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Multiperipherism and Inclusive Reactions

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The general properties of hadronic inclusive reactions are reviewed. They are mainly analyzed in the frame-work of the multiperipheral and Regge-exchange models; recent results from dual models are also outlined. Particular emphasis is given to those features which are independent of the details of the model and are essentially based on general concepts and hypotheses.

São revistas aqui as propriedades gerais das reações hadrônicas inclusivas Sua análise é feita sobretudo no esquema dos modelos multiperiférico e de Regge, embora sejam também esboçados resultados recentes de modelos duais. Particular ênfase é dada aos aspectos que são independentes de detalhes dos modelos e que se baseiam, essencialmente, em conceitos e hipóteses gerais.

1. Introduction

The field of multiparticle hadronic reactions has been rapidly growing in the last few years, and the outcoming results of the new machines, the CERN-ISR and the NAL accelerator, are expected to increase further the interest in it. Even if *exclusive* analyses of multiparticle reactions will probably be needed for the comprehension of hadrondynamics, and results from the practically unexplored region of large transverse momenta may modify drastically the present theoretical schemes, the information coming from the *inclusive* analyses has not been exploited completely, and may still provide new clues on the dynamical mechanism.

Many review articles exist in the literature on the subject, some of them devoteú to exclusive analyses[1], others, with different emphases, to inclusive reactions[2]; a comprehensive review covering several aspects of multiparticle reactions is given by Frazer *et al.*[3].

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In this paper, we shall be mainly concerned with the general features of the inclusive reactions in the framework of the multiperipheral[4] and related models[5]. The multiperipheral scheme has developed, in the last decade, into a self-contained and well-equipped theoretical laboratory. New ideas, such as Regge and Toller poles, received further support and implements in this scheme. Several models have been formulated from the original one[4], but the main properties, such as scaling and logs dependence of the average multiplicity, are independent of the details of the models, and were already exhibited by the original one. The new concept of duality leads naturally from the multiperipheral and Regge-exchange models to dual models[6]; we shall briefly examine the consequences of duality ideas on inclusive reactions.

The material will be organized as follows. We begin, in Part 2, with a recollection of kinematical formulae and definitions of inclusive distributions, especially with the aim of specifying our conventions. Then, we consider briefly the energy-momentum sum rules, which will be relevant in later discussion. Next, we survey the general features of inclusive reactions, the main approaches and the different kinds of models. Part **3** is devoted mainly to multiperipherism for inclusive reactions. We start with a simplified version of the uncorrelated jet model, to illustrate in a simple fashion the important role of the transverse momentum cut-off. Then we analyze the predictions given by the multiperipheral and Regge-exchange models, and discuss some recent results obtained from dual models. For the discussion of other models and a detailed analysis of the phenomenological situation, we refer to existing review papers.

2. Generalities on Inclusive Reactions

2.1 Kinematics

In this section, we collect the essential kinematical formulae which can be found in any review paper on the subject. They are reported here for completeness and for specifying our conventions.

First of all, we notice that the number of independent kinematical variables for an inclusive reaction of order k,

$$a + b \to c_1 + c_2 + \cdots c_k + X, \tag{2-1}$$

where k particles are observed in the final state, is equal to $3k$, as can be easily checked.

A. Kinematics for One-Particle Inclusive Reactions

In this case there are three independent variables; one can make different choices starting from the four-momenta P_a, P_b, P_c (see Fig. 1). In the following, we shall always denote by a and b the target and projectile particle, respectively.

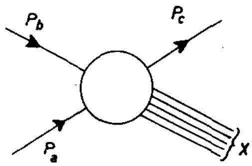


Fig. 1 - Inclusive reaction

i) Set s, t, M_X^2 (Mandelstam variables plus missing mass).

$$
\begin{aligned}
s &= (P_a + P_b)^2, \\
t &= (P_b - P_c)^2, \\
M_X^2 &= (P_a + P_b - P_c)^2.
\end{aligned}
\tag{2-2}
$$

ii) Set s, ξ_c, p_c (longitudinal rapidity and transverse momentum of the observed particle).

Taking the z-axis along the direction of the projectile momentum $\mathbf{P}_b$, we can write in an arbitrary frame of reference:

$$
\begin{aligned}
P_a &= (m_a \cosh \xi_a,\ 0, 0, m_a \sinh \xi_a), \\
P_b &= (m_b \cosh \xi_b,\ 0, 0, m_b \sinh \xi_b), \\
P_c &= (\mu_c \cosh \xi_c,\ p_c \cos \varphi_c,\ p_c \sin \varphi_c,\ \mu_c \sinh \xi_c),
\end{aligned}
\tag{2-3}
$$

where $\mu_c = \sqrt{p_c^2 + m_c^2}$, and where p_c denotes the magnitude of the transverse momentum.

The quantity ξ_i, defined by

$$
\xi_i = \log \frac{P_i^0 + P_i^z}{\mu_i}, \tag{2-4}
$$

is the so-called ***longitudinal rapidity.*** Under a Lorentz transformation along the z-axis, longitudinal rapidities change by an additive constant.

The ***relative rapidities,*** defined by

$$
\begin{aligned}
\xi_{ca} &= \xi_c - \xi_a, \\
\xi_{bc} &= \xi_b - \xi_c,
\end{aligned}
\tag{2-5}
$$

are therefore invariant under this kind of Lorentz transformations. The range of variability of ξ_c for large s can be expressed, in general, as

$$\xi_a + \log \frac{\mu_c}{m_a} \lesssim \xi_c \lesssim \xi_b + \log \frac{m_b}{\mu_c} \cdot \tag{2-6}$$

The following relations will be often used:

$$s = m_a^2 + m_b^2 + 2\, m_a m_b \cosh \xi_{ba}\,, \tag{2-7}$$

$$M_X^2 = s + m_c^2 - 2\, m_a \mu_c \cosh \xi_{ca} - 2\, m_b \mu_c \cosh \xi_{bc}\,. \tag{2-8}$$

Denoting by ξ_i^* the longitudinal rapidity in the C. M. frame, one gets for large s

$$\xi_a^* \approx -\log \frac{\sqrt{s}}{m_a}, \qquad \xi_b^* \approx \log \frac{\sqrt{s}}{m_b} \tag{2-9}$$

and

$$-\log \frac{\sqrt{s}}{\mu_c} \lesssim \xi_c^* \lesssim \log \frac{\sqrt{s}}{\mu_c} \cdot \tag{2-10}$$

The corresponding relations in the laboratory frame are

$$\xi_a = 0, \qquad \xi_b \approx \log (s/m_b^2) \tag{2-11}$$

and

$$\log \frac{\mu_c}{m_a} \lesssim \xi_c \lesssim \log \frac{s}{m_b \mu_c} \cdot \tag{2-12}$$

One sees that the range of the longitudinal rapidity of the observed particle is of the order of $\log s$.

iii) Set s, x_c, p_c **(Feynman's variables).**

The scaling variable x_c can be defined as

$$x_c = \frac{2 P_c^{*z}}{\sqrt{s}}, \tag{2-13}$$

where P_c^{*z} is the longitudinal rnornentum in the C. M. frame. Other definitions are used in the literature, which become equivalent to (2-13) at high s.

This variable is related to the longitudinal rapidity by

$$x_c = \frac{2\mu_c}{\sqrt{s}} \sinh \xi_c^*. \tag{2-14}$$

For large value of s, since from *(2.9)* one gets

$$\sqrt{s} \approx m_b \exp \xi_b^* \approx m_a \exp(-\xi_a^*), \tag{2-15}$$

we can use the expressions

$$x_c \approx \frac{\mu_c}{m_a} [\exp 2\xi_c^* - 1] \exp(-\xi_{ca}), \tag{2-16}$$

$$x_c \approx \frac{\mu_c}{m_b} [1 - \exp(-2\xi_c^*)] \exp(-\xi_{bc}). \tag{2-17}$$

B. Kinematics for Two-Particle Inclusive Reactions

In this case, one needs to specify 6 independent variables. One can take, for instante, either of the following sets:

i) $s, \xi_c, \xi_d, p_c, p_d, \varphi_{cd}$,
i.e., the longitudinal rapidities ξ_c, ξ_d of the particles c, d; their transverse momenta p_c, p_d and the relative angle φ_{cd};

ii) $s, x_c, x_d, p_c, p_d, \varphi_{cd}$,
where the rapidities are replaced by the scaling variables x_c, x_d defined as in (2-13).

Also in this case, one often uses the missing mass variable, defined by

$$M_X^2 = (P_a + P_b - P_c - P_d)^2 \tag{2-18}$$

and which can be expressed as follows, in terms of the rapidities:

$$\begin{aligned} M_X^2 = s &+ m_c^2 + m_d^2 - 2\mu_c(m_a \cosh \xi_{ca} + m_b \cosh \xi_{bc}) \\ &- 2\mu_d(m_a \cosh \xi_{da} + m_b \cosh \xi_{bd}) + 2\mu_c\mu_d \cosh \xi_{dc} \\ &- 2p_c p_d \cos \varphi_{cd}. \end{aligned} \tag{2-19}$$

2.2 Definitions of Inclusive Distributions

We start with the exclusíve reactíon

$$a + b \to c_1 + c_2 + \cdots c_{n}, \tag{2-20}$$

in which, for the sake of simplicity, we assume $c_1, c_2, \ldots c_n$ to be n identical particles with spin zero and mass m. The integral cross section for this process is defined by

$$= \frac{1}{\gamma(s)} \int \delta^4(P_a + P_b - \sum_{i=1}^{n} P_i) \, |M_n(P_1, \ldots, P_n)|^2 \frac{1}{n!} \prod_{i=1}^{n} (dP_i), \qquad (2\text{-}21)$$

where M_n is the relevant matrix element (to get a handy expression we eliminate all usual 2π-factors which are supposed to be inserted in M_n); $1/(n!)$ is the permutation factor. which gives the right counting of the final states, and $\gamma(s)$ is the incident flux

$$\gamma(s) = [(s - m_a^2 - m_b^2)^2 - 4m_a^2 m_b^2]^{1/2}, \qquad (2\text{-}22)$$

which asymptotically reduces to

$$\gamma(s) \approx s. \qquad (2\text{-}23)$$

Finally, dP_i denotes the invariant phase space element

$$dP_i = \delta(P_i^2 - m^2) \, d^4 P_i = \frac{d^3 P_i}{2P_i^0} \qquad (2\text{-}24)$$

which, in terms of ξ_i, p_i, φ_i becomes:

$$dP_i = \frac{1}{2} p_i \, dp_i \, d\xi_i \, d\varphi_i \,. \qquad (2\text{-}25)$$

The total cross section for the process

$$a + b \rightarrow \text{everything}, \qquad (2\text{-}26)$$

which can be defined as inclusive reaction of order zero, is obtained summing over n from 2 to the maximum value allowed by energy conservation:

$$\sigma_T(s) = \sum_n \sigma_n(s). \qquad (2\text{-}27)$$

The differential one-particle inclusive cross section for fixed n (probability of finding one particle out of n in dP_1) is given by

$$\frac{d\sigma_n^{(1)}}{dP_1} \equiv F_n^{(1)}(P_1) = \frac{1}{\gamma(s)} \, n \int \delta^4(P_a + P_b - \sum_{i=1}^{n} P_i) \frac{1}{n!} |M_n|^2 \prod_{i=2}^{n} (dP_i). \qquad (2\text{-}28)$$

Summing over n, one gets the *inclusive one-particle distribution function*

$$\frac{d\sigma^{(1)}}{dP_1} \equiv F^{(1)}(P_1) = \sum_n F_n^{(1)}(P_1). \qquad (2\text{-}29)$$

From the normalization (2-28), one obtains immediately

$$\int F^{(1)}(P_1)\,dP_1 = \sum_n n\,\sigma_n(s) = \langle n\rangle\,\sigma_T\,, \tag{2-30}$$

where $\langle n\rangle$ is the *average* multiplicity.

