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Sach's Quaternion Equations of Gravitation and Bergmann's Theory of General Relativity*

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It is shown that Sach's quaternion equations of gravitation in a space-time manifold with Riemannian structure are immediate consequence of Bergmann's spinor theory of general relativity and Einstein's equations.

Mostra-se que as equações quaterniônicas da gravitação, em uma variedade espaço-temporal provida de uma estrutura Riemanniana, são consequência imediata da teoria espinorial de Bergmann da relatividade geral e das equações de Einstein.

1. Introduction

Since Einstein's formulation of the theory of gravitation[1], efforts have been made to generalize the theory of the gravitational field, so as to include more general geometries than the Riemannian. Some of these attempts used the quaternion approach[2].

Among the quaternion theories of gravitation, the theory developed by P. G. Bergmann[3] is particularly interesting due to its simple mathematical structure and its natural applicability to wave equations in curved space-time. Recently, Sachs[4,5] using a quaternion formalism obtained a pair of quaternion equations, conjugated to each other, which describe the gravitational theory. These equations were obtained by variational methods in which Palatini's technique was used. It should be mentioned that the quaternion equations of gravitation may be formulated in a flat space-time theory of the gravitational field[7] and, in this aspect, they are useful for describing the interaction among the gravitational field and spinor fields in flat space-time.

In this paper, we reobtain some results of Bergmann's paper and, starting from them, we obtain directly Sach's quaternion equations, only with the

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additional use of Einstein's equations. This step is equivalent to Sach's variational method.

2. Some Results of Bergmann's Quaternion Theory of Gravitation

In Bergmann's theory, the fundamental geometric entities are two quaternion fields $\sigma_\mu(x)$ and $\tau_\mu(x)$, $\mu = 0, 1, 2, 3$, defined in the four dimensional space-time manifold conveniently represented by 2 x 2 Hermitian matrices. Here, instead of using Bergmann's field τ_μ, which is the time inverse of o,, we shall use $\bar{\sigma}_\mu$, which is the space inverse of the o„ field. These two fields allow us to construct uniquely the metric $g_{\mu\nu}$ and, consequently, the Riemann-Christoffel tensor R„„ . It is important to remember that the connection between g„ and o,, $\bar{\sigma}_\lambda$, is given by

$$\sigma_\mu | \sigma_\nu = g_{\mu\nu} \dot{\sigma}_0 = (1/2)(\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu), \tag{2-1}$$

where $\dot{\sigma}_0$ is the 2 x 2 unit matrix.

As usual, Bergmann constructs[3] the spinor curvature $\mathscr{R}_{rs}$ by means of the spinor affine connection Γ_ν, as:

$$\mathscr{R}_{\nu\mu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu, \tag{2-2}$$

where ∂_μ means partial derivative.

The correlation between the spinor-affine connection Γ_ρ and the external affine connection, L"„ , is obtained by the annulment of the covariant derivative of the σ^μ field:

$$\sigma^\mu{}_{|\rho} = \sigma^\mu{}_{,\rho} + \sigma^\lambda L^\mu{}_{\lambda\rho} + \Gamma_\rho \sigma^\mu + \sigma^\mu \Gamma_\rho^{\dagger} = 0, \tag{2-3}$$

where $\Gamma_\rho^{\dagger}$ is the Hermitian conjugate of the Γ_ρ and $\sigma^\mu{}_{,\rho}$ means $\partial_\rho \sigma^\mu$

In (2-3), the external affine-connection $L^\mu{}_{\lambda\rho}$ is not necessarily symmetric, but Bergmann was interested in a space-time manifold with a Riemannian structure and used in his paper[3] the Riemann-Christoffel conection

in place of $L^\mu{}_{\lambda\rho}$, as we shall do from now on.

From (2-3), Bergmann obtained the explicit form of Γ,,:

$$\Gamma_\rho = -(1/4)\left[\sigma^\mu{}_{,\rho} + \sigma^\nu \left\{ {\mu \atop \nu\ \rho} \right\}\right] \bar{\sigma}_\mu . \tag{2-4}$$

It is important to note that the Riemann-Christoffel connection has, of course, the usual form in terms of the $g_{\mu\nu}$, but the g„ fields must be expressed in terms of the fundamental fields o, and C„ as in (2-1). From this viewpoint, (2-4) is constructed with the **o, ,** $\bar{\sigma}_\nu$ and its derivatives.

With the aid of (2-2) and (2-4) it is possible to show that[6]:

$$\text{r,} + \bar{\text{r}}\text{,} = 0, \tag{2-5}$$

$$\mathcal{R}_{\nu\mu} + \mathcal{R}_{\mu\nu} = 0, \tag{2-6}$$

$$\mathcal{R}_{\mu\nu} + \bar{\mathcal{R}}_{\mu\nu} = 0. \tag{2-7}$$

From (2-2) and (2-4), we obtain after some rearrangement[3]:

$$\mathcal{R}_{rs} = -(1/4)\,\sigma^\mu\,\bar{\sigma}_\lambda \left[\partial_r \begin{Bmatrix} \lambda \\ \mu\ s \end{Bmatrix} - \partial_s \begin{Bmatrix} \lambda \\ \mu\ r \end{Bmatrix} + \begin{Bmatrix} \theta \\ \mu\ r \end{Bmatrix}\begin{Bmatrix} \lambda \\ \theta\ r \end{Bmatrix} - \begin{Bmatrix} \theta \\ \mu\ r \end{Bmatrix}\begin{Bmatrix} \lambda \\ \theta\ s \end{Bmatrix}\right] =$$

$$= -(1/4)\sigma^\mu\,\bar{\sigma}_\lambda\; R_\mu{}^\lambda{}_{rs} = -(1/4)\sigma^\mu\,\bar{\sigma}^\lambda\; R_{\mu\lambda rs}, \tag{2-8}$$

where R $_{\mu\lambda rs}$ is the Riemann-Christoffel curvature.

