theorem.

4. Concluding Remarks

The new second-order theorem (334 is analogous to those corresponding to the spin-0 and spin-1/2 cases[3] and to the spin-1 case[5]. Theorem (33*e*) is analogous to the corresponding one for spin-1 targets[5]. Taken together, these two theorems confirm for $J = 3/2$ the following second-order conjecture[3]:

Let

$$\mathrm{Tr}\,(\varepsilon'_i T_{ij}^{\{\alpha\beta\}} \varepsilon_j)/\mathrm{Tr}\,(1) = T_1^{\{\alpha\beta\}}(\omega, \omega')\boldsymbol{\varepsilon}' \cdot \boldsymbol{\varepsilon} + T_2^{\{\alpha\beta\}}(\omega, \omega') \times (\boldsymbol{\varepsilon}' \cdot \mathbf{k}\ \boldsymbol{\varepsilon} \cdot k' - \mathbf{k} \cdot \mathbf{k}' \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}'), \tag{38}$$

where the trace is over spin states, and let

$$T_1^{\{\alpha\beta\}}(\omega, \omega') = T_1^{\{\alpha\beta\}}(0,0) + t_1^{\{\alpha\beta\}}\mathbf{k} . \mathbf{k}' + t_2^{\{\alpha\beta\}}\omega\omega' + 0(\omega^3); \tag{39}$$

then

$$t_1^{\{\alpha\beta\}} = 0. \tag{40}$$

The general proof of these new second-order theorems, Eqs. (334 and (33*e*), and of all previous conjectures[3,5,7] for arbitrary spin targets, is under investigation 'and will be reported elsewhere.

References and Notes

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