The inclusive two-particle distributions are obtained in a similar way from

$$\frac{d\sigma_n^{(2)}}{dP_1\,dP_2} \equiv F_n^{(2)}(P_1\,,P_2) = \frac{1}{\gamma(s)}\,n(n-1)\int \delta^4(P_a+P_b-\sum_{i=1}^{n}P_i)\,\frac{1}{n\,!}\,|M_n|^2\prod_{i=3}^{n}(dP_i) \tag{2-31}$$

and

$$F^{(2)}(P_1\,,P_2) = \sum_n F_n^{(2)}(P_1\,,P_2). \tag{2-32}$$

From the above definitions, one gets

$$\iint F^{(2)}(P_1\,,P_2)\,dP_1\,dP_2 = \langle n(n-1)\rangle\,\sigma_T\,. \tag{2-33}$$

The above relations can be easily generalized to the case in which different kinds of particles are present in the final states[7].

Sometimes, instead of the above distributions, it is more convenient to use inclusive densities, defined by

$$f^{(1)}(P_1) = \frac{1}{\sigma_T}\,F^{(1)}(P_1), \tag{2-34}$$

$$f^{(2)}(P_1\,,P_2) = \frac{1}{\sigma_T}\,F^{(2)}(P_1\,,P_2). \tag{2-35}$$

In this connection, also correlation *functions* are often introduced, according to the cluster decomposition used in many-body theory. Since, in the following, we shall limit ourselves mainly to two-particle inclusive reactions, we give here only the two-particle correlation function

$$\rho^{(2)}(P_1\,,P_2) = f^{(2)}(P_1\,,P_2) - f^{(1)}(P_1)\,f^{(1)}(P_2). \tag{2-36}$$

The three particle correlation function $\rho^{(3)}(P_1, P_2, P_3)$ is defined by the relation

$$f^{(3)}(P_1, P_2, P_3) = \rho^{(3)}(P_1, P_2, P_3) + \rho^{(2)}(P_1, P_2) f^{(1)}(P_3) + \text{permutations} + f^{(1)}(P_1) f^{(1)}(P_2) f^{(1)}(P_3), \tag{2.37}$$

from which one can immediately obtain the generalization to the case of correlation functions of order k : $\rho^{(k)}(P_1, P_2, \ldots, P_k)$.

As it will appear in the following, it is very useful to deal with the correlation coefficients[57], i.e., the integrals of the correlation functions:

$$R^{(k)} = \int dP_1\, dP_2 \ldots dP_k \rho^{(k)}(P_1, P_2, \ldots, P_k). \tag{2-38}$$

In the case k = 2, from Eqs. (2-38) and (2-36) one gets:

$$R^{(2)} = \langle n(n-1)\rangle - \langle n\rangle^2 = D^{(2)} - \langle n\rangle, \tag{2-39}$$

where the quantity

$$D^{(2)} = \langle (n - \langle n\rangle)^2\rangle \tag{2-40}$$

is the multiplicity (mean square) fluctuation (or dispersion).

We end this section by defining a generating function,

$$Q(z, s) = \sum_n z^n \sigma_n(s), \tag{2-41}$$

such that

$$Q(1, s) = \sigma_T(s)$$

$$\left|\frac{\partial \log Q}{\partial z}\right|_{z=1} = \langle n\rangle,$$

$$\left|\frac{\partial^2 \log Q}{\partial z^2}\right|_{z=1} = \langle n(n-1)\rangle. \tag{2-42}$$

The analogy of the definition (2-41) with the partition function of a grand canonical ensemble lead to a statistical mechanical analogue (Feynman's liquid) for the description of the multiparticle processes in the rapidity plot[8].

2.3 Energy-Momentm Sum Rules

In this section, we want to discuss briefly some general relations based only on the energy-momentum conservation This imposes relations (sum rules) amongst inclusive distributions of different order; their physical implication will be commented in following sections. Sum rules based on other conservation laws, such as charge[9] and isospin[tO], will not be considered in this paper.

Energy-momentum sum rules, first obtained by Chow and Yang[11], have been expressed in a general way[12]. We follow here a simplified derivation[13].

It is convenient to start from the k-particle inclusive distributtion, which can be written, as simple generalization of (2-31), as

$$F_n^{(k)}(P_1, \dots P_k) = \frac{1}{\gamma(s)} \frac{1}{(n-k)!} \int \delta^4(P_a + P_b - \sum_{i=1}^{n} P_i) \, |M_n|^2 \, dP_{k+1} \dots dP_n \tag{2-43}$$

and

$$F^{(k)}(P_1, \dots P_k) = \sum_n F_n^{(k)}(P_1, \dots P_,) \tag{2-44}$$

The sum rules we want to consider can be expressed in the form

$$(P^\mu - \sum_{i=1}^{k} P_i^\mu) \, f^{(k)}(P_1, \dots P_k) = \int P_{k+1}^\mu \; f^{(k+1)}(P_1, \dots P_{k+1}) \, dP_{k+1} \tag{2-45}$$

in terms of the inclusive densities. This relation can be easily checked by making use of the definitions (2-43),(2-44) and of the symmetry properties of M,.

It is instructive to consider the simplest cases k = 0, k = 1:

$$P^\mu = \int P_1^\mu \, f^{(1)}(P_1) \, dP_1 \, , \tag{2-46}$$

$$(P^p - P_1^\mu) \, f^{(1)}(P_1) = \int P_2^\mu \, f^{(2)}(P_1 \, , P_,) \, dP_2 \, . \tag{2-47}$$

In terms of the correlation function (2.36), one gets from the above relations:

$$P_1^\mu f^{(1)}(P_1) = -\int P_2^\mu \rho^{(2)}(P_1, P_2)\, dP_2 . \tag{2-48}$$

We note that energy-momentum conservation imposes the correlation function to be different from zero at least in some region of the phase space.

By multiplying Eq. (2-47) by P_1^ν and integrating over P, , one gets the Predazzi-Veneziano sum rule[12]:

$$P^\mu P^\nu = \int P_1^\mu P_1^\nu f^{(1)}(P_1)\, dP_1 + \iint P_1^\mu P_2^\nu f^{(2)}(P_1, P_2)\, dP_1\, dP_2 , \tag{2-49}$$

which for transverse momenta ($\mu = \nu = 1, 2$) becomes

$$\int p_1^2 f^{(1)}(P_1)\, dP_1 = -\iint p_1 p_2 \cos\varphi_{12} f^{(2)}(P_1, P_2)\, dP_1\, dP_2 . \tag{2-50}$$

The latter relation implies an *azimuthal* dependence of $f^{(2)}(P_1 . P_2)$ (φ_{12} is the angle between the two transverse momenta $\mathbf{p}_1$, $\mathbf{p}_2$).

Different expressions for energy-momentum sum rules can be found in the literature; however, it can be shown that Eq. (2-45) provides all the independent sum rules for the k-order inclusive density[13].

2.4 Main Features of Inclusive Reactions

We start with two general features of high energy multiparticle reactions, i.e., the smallness of average transverse momentum and the low multiplicity of the particles produced.

i) Distribution of transverse momenta. The number of particles produced decreases very rapidly with increasing of the transverse momentum. In particular, the inclusive distributions, for fixed values of the scaling variable x, decrease exponentially, at least in the region $0.15 < p < 1$ GeV/c. They all have reached scaling above 1000 GeV/c. but the slopes of the distributions differ for different kinds of particles (see Fig. 2).

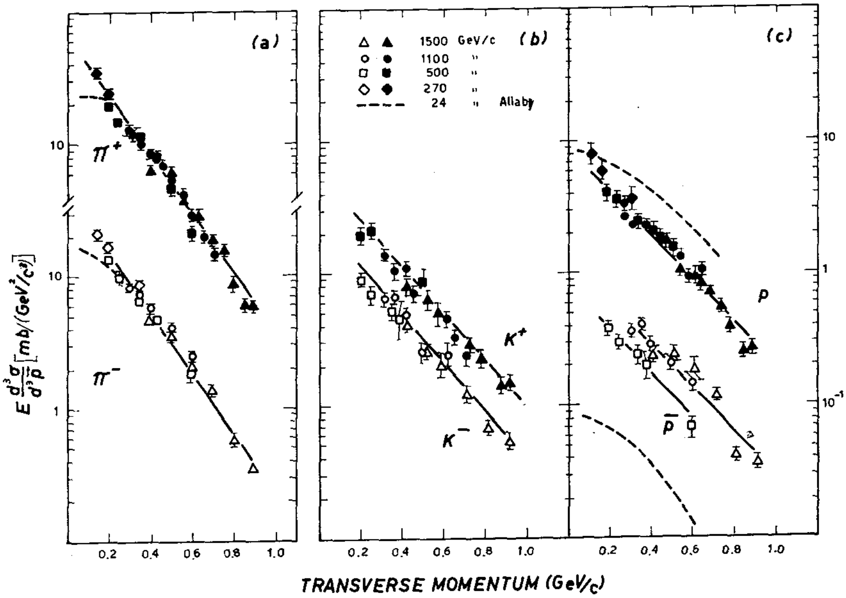


Fig. 2 - Inclusive one-particle cross sections **versus** transverse momentum at fixed RN-R **CERN-ISR energies. The** solid lines **represent exponential** fits.

The following values for the average transverse momenta are estimated from the CERN-ISR data[14]:

$$(p) \approx 350\,\mathrm{MeV/c} \quad \text{for} \quad \pi^{\pm}$$
$$\langle p\rangle \approx 450\,\mathrm{MeV/c} \quad \text{for} \quad K' \;,$$
$$\langle p\rangle \approx 500\,\mathrm{MeV/c} \quad \text{for} \quad p, \bar{p}.$$

ii) Multiplicity of particle produced. The average number of particles produced in a high energy reaction is rather small, in comparison with the number which could be created. The multiplicity increases slowly with energy; its exact dependence has not yet been definitely established, but a form of the type

$$\langle n_{ch} \rangle A + B \log s \tag{2-51}$$

for the average number $\langle n_{ch} \rangle$ of charged particles, is favoured for energies above 80 GeV (Ref. 14). A compilation of charged multiplicity versus energy is shown en Fig. 3.

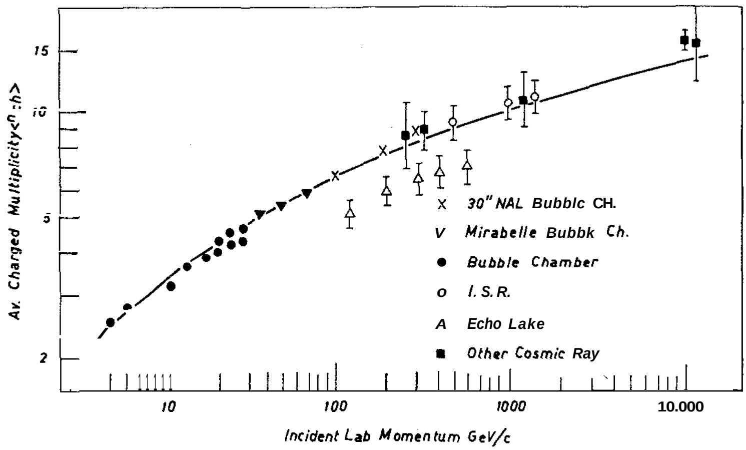


Fig. 3 - Average charged multiplicity versus incident lab. momentum. The solid line represents a fit of the type $A + B \log s + C s^{-1/2}$.