Thus, in the case of a flat Riemannian manifold, we obtain:

$$\text{W,,,} = 0. \tag{2-9}$$

If we contract (2-9), with the anti-symetrized part of $\sigma^r\,\bar{\sigma}^s$, we have:

$$\mathcal{R}_{rs}\,(\sigma^r\,\bar{\sigma}^s - \sigma^s\,\bar{\sigma}^r) = 0 \tag{2-10}$$

It is interesting to remember that the projected form of equation (2-10), on a local Cartan frame, was postulated by Teitler[8] as a general conservation law when he interpreted the projected $\mathcal{R}_{rs}$ as a gravitational flux quantity. Teitler's theory is, in reality, a Bergmann's flat theory of gravitation, as we have shown in a previous paper[7].

We would like to inention that, similarly to Bianchi's tensor identities, we easily obtain the Bianchi's quaternion identities:

$$\mathcal{R}_{rs|\rho} + \mathcal{R}_{\rho r|s} + \mathcal{R}_{s\rho|r} = 0 \tag{2-8'}$$

and

$$\mathcal{R}'_{rs|\rho} + \mathcal{R}'_{\rho r|s} + \mathcal{R}'_{s\rho|r} = 0. \tag{2-8''}$$

Now we are interested in the inverted form of (2-8) and to obtain it, we construct $\sigma^\mu{}_{|rs} - \sigma^\mu{}_{|sr}$.

From (2-3), we have:

$$\sigma^\mu{}_{|rs} - \sigma^\mu{}_{|sr} = 0. \tag{2-11}$$

But if we use (2-4j and (2-8) in (2-1lj, we obtain[6] the explicit form[3] :

$$\sigma^{\lambda} R^{\mu}{}_{\lambda rs} - \mathscr{R}_{rs} \sigma^{\mu} - \sigma^{\mu} \mathscr{R}^{\dagger}{}_{rs} = 0; \qquad (2\text{-}12)$$

this equation is the inverse of *(2-8)*.

If we use *(2-7),* the adjoint of *(2-12)* is[6]:

$$\bar{\sigma}^{\lambda} R^{\mu}{}_{\lambda rs} + \bar{\sigma}^{\mu} \mathscr{R}_{rs} + \mathscr{R}^{\dagger}{}_{rs} o'' = 0. \qquad (2\text{-}12')$$

3. Sach's Equations

If we contract μ with s in *(2-12)* and (2-12′), we obtain:

$$\sigma^{\lambda} R_{\lambda r} = \mathscr{R}_{rs} \sigma^{s} + \sigma^{s} \mathscr{R}^{\dagger}_{rs}, \qquad (3\text{-}1)$$

and

$$\bar{\sigma}^{\lambda} R_{\lambda r} = -\bar{\sigma}^{s} \mathscr{R}_{rs} - \mathscr{R}^{\dagger}_{rs} \bar{\sigma}^{s}, \qquad (3\text{-}1')$$

where R,, is Ricci's tensor.

If we remember Einstein's equations',

$$R_{\lambda r} = (R/2) g_{\lambda r} + T_{\lambda r}, \qquad (3\text{-}2)$$

(where R is the scalar Riemannian curvature and T, is the energy-momentum tensor of the non-gravitational fields) and couple *(3-2)* with *(3-1)* and *(3-I'),* we have:

$$\sigma^{\lambda} T_{\lambda r} = -(R/2)\, o, + \mathscr{R}_{rs} \sigma^{s} + o^{s} \mathscr{R}^{\dagger}{}_{rs} \qquad (3\text{-}3)$$

and

$$\bar{\sigma}^{\lambda} T_{\lambda r} = -(R/2)\, \bar{\sigma}_{r} - \bar{\sigma}^{s} \mathscr{R}_{rs} - \mathscr{R}^{\dagger}{}_{rs} \bar{\sigma}^{s} \qquad (3\text{-}3')$$

writing:

$$\sigma^{\lambda} T_{\lambda r} = T_{r}, \ \bar{\sigma}^{\lambda} T_{\lambda r} = \bar{T}_{r}, \qquad (3\text{-}3'')$$

we obtain:

$$T_{r} = -(R/2)\, \sigma_{r} + \mathscr{R}_{rs} \sigma^{s} + o^{s} \mathscr{R}^{\dagger}{}_{rs}, \qquad (3\text{-}4)$$

$$\bar{T}_{r} = -(R/2)\, \bar{\sigma}_{r} - \bar{\sigma}^{s} \mathscr{R}_{rs} - \mathscr{R}^{\dagger}{}_{rs} \bar{\sigma}_{s}. \qquad (3\text{-}4')$$

By analogy, we call the Hermitian quaternion T_r the energy-momentum quaternion of the non-gravitational fields present in the manifold.

The equations *(3-4)* and (3-4′) constitute a pair of quaternion equations of gravitation. They are straightforward consequences of Bergmann's results and coincide with Sach's equations of gravitation. Then we have proved that Sach's quaternion equations of gravitation, in a space-time manifold with Riemannian structure, are an immediate consequence of Bergmann's spinor theory of general relativity and Einstein's equations.

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