Besides these basic properties of multiparticle reactions, the hypotheses of scaling and limiting fragmentation reflect general features of the single particle spectra. They can be formulated as follows:

iii) Limiting fragmentation[15]: in the laboratory frame, the inclusive distribution $F^{(1)}(P_c, s) = F^{(1)}(P_c^z, p_c, s)$ approaches a finite limit

$$F^{(1)}(P_c^z, p_c, s) \to F^{(1)}(P_c^z, p_c), \tag{2-52}$$

when P_c^z is held fixed and $s \to \infty$. The particle c is considered a ***fragment of the target.*** Similar statement holds in the rest frame of the projectile, and then *c* is considered a ***fragment of the projectile.***

iv) Scaling hypothesis[16]: in the C. M. frame, the inclusive distribution depends, in the asymptotic limit for large s, only on the scaling vanable x_c (2-13):

$$F^{(1)}(P_c^{*z}, p_c, s) \to F^{(1)}(x_c, p_c). \tag{2-53}$$

For fixed value $x_c \neq 0$, this is equivalent to the limiting fragmentation hypothesis; however, Eq. (2-53) applies also to the ***central region*** $x_c \approx 0$

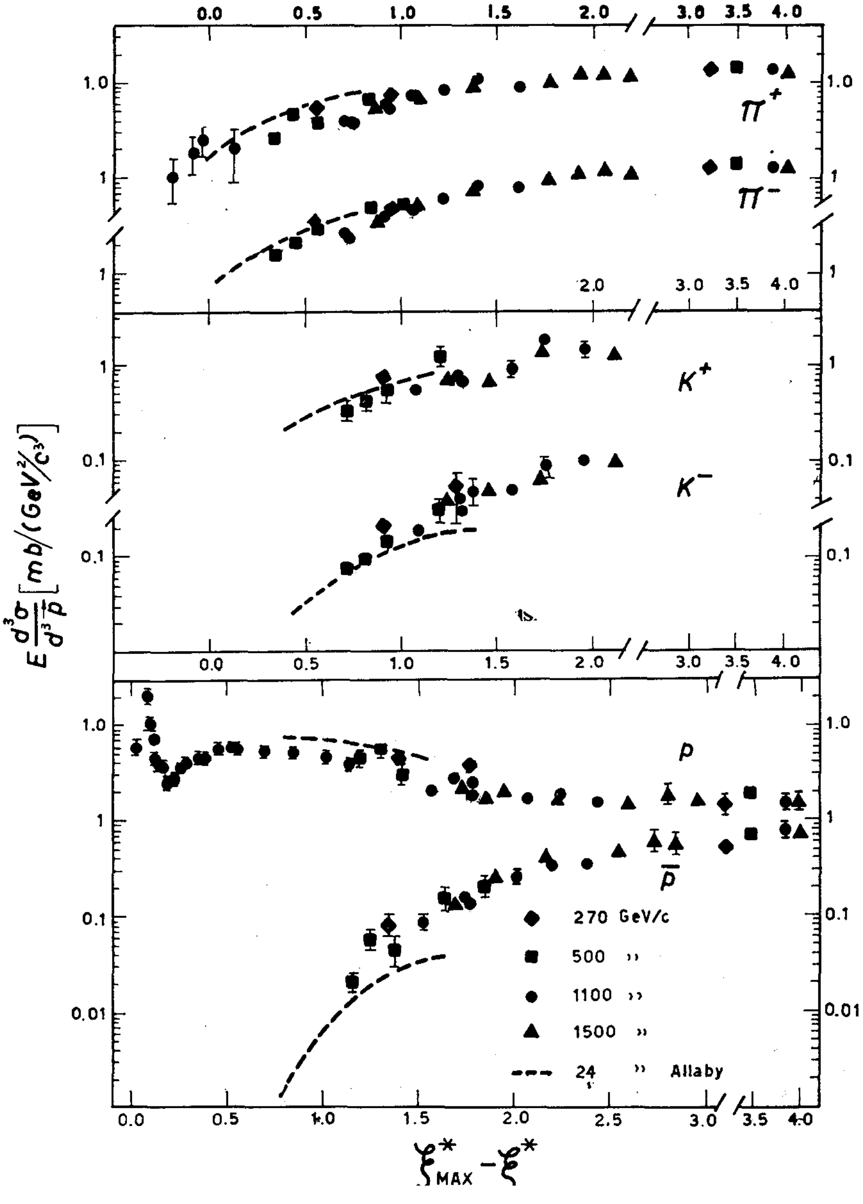


Fig. 4 - Inclusive one-particle cross sections from CERN-ISR data versus the longitudinal rapidity $\xi^*_{max}-\xi^*$.

which does not correspond to any finite momentum in the laboratory or in the projectile frame.

Similar statements are extended to higher order inclusive distributions. When the above. hypotheses are satisfied, we shall refer, indifferently, to scaling or limiting distributions.

One can separate the single particle spectra in the rapidity plots in different regions, making the hypothesis of a certain correlation length L: one assumes that two particles are uncorrelated if their relative rapidity is larger than L. One can then define the three following regions: target *fragmentatioii* when $\xi_{ca} \lesssim L$; *projectile fragmentation* when $\xi_{bc} \lesssim L$; central region when both inequalities $\xi_{ca} > L$ and $\xi_{bc} > L$ hold. It can be shown that the predictions of limiting fragmentation and scaling can be derived from this correlation length hypothesis. In accordance with recent experimental indications, it is convenient to make use of a correlation length of the order of 2.

In Fig. 4, recent data on single particle inclusive distributions are presented. They indicate that limiting distributions are reached at the ISR energies; the way of approaching the limit depends on the type of particle observed.

2.5 General Approaches to Inclusive Reactions

We discuss briefly here the problem of constructing inclusive cross sections starting from a given model. Two general approaches have been followed up to now:

i) the Direct Approach, which requires the knowledge of the n-particle production amplitudes; the inclusive distributions are then obtained by summing exclusive cross sections over the appropriate unobserved quantities;

ii) the Mueller's Approach[17], which is based on the knowledge of forward elastic multiparticle amplitudes; the inclusive distributions are obtained by taking appropriate discontinuities of these amplitudes.

Of course, if one can derive from a given model a unitary S-matrix, the results obtained following either of the two approaches must coincide. However, the two approaches can lead to different results for the inclusive cross sections, if an approximate form of the S-matrix is used. Which of the two ways is more convenient to employ depends essentially on the specific model that is considered. Different examples will be presented in this paper.

The rest of this section is devoted to the specification of Mueller's prescriptions for obtaining inclusive cross seetions from forward multiparticle amplitudes. These prescriptions can be considered a generalization of the optical theorem which relates the total cross section for the process a $+ b \to$ anything (inclusive cross section of order zero) to the imaginary part of the forward amplitude of the process a $+ b \to a + b$.

Let us consider the one-particle inclusive process

$$a + b \to c + X \tag{2-54}$$

represented in Fig. 1. The corresponding cross section is related, according to Mueller's Ansatz, to a certain discontinuity of the forward amplitude for the process

$$a + b + \bar{c} \to a + b + \bar{c}. \tag{2-55}$$

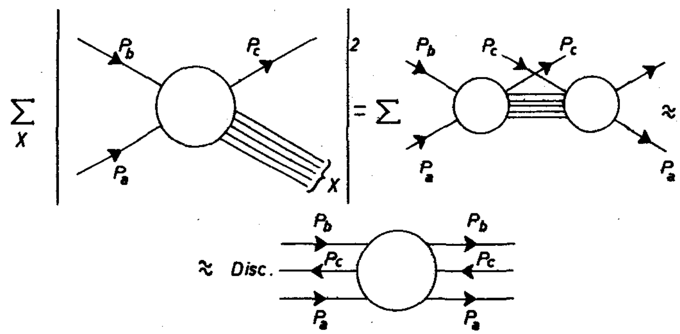


Fig. 5 - Inclusive one-particle cross section as discontinuity of forward elastic 3-body amplitude.

The situation is depicted in Fig. 5; in formulae, we write

$$F^{(1)}(P_c) = \frac{1}{\gamma(s)} A(P_a, P_b, -P_c), \tag{2-56}$$

where A is the discontinuity of the connected part of the forward 3-body amplitude $M(s, t, M_X^2)$, taken across the cut of the variable $M_X^2 = (P_a + P_b - P_c)^2$, which is just the missing mass for the process (2-54) We note that, in contrast with the optical theorem, the forward amplitude is not evaluated in its physical region, since in going from *(2-54)* to (2-55) one needs to invert the four momentum of particle c and replace c by its anti-particle $\bar{c}$. The discontinuity has to be evaluated from the non-forward amplitude $M(P_a, P_b; P'_a, P'_b; P_c)$, where P'_a, P'_b refer to the final state, in the limit $P'_a = P_a$, $P'_b = P_b$. Specifically[18]:

$$A(P_a, P_b, -P_c) = M(s + i\varepsilon, s' - i\varepsilon, t, M_X^2 + i\varepsilon) - M(s + i\varepsilon, s' - i\varepsilon, t, M_X^2 - i\varepsilon), \tag{2-57}$$

where s and $s' = (P'_a + P'_b)^2$ are to be taken above and below, respectively, the physical cut of the process a + b → a + b, while no ambiguity exists for t, since it is below threshold.

The above prescription can be extend to the cases of higher order inclusive distributions. In general, one can write

$$F^{(k)}(P_1, \ldots P_k) = \frac{1}{\gamma(s)} A(P_a, P_b, -P_1, \ldots -P_k) \tag{2-58}$$

where the discontinuity A of the forward $(2 + k)$-body amplitude has to be evaluated with the due care.

Although no general proof exists for the Mueller's extension of the optical theorem, its validity has been established in the framework of quantum field theory[19].

2.6 Models

Several models of multiparticle reactions have been developed, from different view points and with different scopes. Some are based on general concepts, others refer to specific theoretical schemes. Some have a more phenomenological character, and describe specific and detailed properties of experimental data; others exploit mainly theoretical ideas. Common features are shared by different models, while different aspects are emphasized by similar ones.

In the following, we list general classes of models, each class being defined in a'rather loose way.

i) Diffractive Fragmentation Models. We include in this class the eikonal treatment of high-energy hadronic reactions[20], and all kinds of the so-called "fireball", "jet" and "nova" models[21]. In general, these models describe a reaction in terms of the excitation of either or both incident hadrons;

the word "diffractive" is usually employed in the broad sense that the excited states carry the same internal quantum numbers of the corresponding incident hadron. Single and double excitation models[22] have met recently some success in describing the properties of the inclusive distributions.

ii) Statistical **Thermodynamical** Models. The first version of these models is due to Fermi[23]; refined versions have been developed recently[24]. The reaction products are considered to be originated by a state of statistical equilibrium of "fireballs", where this word means here both particles and resonances. The most successful outcome of this kind of models is the prediction of the transverse momentum distributions.

iii) Field **Theory** Models. In these models, the asymptotic behaviour of scattering amplitudes is derived in perturbation theory, by summing the leading terms of an infinite set of diagrams[25]. It is interesting to note that, by a convenient choice of the set of diagrams, one can derive, in the framework of quantum field theory, a simple eikonal form for the scattering amplitude.

iv) **Multiperipheral and Regge-Exchange** Models. In this class we include both all versions of multiperipheral models, which originated from the simple ladder model for the multiparticle production processes[4], and all kinds of multi-Regge models[5]. All these models have the common features that the production matrix element can be factorized in a specific way, and that the absorptive part of the elastic amplitude satisfies an integral equation with definite symmetry properties. General predictions of these models are the scaling behaviour of the distribution functions and the log s dependence of the average multiplicity.

v) Dual Models. Duality requirements and specific dual models have been applied to one- and two-particle inclusive distributions[26]. One of the most interesting outcome is the strong, universal cut-off in transverse momentum.

The above list is by no means complete. For detailed analyses of the various models, we refer to the quoted review papers[2,3]. The multiperipheral ideas and related models for inclusive reactions will be discussed in the next part of this paper.

3. Multiperipherism, Duality and Related Models

3.1. A Simple Version of Uncorrelated Jet Model

As pointed out in Sec. 2.4, one of the prominent features of high energy multiparticle reactions is the smallness of the average transverse momentum.

Before going to the multiperipheral model, we shall examine the consequences of a transverse momentum cut-off in a very simple model.

We assume that the n-particle production matrix element M, can be factorized in the form

$$|M_n(P_1, \dots, P_n)|^2 = g^{2n} \prod_{i=1}^{n} N(p_i), \tag{3-1}$$

where $N(p_i)$ is taken to be a universal function depending only on the magnitude of the transverse momentum p_i. It is not necessary to specify the form of the function $N(p_i)$, but only to assume that it decreases fast enough with increasing p_i to make certain integrals convergent Eq. **(3-1)** represents the simplest version of uncorrelated jet models, also called P-factorized models[27], in which each particle is emitted independently, and the only correlation is due to the overall energy-momentum conservation In this section, we outline the derivation of the inclusive distribution functions given by Bassetto *et al.*[27], following the direct approach (see Sec. 2.5).

A. Calculation Aids

It is useful to introduce the generating function of the four-vector R:

$$\Phi(R) = \sum_{n \geq 2} \frac{1}{n!} \Phi_n(R), \tag{3-2}$$

$$\Phi_n(R) = \int \delta^4(R - \sum_{i=1}^{n} P_i) |M_n|^2 \prod_i (dP_i). \tag{3-3}$$

Clearly, @(R) vanishes unless R is time-like and R, > 0.

It foilows that the total cross section is simply given by

$$\sigma_T(s) = \frac{1}{\gamma(s)} \Phi(P_a + P_b). \tag{3-4}$$

Making use of the specific form (3-1) for the production matrix element, one can write the single and two-particle inclusive densities in terms of the function *@(R):*

$$f^{(1)}(P_c) \approx g^2 N(p_c) \frac{\Phi(P_a + P_b - P_c)}{\Phi(P_a + P_b)}, \tag{3-5}$$

$$f^{(2)}(P_c, P_d) \approx g^4 N(p_c) N(p_d) \frac{\Phi(P_a + P_b - P_c - P_d)}{\Phi(P_a + P_b)} \tag{3-6}$$

The evaluation of the generating function @(R) is more easily carried out in terms of its Fourier transform

$$\tilde{\Phi}(V) = \int \mathrm{d}^4\mathrm{R}\ \mathrm{e}^{\mathrm{iVR}}\ @(\mathrm{R}), \tag{3-7}$$

which is an analytic function for Im $V_0 > 0$. Due to the symmetry properties (3-1) of the matrix element, both functions $\Phi(R)$ and $\tilde{\Phi}(V)$ are invariant under rotations about and boosts along the z-axis (defined, e.g., in the laboratory frame by P,). Then $\Phi(R)$ depends only on the quantities

$$r = \sqrt{R_x^2 + R_y^2}\,, \qquad r_L = \sqrt{R_0^2 - R_z^2} \tag{3-8}$$

and $\tilde{\Phi}(V)$ on the analogous ones

$$v = \sqrt{V_x^2 + V_y^2}\,, \qquad v_L = \sqrt{V_0^2 - V_z^2}\,. \tag{3-9}$$

Eq. (3-7) can be written more explicitly as

$$\tilde{\Phi}(v, v_L) = \int_0^\infty r dr \int_0^{2\pi} d\varphi e^{-ivr\cos\psi} \int_0^\infty r_L dr_L \int_{-\infty}^{+\infty} d\xi\, e^{iv_L \cosh\xi}\Phi(r, r_L), \tag{3-10}$$

where the quantity ξ is the analogue of a relative longitudinal rapidity, and φ is the angle between v and r. Integrations over ξ and φ give:

$$\tilde{\Phi}(v, v_L) = 4\pi \int_0^\infty r dr\, J_0(vr) \int_0^\infty \mathrm{r}\ \ dr_L\, K_0(-\mathrm{i}\, v_L\, \mathrm{r}\ )\mathrm{r}\quad , r_L), \tag{3-11}$$

so that by inverse transform one gets, putting $v_L = \mathrm{iz}$,

$$\Phi(r, r_L) = \frac{1}{4\pi^2 i} \int_{c-}^{c+i\infty} \mathrm{z\,dz}\, I_0(zr_L) \int_0^\infty \mathrm{v}\, dv\, J_0(rv)\, \tilde{\Phi}(v, iz). \tag{3-12}$$

The transform $\tilde{\Phi}(v, \mathrm{v}_)$ is obtained by

$$\tilde{\Phi}(v, v_L) = \sum_n \frac{1}{n!} \tilde{\Phi}_n(v, v_L), \tag{3-13}$$

where

$$\tilde{\Phi}_n(v, v_L) = g^{2n} \int d^4R\, e^{iVR}\, \delta^4(R - \sum_i P_i) \prod_i (N(p_i)\, dP_i) = [g^2\varphi(v, v_L)]^n \tag{3-14}$$

and

$$\varphi(v, v_L) = \int dP\, e^{iVR}\, N(p) = 2\pi \int_0^\infty p\, dp\, J_0(vp)\, K_0(-iv_L\sqrt{p^2 + m^2})\, N(p). \tag{3-15}$$

We are interested in the asymptotic expressions of $\Phi(r, r_L)$ valid for large r_L; the leading contributions in this limit come from small values of v, v_L in (3-12), (3-15). Specifically, one uses for $\varphi(v, v_L)$ the approximated form

$$\varphi(v, iz) \approx -2h \log (zm)\left(1 - \frac{v^2}{4\Lambda^2}\right) + 2\lambda h, \tag{3-16}$$

where

$$h = \pi \int_0^{\infty} p\, dp\, N(p), \tag{3-17}$$

$$\frac{h}{\Lambda^2} = \pi \int_0^{\infty} p^3\, dp\, N(p), \tag{3-18}$$

$$\lambda h = -\pi \int_0^{\infty} N(p) \log\left(\frac{C}{2}\sqrt{1 + \frac{p^2}{m^2}}\right). \tag{3-19}$$

In the above formulae, we have replaced, for the sake of simplicity, the quantity g^2h by h.

Inserting (3-16) into (3-14), one gets finally from (3-12) and (3-13) the approximate expression for the generating function:

$$\Phi(r, r_L) \approx e^{2\lambda h}\frac{\Lambda^2}{4\pi h m^2}\left(\frac{1}{\Gamma(h)}\right)^2\left(\frac{r_L}{2m}\right)^{2h-2}\left(\log\frac{r_L}{2m}\right)^{-1}\exp\left(-\frac{r^2\Lambda^2}{2h\log\frac{r_L}{2m}}\right). \tag{3-20}$$

B. Inclusive Distributions

We can now evaluate, using (3-4) and (3-20), the asymptotic expression of the total cross section. Since in this case $R = P_a + P_b$, one has in the C. M. frame $r = 0$, $r_L \approx \sqrt{s}$, so that:

$$\sigma_T(s) \approx \frac{1}{s}\Phi(0, \sqrt{s}) \approx e^{2\lambda h}\frac{\Lambda^2}{4\pi h m^2}\left(\frac{1}{\Gamma(h)}\right)^2\frac{1}{s}\left(\frac{s}{4m^2}\right)^{h-1}\left(\log\frac{\sqrt{s}}{2m}\right)^{-1}. \tag{3-21}$$

We see that, in order to obtain a reasonable behaviour for ***o,*** (decreasing at most as $(\log s)^{-1}$, we have to assume $h \approx 2$. The average multiplicity is obtained, according to (2-42), in the form:

$$(n) \quad = h \frac{\partial}{\partial h} \log \Phi(s, h) \approx h \log s + const, \tag{3-22}$$

Where we have neglected terms of the order of $(\log s)^{-1}$.

The one-particle inclusive density is obtained from (3-5). Since now $R = P_a + P_b - P_c$, one gets

$$r = p_c, \quad r_L = \sqrt{M_X^2 + p_c^2}, \tag{3-23}$$

and making use of (2-7), (2-8) and neglecting terms of the order $(\log s)^{-1}$:

$$f^{(1)}(P_c) \approx g^2 N(p_c) \frac{M_x^2}{s} \approx g^2 N(p_c) \left\{ 1 - \frac{\mu_c}{m_a} \exp(-\xi_{ca}) - \frac{\mu_c}{m_b} \exp(-\xi_{bc}) \right\} \cdot \tag{3-24}$$

We distinguish the following regions of tbe rapidity plot:

i) **Central Region** ($\xi_{bc} \gg 1$, $\xi_{ca} \gg 1$). *It is also called "pionization"* region and it wrresponds in the C. M. frame to $x_c \approx 0$. The inclusive density depends only on the transverse momentum

$$f^{(1)}(P_c) \approx g^2 N(p_c); \tag{3-25}$$

ii) Target Fragmentation ($\xi_{bc} \gg 1$, ξ_{ca} small). It corresponds to a longitudinal rapidity **i;,** closed to **i;,,** i.e., to small values of ξ_c in the laboratory frame ($\xi_a = 0$). We see that one gets a finite limit in the laboratory frame for $s \to \infty$, i.e.,

$$f^{(1)}(P_c) \approx g^2 N(p_c) \left(1 - \frac{\mu_c}{m_a} e^{-\xi_c} \right); \tag{3-26}$$

iii) Projectile Fragmentation ($\xi_{ca} \gg 1$, ξ_{bc} small). This region corresponds to small ξ_c values in the rest frame of b ($\xi_b = 0$). One gets, for $s \to \infty$, the limit:

$$f^{(1)}(P_c) \approx g^2 N(p_c) \left\{ 1 - \frac{\mu_c}{m_b} e^{\xi_c} \right). \tag{3-27}$$

We note that the asymptotic expressions of the inclusive distributiona exhibit the scaling properties, provided that terms of the order of $(\log s)^{-1}$ are neglected. According to what was said in Sec 2.4, the inclusion of such terms would correspond to iniinite correlation lengths.
From the explicit form of the inclusive density (3-25), it is easy to see that the quantities h and $1/\Lambda^2$, defined by (3-17) and (3-18). can be interpreted

as average multiplicity per unity of rapidity, and average square transverse momentum, respectively, in the central region.

Similar analysis can be carried out for the two-particle inclusive densities. With the same approximation (neglecting terms $\sim (\log s)^{-1}$), one obtains

$$f^{(2)}(P_c, P_d) \approx g^4 N(p_c) N(p_d) (M_X^2/s) \tag{3-28}$$

where the following approximate form of (2-19) is used:

$$M_x^2 \approx s\left\{1 - \mu_c\left[\frac{1}{m_a}\exp(-\xi_{ca}) + \frac{1}{m_b}\exp(-\xi_{bc})\right]\right.$$
$$- \mu_d\left[\frac{1}{m_a}\exp(-\xi_{da}) + \frac{1}{m_b}\exp(-\xi_{bd})\right]$$
$$\left. + \frac{\mu_c \mu_d}{m_a m_b}[\exp(\xi_{cd}) + \exp(-\xi_{cd})]\exp(-\xi_{ba})\right\}. \tag{3-29}$$

Let us consider particular regions for ξ_c, ξ_d :

i) Central Region: both particles c and d are in the central region, so that all relative rapidities appearing in (3-29) are large and one can write:

$$f^{(2)}(P_c, P_d) \approx g^4 N(p_c) N(p_d); \tag{3-30}$$

ii) **Fragmentation** Region: particles c and d are fragments of particles a and b, respectively. One obtains in this case:

$$f^{(2)}(P_c, P_d) \approx g^4 N(p_c) N(p_d)\left(1 - \frac{\mu_c}{m_a} e^{-\xi_{ca}}\right)\left(1 - \frac{\mu_d}{m_b} e^{-\xi_{bd}}\right) \approx f^{(1)}(P_c) f^{(1)}(P_d). \tag{3-31}$$

We obtain, also in this case, *factorization*, so that the correlation function $\rho^{(2)}(P_c, P_d)$, defined by (2-36), vanishes;

iii) Double **Fragmentation Region.** We take both particles c and d in the rapidity range closed to the same particle, say, *a*. We have then (both ξ_{bc} and ξ_{bd} being large):

$$f^{(2)}(P_1, P_2) \approx g^4 N(p_c) N(p_d)\left(1 - \frac{\mu_c}{m_a} e^{-\xi_{ca}} - \frac{\mu_c}{m_a} e^{-\xi_{da}}\right). \tag{3-32}$$

In this case there is no longer factorization, since a kinematical correlation is obviously present.

We would like to remark the following fact: the inclusive densities obtained in this section, since derived by the direct approach from the matrix element M_n, are expected to satisfy automatically the energy-momentum sum rule. One can check, however, that in order to saturate the sum rule (2-50), one need to include in (3-28) non-leading contribution coming from the exponential present in (3-20). This seems to be a quite general feature, occurring in other models: since the sum rules require the knowledge of the inclusive densities in all the phase space, non-leading *contributions* are usually needed for their saturation.

Before closing this section, we would Iike to mention a limiting case of (3-1) which shows the importance of the cut-off function $N(p_i)$. Let us suppose to eliminate the p_i-dependence in (3-1), so that $|M_n|^2 \sim g^{2n}$, and Φ_n reduces simply to the n-particle phase-space. The evaluation of Φ_n can be performed by means of a Laplace transform[28]; summing over n, one obtains[29]

$$\Phi(P_a + P_b) \approx \beta\, g^{4/3}\, s^{-4/3} \exp(\alpha g^{2/3}\, s^{1/3}), \tag{3-33}$$

where a, β are two positive constants. The average multiplicity $\langle n \rangle$ obtained from (3-33) grows with s as $s^{1/3}$; the upper bound for (n), in the case of identical particles, is $\sim s^{1/2}$, and it would correspond to creation of particles at rest.

3.2 Multiperipheral Models

The inclusive reactions are examined in this section in the framework of multiperipherism. The simplest version of multiperipheral model was first formulated a decade ago[4] and its properties of scaling and logarithmic increase of the average multiplicity were put fonvard at the same time. After this pioneering work, multiperipheral dynamics has been extensively exploited, and more elaborated versions, among which we mention the multi-Regge model by Chew and Pignotti[5], have been proposed.

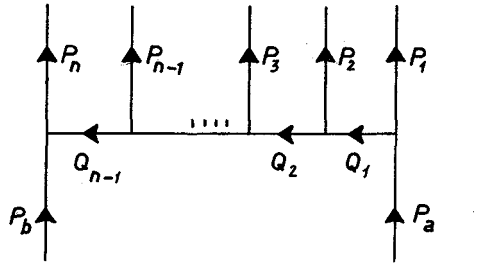


Fig. 6 - Multiperipheral graph for production amplitude.

This class of models is characterized by general properties which will be discussed in the following. For the sake of simplicity, they will be formulated here in terms of the simplest version, in which the n-particle production process is represented by the "multiperipheral graph" of Fig. 6. With reference to this graph, one usually defines the two sets of variables: the momentum *transfers*

$$u_i = Q_i^2, \quad Q_i = P_a - P_1 - \dots P_i, \quad (i = 1, \dots, n-1), \qquad (3\text{-}34)$$

and the sub-energies

$$s_{ij} = (P_i + P_{i+j} + \dots P_j)^2, \quad (i < j;\ i = 1, \dots n-1;\ j = 2, \dots n). \qquad (3\text{-}35)$$

The main properties can be expressed as follows:

i) **Q-Factorization** - The matrix element M_n is supposed to be factorizable according to

$$M_n(P_1, \dots, P_n) = cg(u_1)\, g(u_2) \dots g(u_{n-1}). \qquad (3\text{-}36)$$

This property, which can be expressed in a much more general form, is an essential ingredient of the models; it allows to build the M_{n+1} matrix element by multiplying M, by a simple factor, i.e., by adding a link to the chain represented in Fig. 6. Since each factor describes the dynamics locally in the multiperipheral graph, this property implies a finite (possibly small) correlation length.

ii) Momentum Transfer **Cut-off.** The function $g(u_i)$ is assurned to decrease rapidly with increasing $|u_i|$. This damping in the momentum transfer dependente is necessary in order to limit the transverse momenta p,.

iii) Strong Ordering. The particies are ordered in the multiperipheral chain according to increasing longitudinal momenta or, equivalently, longitudinal rapidities. (According to our convention, the rapidities are increasing from right to left in Fig. 6). The matrix element *(3-36)* is assumed to be small outside the phase space region corresponding to the "right" order indicated. Since we are dealing with identical particles, it will be necessary, in order to obtain the cross sections, to sum over the n! permutations of (P, , . . . , P_n); for each of these terms, the matrix element will be dominant only in a small region of the phase space.

Next we examine, in the framework of the simple multiperipheral model outlined above, the forward scattering amplitude and the inclusive distribution functions.

A. Forward Scattering Amplitude

We are interested here in the absorptive part $A(P_a, -Q)$ of forward (off-shell) scattering amplitude. In the multiperipheral model, it can be obtained in terms of the sum

$$A(P_a, -Q) = \sum_{n \geq 1} A_n(P_a, -Q), \tag{3-37}$$

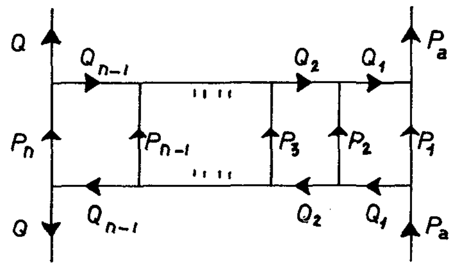


Fig. 7 - Ladder graph for forward elastic amplitude.

where A_n is the contribution of the n-ladder graph of Fig. 7. In terms of the matrix elements (3-36), corresponding to the production of n particles, one can write:

$$A_n(P_a, -Q) = \int \delta^4(P_a - Q - \sum_{i=1}^{n} P_i) \, |M_n(P_1, \ldots, P_n)|^2 \, dP_1 \ldots dP_n. \tag{3-38}$$

Defining, for the sake of simplicity,

$$|M_n(P_1, \ldots, P_n)|^2 = (2\pi)^{4-3n} g^{2n} \Delta(u_1) \ldots \Delta(u_{n-1}), \tag{3-39}$$

the following recursion relation can easily be verified:

$$A_{n+1}(P_a, -Q) = \frac{g^2}{(2\pi)^3} \int \delta[(Q' - Q)^2 - m^2] \, \theta(Q'^0 - Q^0) \, \Delta(Q'^2) \, A_n(P_a, -Q') \, d^4Q'. \tag{3-40}$$

In the original version[4], the functions $\Delta(u_i)$ in (3-39) are given explicitly by

$$\Delta(u_i) = \frac{1}{(u_i - m^2)^2}. \tag{3-41}$$

However, it is not necessary to specify the dependence of $\Delta(u_i)$, and Eq. (3-40) can be adapted to various specific cases.

Summation over n gives the well-known integral equation

$$A(P_a, -Q) = \pi g^2\, \delta[(P_a - Q)^2 - \mathrm{m}^2]\, \theta(P_a^0 - Q^0) +$$
$$+ \frac{g^2}{(2\pi)^3} \int \delta[(Q' - Q)^2 - m^2]\, \theta(Q'^0 - Q^0)\, \Delta(Q'^2)\, A(P_a, -Q')\, d^4Q', \qquad (3\text{-}42)$$

which is discussed extensively in many review papers[30], and is described symbolically in Fig. 8.

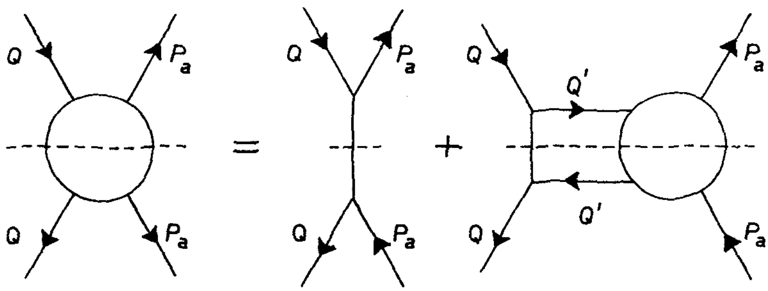


Fig 8 - Graphical representation of the multiperipheral integral equation.

We need now to recall some general properties of the solutions of Eq. (3-42). This equation can be partially diagonalized performing a harmonic analysis of the absorptive amplitude. Since we are dealing with a forward amplitude, the appropriate little group is the Lorentz group O(3,1) itself. Choosing a generic frame of reference, we can write

$$A(P_a, -Q) = -i\sqrt{4\pi} \int_{r-im}^{c+i\infty} a(\lambda, \mathrm{u})\, B_{00}^{\lambda}(L(\alpha)q)\, d\lambda$$
$$= -i\sqrt{4\pi} \int_{c-i\infty}^{c+i\infty} a(\lambda, u) \sum_{jm} \mathscr{D}_{00\,jm}^{0\lambda}(\alpha)\, B_{jm}^{\lambda}(q)\, d\lambda, \qquad (3\text{-}43)$$

where q is a unit four-vector defined in the rest frame of particle a by

$$q = -\frac{Q}{\sqrt{-u}} = (\sinh \mathbf{o},\ q\ \cosh\sigma). \qquad (3\text{-}44)$$

We indicate by $\mathscr{D}_{jmj'm'}^{0\lambda}(\alpha)$ the unitary irreducible representations of the Lorentz group, and by $B_{jm}^{\lambda}(q)$ the relative bases; α is the element of the group corresponding to the Lorentz transformation which relates the chosen frame to the rest frame of particle a. For more details we refer to specific papers[27,31].

In the rest frame of a, Eq. (3-43) reduces essentially to a Laplace transform:

$$A(P_a, -Q) \equiv A(\sigma, u) = -i\sqrt{4\pi}\int_{c-i\infty}^{c+i\infty} a(\lambda, u)\, B^{\lambda}_{00}(q)\, d\lambda =$$

$$= -i\int_{c-i\infty}^{c+i\infty} a(\lambda, u)\, \frac{\exp(\lambda\sigma)}{\sqrt{\pi}\cosh\sigma}\, d\lambda. \tag{3-45}$$

It can be shown, in the frame of the present model, that the asymptotic behaviour of $A(\sigma, u)$ for large o is dominated by the leading singularities of the "partial wave amplitude" $a(\lambda, u)$ in the I-complex plane, which are isolated poles with factorized residues[32]. They are called Toller or Lorentz poles[33]; a single Toller pole is equivalent to an infinite sequence of Regge poles equally spaced[34].

Starting from Eq. (3-45), one can obtain the absorptive part of the on-shell forward amplitude. With a re-definition of a "partial wave amplitude" $a(\lambda)$, we can write for large s:

$$A(P_a, P_b) \equiv A(\xi) \approx -i\int_{c-i\infty}^{c+i\infty} a(\lambda)\, \frac{\exp(\lambda\xi)}{\sqrt{\pi}\sinh\xi}\, d\lambda, \tag{3-46}$$

where ξ coincides, in the laboratory frame, with the relative rapidity ξ_{ba}, related to s by (2-7).

Supposing that the leading singularity of $a(\lambda)$ is an isolated pole at I = a + 1, one can obtain from (3-46) the asymptotic expression for the total cross section $\sigma_T(a + b \rightarrow$ everything):

$$\sigma_T = \frac{1}{\gamma(s)} A(\xi_{ba}) \approx \frac{1}{s}\gamma_a\gamma_b \exp(a\xi_{ba}) \approx \gamma_a\gamma_b\, s^{a-1} \tag{3-47}$$

The cross section tends to a constant if a = 1: the Toller pole then corresponds to the Pomeranchuk singularity.

B. Inclusive Distributions

In this section we shall indicate how one can obtain, from the multiperipheral model, the explicit form of the ipclusive distributions.

We consider first the case of single-inclusive reactions. One can obtain $F^{(1)}_n(P_a, P_b, P_c)$, with n fixed, from the matrix element (3-39) corresponding to Fig. 6, summing over all final momenta except P_c. If we re-label the momenta as in Fig. 9, we can write (with $u'_j = Q'^2$):

$$F_n^{(1)}(P_a, P_b, P_c) = (2\pi)^{4-3n} g^{2n} \frac{1}{\gamma(s)} \sum_{r=0}^{n-1} \int \prod_{i=1}^{r} \{\Delta(u_i)\, dP_i\} \prod_{j=1}^{s=n-r-1} \{\Delta(u'_j)\, dP'_j\} \cdot$$
$$\cdot\, \delta^4(P_a - Q_r - \sum_i P_i)\, \delta^4(P_b - Q'_s - \sum_j P')\, \delta^4(Q_r + Q'_s - P_c)\, d^4Q_r\, d^4Q'_s, \qquad (3\text{-}48)$$

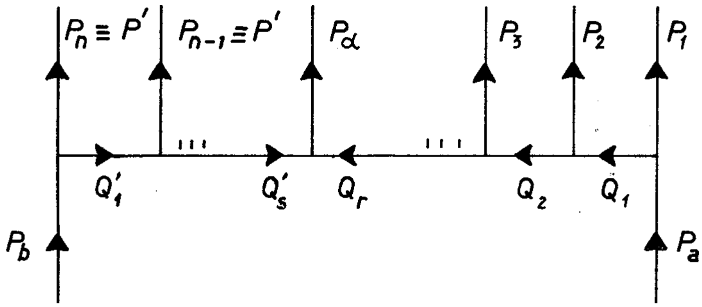


Fig. 9 - Multiperipheral graph for one-particle inclusive distributions.

where the sum over r takes into account that the observed particle can be emitted at any point of the multiperipheral chain.

Finally, the one particle inclusive distribution function is obtained, by summing over n, in the form:

$$F^{(1)}(P_a, P_b, P_c) =$$
$$= \frac{1}{\gamma(s)} \frac{g^2}{(2\pi)^7} \int A(P_a, -Q)\, \Delta(Q^2)\, A(P_b, Q - P_c)\, \Delta[(Q-P_c)^2]\, d^4Q +$$
$$+ \frac{1}{\gamma(s)} \frac{g^2}{(2\pi)^3} A(P_b, P_a - P_c)\, \Delta[(P_a - P_c)^2] +$$
$$+ \frac{1}{\gamma(s)} \frac{g^2}{(2\pi)^3} A(P_a, P_b - P_c)\, \Delta[(P_b - P_c)^2]. \qquad (3\text{-}49)$$

The three terms in which $F^{(1)}$ is decomposed in Eq. (3-49) are the contributions represented by the three graphs of Fig. 10. One can easily convince oneself that the second and the third term give contribution to the inclusive cross section only in the fragmentation region of particles a and b, respectively.

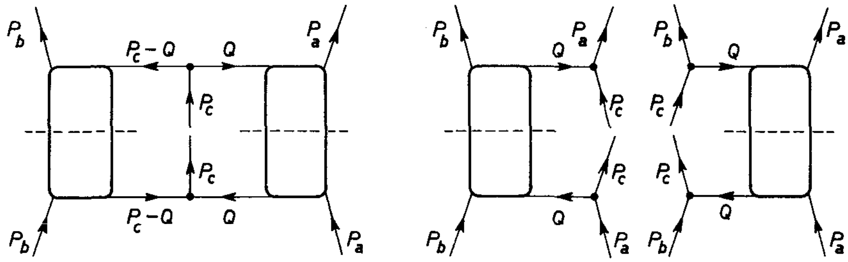


Fig. 10 - Multiperipheral amplitudes contributing to one-particle inclusive distributions in the central and fragmentation regions.

We are interested here mainly in the central region of the rapidity plot defined by $\xi_{ca} \gg 1, \xi_{bc} \gg 1$, so that we shall keep only the first term in Eq. (3-49).

Replacing the two absorptive amplitudes appearing in this term by harrnonic expansions of the type given in (3-43), one obtain the following expression for $F^{(1)}$ valid in the central region:

$$F^{(1)}(P_a, P_b, P_c) =$$

$$-\frac{1}{\gamma(s)}\frac{g^2}{(2\pi)^7}\int_{c-i\infty}^{c+i\infty} d\lambda \int_{c-i\infty}^{c+i\infty} d\lambda' \sum_{jm} \mathscr{D}^{0\lambda}_{00jm}(\alpha_{ca})\, F^j(\lambda, \text{A})\, \mathscr{D}^{0\lambda'}_{jm00}(\alpha_{bc}). \tag{3-50}$$

For convenience we have chosen the rest frame of particle c, so that α_{ca}, α_{bc} are the elements of the Lorentz group which relate the rest frames of particles a and b, respectively, to the rest frame of particle c. The explicit expressions of α_{ca}, α_{bc} are given in the quoted paper[27]. The function $F_j(1, \text{X})$ is defined by:

$$F_j(\lambda, \text{X}) = -\frac{\pi}{\text{m}^2}\iint \Delta(u)\,\Delta(u')\,a(\lambda, \text{u})\,a(\lambda', \text{u}')\,b_j^\lambda(\sigma)\,b_j^{\lambda'}(\sigma')\,\cdot$$

$$\cdot\,[T(u, u', m^2)]^{1/2}\,du\,\text{du}', \tag{3-51}$$

where $a(\lambda, \text{u})$, a (λ', u') are the two transformed absorptive amplitudes and the b_j^λ are related to the basis functions B^λ_{jm} appearing in Eq. (3-43).

Extracting from Eq. (3-51) the leading singularities in A, λ', which consist in two poles with factorized residues at A = a + 1, X = a + 1, with a $\simeq$ 1, we get

$$F^{(1)}(P_a, P_b, P_c) \approx \frac{1}{s}\sum_{jm} \gamma_a\, \mathscr{D}^{0,a+1}_{00jm}(\alpha_{ca})\,\gamma_{c,j}\,\mathscr{D}^{0,a+1}_{jm00}(\alpha_{bc})\,\gamma_b. \tag{3-52}$$

In the central region, both relative rapidities ξ_{ca}, ξ_{bc} are large, so that one can use the asymptotic expressions for the $\mathscr{D}$-functions given in the quoted paper[27] and Eq. (3-52) reduces to:

$$F^{(1)}(P_c) \approx \frac{1}{s} \gamma_a \gamma_b \exp(a\xi_{ba}) N_c(p_c). \tag{3-53}$$

From the asymptotic expression (3-47) of the total cross section, one gets then for the inclusive density:

$$f^{(1)}(P_c) \approx N_c(p_c), \tag{3-54}$$

where N_c is a universal function depending only on the nature of the particle c, and not on particles a and b. The above formula coincides with Eq. (3-25) of the previous section; in the present case, however, it is an output of the multiperipheral hypotheses.

It is interesting to remark that the same expression (3-54)could be obtained following the Mueller's approach. One would start directly from a 3-body forward elastic amplitude, and analyze it in terms of the irreducible representations of the Lorentz group. In the central region, one should perform a double harmonic expansion and, making the hypothesis that the asymptotic behaviour is dominated by Toller pole exchange, one would obtain Eq. (3-52). This expression for $F^{(1)}$ can then be interpreted directly as the absorptive part of the amplitude described in Fig. 11.

For the asymptotic expression of $F^{(1)}$ in the fragmentation regions, it is sufficient to start from a single harmonic analysis; the leading terms correspond to a single Toller pole exchange, as represented in Fig. 11. The inclusive distribution shows the limiting fragmentation behaviour.

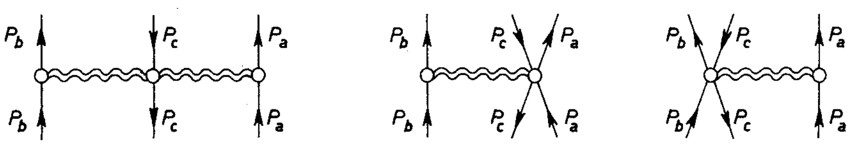


Fig. 11 - Toller-pole amplitudes contributing to one-particle inclusive distributions in the **central and fragmentation regions.**

If we note that keeping only the leading contribution from the Toller pole is equivalent to keep the dominant Regge pole in the corresponding Regge trajectory sequence, the above results reduce, in this approximation, to those obtained first by Mueller[17]. Some of these results will be discussed, with more details, in a later section.

We shall limit ourselves here to the two-particle inclusive distributions in the central (pionization) region, in which the general analysis in terms of Toller pole exchanges can give new interesting information. Following the Mueller's approach, the relevant inclusive distribution is obtaineú from the 8-point function represented in Fig. 12; we assume that all the relative rapidities ξ_{bd}, ξ_{dc}, ξ_{ca} are large enough to make a three Toller pole expansion meaningful.

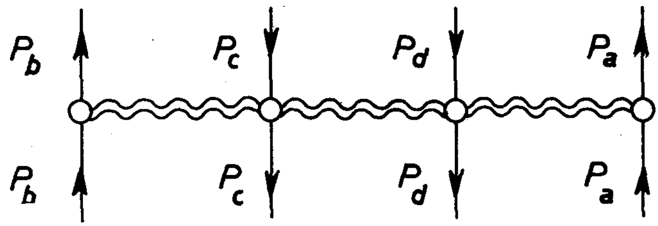


Fig. 12 - Three Toller-pole amplitude contributing to two-particle inclusive distributions in the central region.

The inclusive distribution function can then be written as follows:

$$F^{(2)}(P_c, P_d) \approx \frac{1}{s} \sum_{M} \sum_{jmj'm'} \gamma_a \mathscr{D}^{0,a+1}_{00jm}(\alpha_{ca}) \gamma^M_{c,j} \mathscr{D}^{M,a'_M+1}_{jmj'm'}(\alpha_{dc}) \gamma^M_{d,j} \mathscr{D}^{0,a+1}_{j'm'00}(\alpha_{bd}) \gamma_b \tag{3-55}$$

We note that the M-quantum number of the Toller pole exchanged between c and d could be, in general, different from zero. If we keep only the leading Toller pole corresponding to the vacuum quantum number ($a = a' \simeq 1$), we obtain the asymptotic expression

$$F^{(2)}(P_c, P_d) \approx \frac{1}{s} \gamma_a \gamma_b N_c(p_c) N_d(p_d) \exp(a\xi_{ba}) \tag{3-56}$$

and the inclusive density reduces simply to

$$f^{(2)}(P_c, P_d) \approx N_c(p_c) N_d(p_d). \tag{3-57}$$

However, if c and d are still in the central region, but ξ_{dc} is not very large (we take $\xi_{dc} > 0$), we have to take into account correction terms. The simple expression (3-57) is then replaced by

$$f^{(2)}(P_c, P_d) \approx N_c(p_c) N_d(p_d) + H_{cd}(p_c, p_d, \xi_{dc}, \varphi_{dc}). \tag{3-58}$$

The function H_{cd} represents such correction and corresponds to the correlation function $\rho^{(2)}$ (in the central region), defineú by (2-36). We remark that it depends explicity on the relative azimuthal angle φ_{dc} between the transverse momenta of the two observed particle. We know that a φ_{dc} dependence of the inclusive density is needed in order to saturate the transverse momentum sum rule (2-50), and we have already pointed out that for such saturation non-leading terms are necessary. The function H_{cd} receives contributions from different terms. If one keeps the other terms in Eq. (3-55) corresponding to $M \neq 0$, one gets[27,35]:

$$H_{cd} \approx \sum_M b_c^M(p_c)\, b_d^M(p_d) \exp\{(a'_M - a)\,\xi_{dc}\} \cos M\varphi_{dc}\,. \tag{3-59}$$

In principle, looking at the energy dependence of the azimuthal correlation function, one could detect the presence of Toller poles with $M \neq 0$. However, since such possible poles are believed to have a rather low intercept, such effect would be masked by contributions coming from the non-leading terms of the vacuum pole $a \approx 1$. These terms are in fact estimated to be the main source of the azimuthal correlation in Eq. (3-58)[36]. Correlations originated by more complicated singularities are expected to be small in the central region and could be discriminated, since they should decrease much slower than the other contributions with increasing ξ_{dc}.

C. Average Multiplicity

As already remarked, a general feature of multiperipherism is the prediction that the average multiplicity $\langle n\rangle$ increases with $\log s$. This is a simple consequence of the fact the leading part of the inclusive density $f^{(1)}$ is independent of the longitudinal rapidity in the central region, and this region increases linearly with the rapidity range $\xi_{ba} \approx \log s$.

The $\log s$ dependence of $\langle n\rangle$ is, in fact, a crucial test for all kind of multiperipheral models. It has been shown by Bassetto *et al.*[37] that not only the models based on matrix elements M_n of the type (3-39), but a much more general class for which a factorization property is applied to a bound of $|M_n|$, exhibit a $\log s$ dependence for the average multiplicity.

To be more specific, the condition imposed on $|M_n|$ can be formulated in the following way:

$$|M_n(P_1,\ldots,P_n)|^2 \leq \sum_\pi [f_n(\mathscr{P}_\pi x)]^2, \tag{3-60}$$

where x indicates a set of $(3n-4)$ kinematical invariants built from $P_1,\ldots,P_n$, and $\mathscr{P}_\pi x$ the set of invariants obtained from x by means of the permutation π of the final particles. The function f_n is assumed to be

Q-factorized. For general purposes, it is sufficient to assume for it a simple factorization of the multi-Regge model type:

$$f_n(x) = c \prod_{i=0}^{n} \left[g(u_i) \left(\frac{s_{i,i+1}}{4m^2} \right)^{\alpha} \right], \tag{3-61}$$

where u_i, $s_{i,i+1}$ are the momentum transfers and subenergies defined in (3-34), *(3-35)*; $g(u_i)$ is a suitably decreasing function of $|u_i|$ and the exponent α is of the order of 1.

Adding to the conditions (3-60), (3-61), the incontrovertible assumption 'that the total cross section does not decrease faster than any negative power of s, Bassetto *et al.*[37] obtained the result that the average multiplicity can increase at most as $\log s$. The importance of this result lies in the fact that an experimental disproof of such prediction would invalidate all kinds of models based on multiperipheral ideas. On the other hand, one has to remember that this prediction has an asymptotic character, since the bound in Eq. *(3-61)* contains an arbitrary numerical factor.

Since the recent data from the I. S. R. and Batavia experiments[14] show that scaling is well satisfied and indicate the presence of a central plateau, one would think that an asymptotic regime for which $\langle n \rangle$ − logs has already been reached. As shown in Fig. 4, the data on multiplicity can be fitted, at least above *80* GeV/c, with a $\log s$ dependence.

Finally, we want to remark that the bound *(3-60)* gives also restrictions on the moments of the multiplicity distribution[37]:

$$\langle n^p \rangle = \sum_n n^p \frac{\sigma_n}{\sigma_T} \leq k(\log s)^p \tag{3-62}$$

The simplest versions of the multiperipheral model give a Poisson distribution for the multiplicity (i.e., for the "topological" cross sections σ_n):

$$\frac{\sigma_n}{\sigma_T} \sim e^{-\langle n \rangle} \frac{\langle n \rangle^n}{n!}, \tag{3-63}$$

so that the integrated correlations vanish.

For all kinds of generalized multiperipheral models, the topological cross sections σ_n have to satisfy the contraints *(3-62).*

A different prediction is provided by the diffractive models; since they give the behaviour

$$\frac{\sigma_n}{\sigma_T} \sim \frac{1}{n^2}, \tag{3-64}$$

one gets for the moments:

$$\langle n^p \rangle \sim s^{\frac{p-1}{2}} \quad (p \geq 2). \tag{3-65}$$

Experimentally[59], the multiplicity distribution is broader than Poisson's, but in disagreement with the behaviour (3-64).

3.3 The Mueller-Regge Model

In the previous section we have shown how the asymptotic expressions of the inclusive distributions can be obtained, in terms of Toller or Regge poles, starting from the multiperipheral models. We give here a more complete survey of the single-particle inclusive distributions, obtained by Mueller's approach from the Regge limits of the forward elastic 3-body amplitude. In the original paper[17], this amplitude was expanded into irreducible representations of 0(2,1) and analyzed in terms of the leading (Pomeranchuk) trajectory. In the following, we start directly from the asymptotic expressions of the amplitude relative to the process (2-55), and take into account also non-leading trajectories; this will allow us to discuss briefly the rate of approach to scaling.

As done in Sec. 3.1, we distinguish different regions in the rapidity plot.

i) **Fragmentation Region.** We consider the fragmentation region of the target a, defined by small ξ_{ca} and large ξ_{bc}, i.e., ξ_{ca} fixed and $\xi_{bc} \sim \log s$. In terms of the Mandelstam vanables, this corresponds to the limit $s \to \infty$, $t \to -\infty$, with both u and $(M_X^2/s) < 1$ fixed. In the laboratory frame (rest frame of a), the limit corresponds to keeping $\mathbf{P}_c$ fixed, so that we can write for the inclusive distribution function:

$$F^{(1)}(s, \xi_c, p_c) \approx \sum_i g_i(\xi_c, p_c)\, s^{\alpha_i(0)-1} \tag{3-66}$$

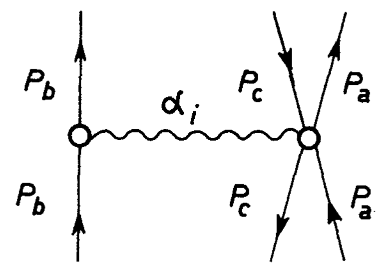


Fig. 13 - One-Regge amplitude contributing to single-particle inclusive distributions in the target fragmentation region.

Each term is a contribution of a single Regge-pole, as represented in Fig. 13, and the sum is extended to the different Regge poles which can be exchanged. If we keep only the leading (Pomeranchuk) trajectory with intercept $\alpha(0) = 1$, and the dominant meson Regge trajectory for which $\alpha_M(0) \approx 1/2$, Eq. (3-66) reduces to

$$F^{(1)}(s, \xi_c, p_c) \approx g_P(\xi_c, p_c) + g_M(\xi_c, p_c)\, s^{-1/2} \tag{3-67}$$

We see that the Pomeranchuk contribution is independent of s, so that for $s \to \infty$ one obtains the limiting fragmentation. The rate of approach to this limit depends on the nature of the particles involved; scaling is expected to occur at lower energy for those processes in which the contribution from meson Regge trajectory is negligible, so that the second term in Eq. (3-67) disappears.

If we include the hypothesis of factorization of the Regge pole residues, i.e.,

$$g_i(\xi_c, p_c) = \gamma_b^i\, \gamma_{ac}^i(\xi_c, p_c), \tag{3-68}$$

we obtain for the inclusive density (using Eq. (3-47) for the total cross section):

$$f^{(1)}(\xi_c, p_c) \approx \frac{\gamma_{ac}^P(\xi_c, p_c)}{\gamma_a} + \frac{\gamma_b^M\, \gamma_{ac}^M(\xi_c, p_c)}{\gamma_a\, \gamma_b}\, s^{-1/2} \tag{3-69}$$

In the limit $s \to \infty$, the inclusive density becomes independent of the nature of the projectile.

Similar considerations hold in the region of projectile fragmentation.

ii) Central Region. This region corresponds to both relative rapidities ξ_{bc}, ξ_{ca} large, and, in terms of the Mandelstam variables, to the limit $s \to \infty$, $t \to -\infty, u \to -\infty, (M_X^2/s) \to 1$. Making use of Fenman's scaling variable x_c, this region can be defined by $|x_c| < s^k$, $-\frac{1}{2} \leq k < 0$, and one can write

$$t \approx \frac{1}{2} s(x_c - \bar{x}_c), \quad u \approx -\frac{1}{2} s(x_c + \bar{x}_c), \quad \bar{x}_c = \left(x_c^2 + \frac{4\mu_c^2}{s}\right)^{1/2} \tag{3-70}$$

One gets, immediately,

$$\frac{ut}{s} \approx \mu_c^2. \tag{3-71}$$

In this case the inclusive distribution function is obtained from a two-Regge limit amplitude, and one gets:

$$F^{(1)}(s, x_c, p_c) \approx \frac{1}{s} \sum_{ij} t^{\alpha_i(0)}\, u^{\alpha_j(0)}\, g_{ij}(p_c). \tag{3-72}$$

A generic term in the sum is represented in Fig. 14. Keeping also here only the Pomeranchuk term with $\alpha(0) = 1$, and the dominant meson contribution relative to $\alpha_M(0) \approx 1/2$, Eq. (3-72) becomes:

$$F^{(1)}(s, x_c, p_c) \approx \mu_c^2\, g_P(p_c) + \mu_c^2\, g_{PM}(p_c)\,(t^{-1/2} + u^{-1/2}). \qquad (3\text{-}73)$$

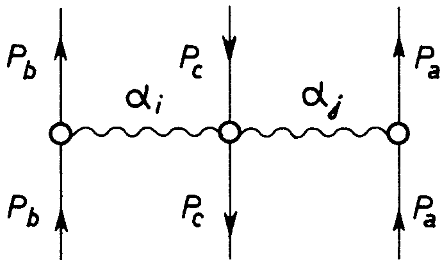


Fig. 14 - Two-Regge amplitude contributing to single particle inclusive distributions in the central region.

We see that Feynman's scaling law is satisfied by the Pomeranchuk contribution; in general, scaling is approached slower in this region, since the second term in the r.h.s. of Eq. (3-73) goes like $s^{-1/4}$, as can be immediately checked by (3-70)

The hypothesis of factorization gives, in the present case,

$$f^{(1)}(s, x_c, p_c) \approx \mu_c^2\, \gamma_c(p_c) + 0(s^{-1/4}), \qquad (3\text{-}74)$$

so that the inclusive density, neglecting terms $\sim s^{-1/4}$, is independent of the nature of both particles a and b.

iii) Triple-Regge Region - This region corresponds to both M_X^2 and s/M_X^2 large with u (or t) fixed. For fixed u, this is the phase space boundary of the target fragmentation region (for fixed t, of the projectile fragmentation region). In fact, in the fragmentation regions, one can write:

$$M_X^2 \approx s(1 - |x_c|), \qquad (3\text{-}75)$$

and a small ratio M_X^2/s for M_X^2 large is obtained for $|x_c| \approx 1$, i.e., near the boundary. A triple-Regge asymptotic form is required in this case for the six-point amplitude, which is represented in Fig. 15. Making use of the factorization property of the Regge pole residues, the inclusive density can be written as[3]*:

$$f^{(1)}(P_c) \approx \frac{\gamma_{ca}^2(u)}{\gamma_a}\, g_{RR}^P(u) \left(\frac{s}{M_X^2}\right)^{2\alpha_{ca}(u) - \alpha(0)}, \qquad (3\text{-}76)$$

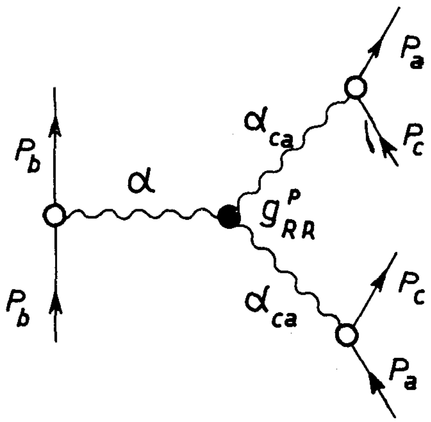


Fig. 15 - Triple Regge limit of a forward elastic 3-body amplitude.

where $\alpha(0)$ is the Pomeranchuk intercept, $\alpha_{ca}(u)$ the leading Regge trajectory coupled to $\bar{c}a$, $\gamma_{ca}(u)$ the corresponding residue function, and $g^P_{RR}(u)$ represents the coupling among three Regge trajectories. In particular, with appropriate choice of the particle c, $\alpha_{ca} \equiv \alpha$ and $g^P{}_{RR}$ coincides with the three-Pomeranchuk coupling g^{μ}_{PP}.

Much interest has been devoted to these couplings and to the inclusive density (3-76), which could provide information about them. The importance of the determination of g^P_{RR} and &, lies on the following facts. If the Pomeranchuk trajectory satisfíes $\alpha(0) = 1$, then $g^P_{PP}(0) = 0$, othenvise unitarity is violated. This can be seen by considering the double diffraction dissociation, which is represented in Fig. 16. It can be shown[39] that the probability for this process increases without bound with s (like log log s). Moreover, a multiple Pomeranchuk exchange in the multiperipheral model would lead to a total cross-section violating the Froissart bound[40]: this would require the vanishing of the two-Pomeranchuk coupling: $g^R_{PP} = 0$.

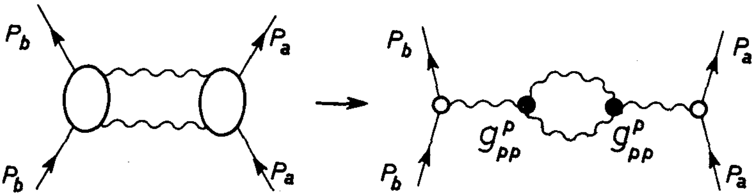


Fig. 16 - Double diffraction dissociation.

On the other hand, a non-vanishing g^{R}_{PP} is expected by factorization and duality[41]: $g^{R}_{PP}\, g^{R}_{RR} \sim (g^{P}_{RR})^2$. In connection with these problems, finite mass sum rules[42] have been written for inclusive distributions, which relate triple Regge couplings (which appear in (3-76) at high M_X^2) to integrals over low missing masses.

The above analysis of single inclusive reactions can be easily extended to higher order inclusive distributions. Here we would like to mention the correlation which appears in the two particle inclusive spectra, when non-leading Regge trajectones are taken into account. We consider the case in which the two observed particles c and d are in the central region and the relative rapidity $\xi_{cd} > 0$ is large. The eight-point amplitude is then dominated by three Regge exchange terms, and is represented in Fig. 17. analogous to Fig. 12 in which Toller poles were considered. The

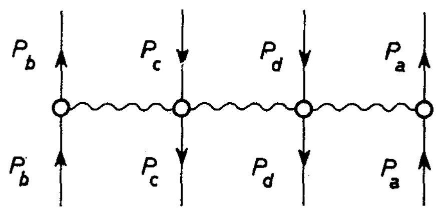


Fig. 17 - Three-Regge limit of eight-point amplitude.

inclusive density, assuming $\alpha(0) = 1$ and taking into account a meson trajectory with $\alpha_M(0) \approx 1/2$, can be written as[43]:

$$f^{(2)}(P_c, P_d) \approx \gamma_c\, \gamma_d + \frac{g(p_c, p_d)}{\gamma_a\, \gamma_b}\, e^{-(1-\alpha_M(0))\xi_{cd}}. \tag{3-77}$$

One can check that the second term on the r.h.s. of the above equation gives contribution to the correlation function $\rho^{(2)}$. We can then interpret the quantity

$$[1 - \alpha_M(0)]^{-1} \approx 2, \tag{3-78}$$

as the correlation length L defined in Sec. 2.4, since for $\xi_{cd} > L$ the correlations become negligible.

Finally, we want to add a few words on the so-called "two-componenf mode"[60], in which one describes the production mechanism as the superposition of a short-range component (e.g., of multiperipheral type) dominating the pionization region (higher multiplicities), and a diffractive component, mainly affecting the fragmentation regions (low niultiplicities). Neglecting the **interference** between the two components, one can write for the total cross **section** and distribution functions

a, $= \sigma_M +$ a,, (3-79)

$$f^{(n)} = \frac{\sigma_M}{\sigma_T} f_M^{(n)} + \frac{\sigma_D}{\sigma_T} f_D^{(n)}, \tag{3-80}$$

where the subscripts M and D stand for "multiperipheral" and "diffractive". By integration, one gets from (3-80) the average multiplicity

$$\langle n \rangle = \frac{\sigma_M}{\sigma_T} \langle n_M \rangle + \frac{\sigma_D}{\sigma_T} \langle n_D \rangle, \tag{3-81}$$

where $\langle n_D \rangle$ is usually chosen constant and small.

The **two-particle** correlation function (2-36) **is** given by

$$\rho^{(2)}(P_1, P_2) = \frac{\sigma_M}{\sigma_T} \rho_M^{(2)}(P_1, P_2) + \frac{\sigma_D}{\sigma_T} \rho_D^{(2)}(P_1, P_2) + \\ + \frac{\sigma_M \sigma_D}{\sigma_T^2} [f_M^{(1)}(P_1) - f_D^{(1)}(P_1)] [f_M^{(1)}(P_2) - f_D^{(1)}(P_2)]. \tag{3-82}$$

By integration, one obtains the correlation **coefficient** $R^{(2)}$:

$$R^{(2)} = \frac{\sigma_M}{\sigma_T} R_M^{(2)} + \frac{\sigma_D}{\sigma_T} R_D^{(2)} + \frac{\sigma_M \sigma_D}{\sigma_T^2} (\langle n_M \rangle - \langle n_D \rangle)^2. \tag{3-83}$$

Noting that the short-range component gives $R_M^{(2)} \sim$ logs *(as* can be seen by direct evaluation from Eq. (3.77)) and taking $R_D^{(2)} \sim$ ***const,*** one obtains the behaviour $R^{(2)} \sim (\log s)^2$.

3.4 Dual Models

In the previous **sections** we have seen how the multiperipheral and the Mueller-Regge models give a satisfactory description of the general features of the inclusive reactions. We **shall** not discuss the phenomenological application of these **models**[44], nor the attempts to include unitary **corrections** and Regge cuts into the multiperipheral **scheme**[45]. We shall, instead, outline the general **properties** of inclusive **distributions** expected from the dual models, which are, to a **certain extent**, related to the models considered previously.

Dual models present the advantages of incorporating reasonable high and low energy behaviour, and indicating an appealing scheme for the Pomeranchuk singularity.

Before going to inclusive reactions, we discuss briefly the general properties of the dual models.

A. Elastic Two-Body Amplitudes and Total Cross Sections

The simplest form of dual model was proposed by Veneziano for a four-meson amplitude[46]. In the case of four identical scalar mesons, the Veneziano amplitude is given by

$$M(s,t,u) = \beta[V(s,t) + V(s,u) + V(t,u)] \qquad (3\text{-}84)$$

arid

$$V(s,t) \equiv B(-\alpha(s), -\alpha(t)) = \int_0^1 dx\, x^{-1-\alpha(s)}(1-x)^{-1-\alpha(t)}, \qquad (3\text{-}85)$$

where β is a constant; $\alpha(s)$ and $\alpha(t)$ are linear Regge trajectories. The amplitude *(3-84)* corresponds to an infinite sum of poles and exhibits asymptotic Regge behaviour in all channels. The property of *duality* requires that each term in *(3-84)* is invariant under a cyclic or anticyclic permutation of the external momenta. as represented in Fig. 18 for $V(s,t)$. As originally

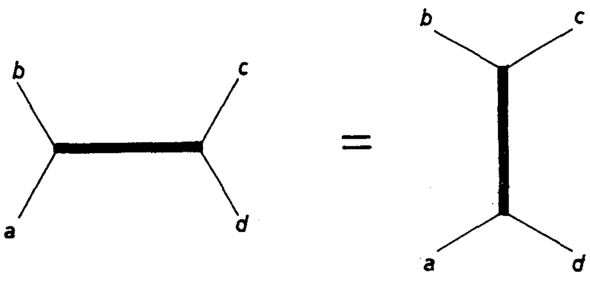


Fig. 18 - Duality property of a four-point amplitude.

expressed in terms of finite energy sum rules, resonances in one channel generate Regge trajectories in the crossed channel. It was conjectured independently by Freund and Harari[47] that, while "normal" trajectories (on which lie physical particles) are built by resonances, the Pomeranchuk trajectory is built by the non-resonant background part of the amplitude.

These properties can be nicely expressed in a pictorial way in terms of the so-called duality diagrams[48], in which a meson line is represented by two quark-antiquark ($q\bar{q}$) lines (see Fig. 19). We introduce in a graphical way, in Fig. 20, the notion of "twisted" propagator. Let us now consider the loop diagram in Fig. 21; in terms of quark-antiquark lines, we see that it

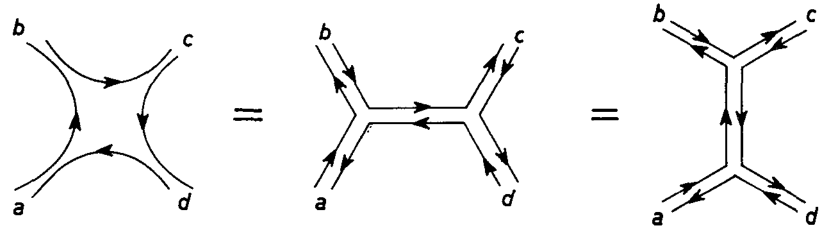


Fig. 19 - Duality diagram for a four-point amplitude

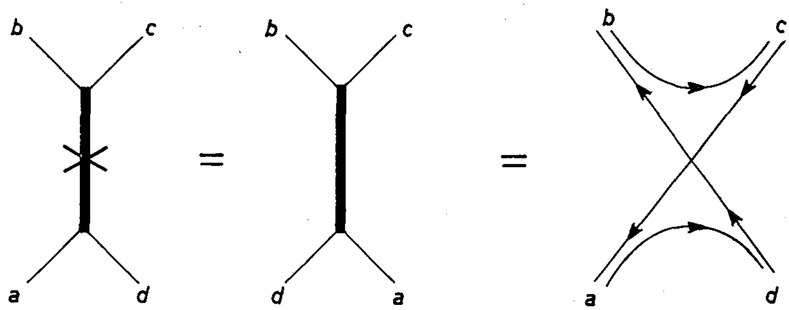


Fig. 20 - Diagram with a twisted propagator.

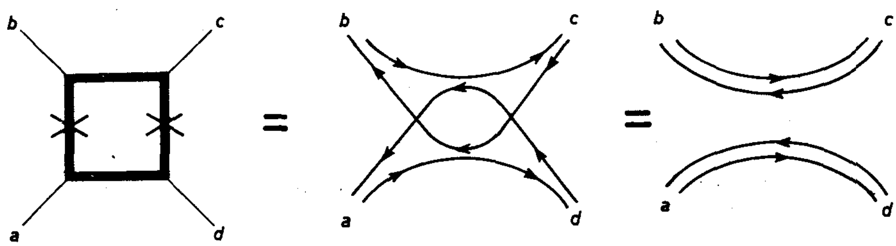


Fig. 21 - Duality diagram for the Pomeranchuk singularity

corresponds to exotic quantum numbers in the s-channel ($qq\bar{q}\bar{q}$) and vacuum quantum numbers in the t-channel. A detailed analysis of this diagram[49] shows that its leading singularity in the angular momentum variable is a logarithmic cut with a fixed intercept $\approx 1/3$ and a slope about 1/2 of that of the input Regge trajectory. It is then very tempting to identify this singularity with the Pomeranchuk and describe diffraction processes in terms of the graph of Fig. 21.

Dual amplitudes have been written for multiparticle processes, in the so-called "tree" approximation[50]. Of course, unitarity is violated by these models, since they contain zero width resonances. For phenomenological purposes, one adds an imaginary part to the Regge trajectories; a general program of unitarization has been proposed, based on the idea that unitarity is ensured by adding higher order loop contributions[51] to the tree approximation. However, even in the tree graph amplitudes a little bit of unitarity appears through factorization. From the fact that the general amplitude factorizes completely and that interference among resonances is excluded, it follows that the optical theorem can be applied to the amplitude (3-84)[52]:

$$\sigma_T(s) \approx \frac{1}{s}(\mathrm{Im}\ M(s, t = 0)). \tag{3-86}$$

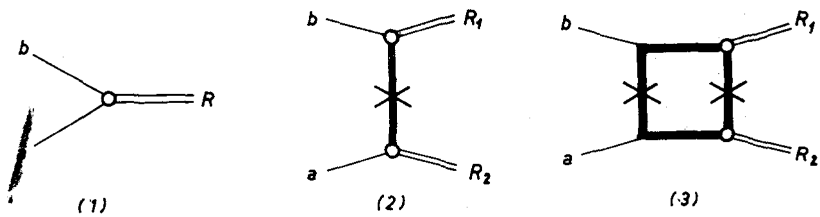
Of course, the above relation does not hold locally, but on the average. in the same sense that the averaged amplitude in the resonance region extrapolates, according to duality, the Regge behaviour. The physical meaning is clearly understood if one thinks of the process a + b → c, . . . c, , which contributes to the total cross section, as accurring through an intermediate s t e ~ ~ ~

$$a + b \to R \to c_1 + \cdots + c_n, \tag{3-87}$$

where R stands for a compound or one-resonance state. The cross section relative to (3-87) will be the produci of the cross section $\sigma(a + b \to R)$ for producing the resonance R times the probability that R decays into the specific final channel. Summing over all final states, the second factor gives one, and the total cross section is simply a sum of $\sigma(a+b\to R)$ over all different resonances R. However, in (3-87), only one resonance R was assumed; the total cross section gets contributions also from those intermediate states which cannot be reduced to a single resonance. It is a consequence of (planar) duality that all possible intermediate resonances collapse either into one or two, so that one has to take into account, besides Eq. (3-87), the process

$$a + b \to R, + R_2 \to (c_1 + \dots c_r) + (c_{r+1} + \dots c_,). \tag{3-88}$$

Besides these terms, one should add the contribution from diffraction dissociation processes. We obtain in this way a three-component model for the production amplitudes, contnbuting to the total cross section,



Fip. 22 - Three-component dual model for the total cross sections.

which are described in Fig. 22. We note that the corresponding extension of the optical theorem (3-86) would require the addition of loop contributions to the elastic two-body amplitude.

B. Inclusive Distributions

The previous considerations about the total cross section would indicate that dual amplitudes are expected to give more reliable results for inclusive rather than exclusive reactions, since in the former the critical dependence due to narrow-width resonances is washed out by summing over all unobserved final states.

We know that inclusive cross sections can be obtained following either the "direct" or the Mueller's approach. Since the dual amplitudes are non-unitary, one expects that the cross sections evaluated from Mueller's approach do not satisfy, in general, the energy-momentum sum rules of Sec. 2.3. Therefore, in principle, the direct approach appears to be preferable. In practice, the inclusive cross sections are computed as discontinuities of dual tree and loop amplitudes, but the direct approach indicates clearly which diagrams are to be included.

Starting from the three-component model for the total cross section, Tye and Veneziano[53] proposed a model for single-particle inclusive distributions, which consists of 13 components: they are indicated in Fig. 23. If one is interested only in the quantum number structure of the inclusive distributions, it is sufficient to take into account the first 9 components of Fig. 23.

In the frame of this model, a detailed analysis of the way in which scaling is approached has been carried out[53]. The rate of approach to scaling depends, in general, on the quantum numbers of the systems ab, $ab\bar{c}$, a?, $b\bar{c}$: the conditions differ for each of the 9 components of the inclusive distri-

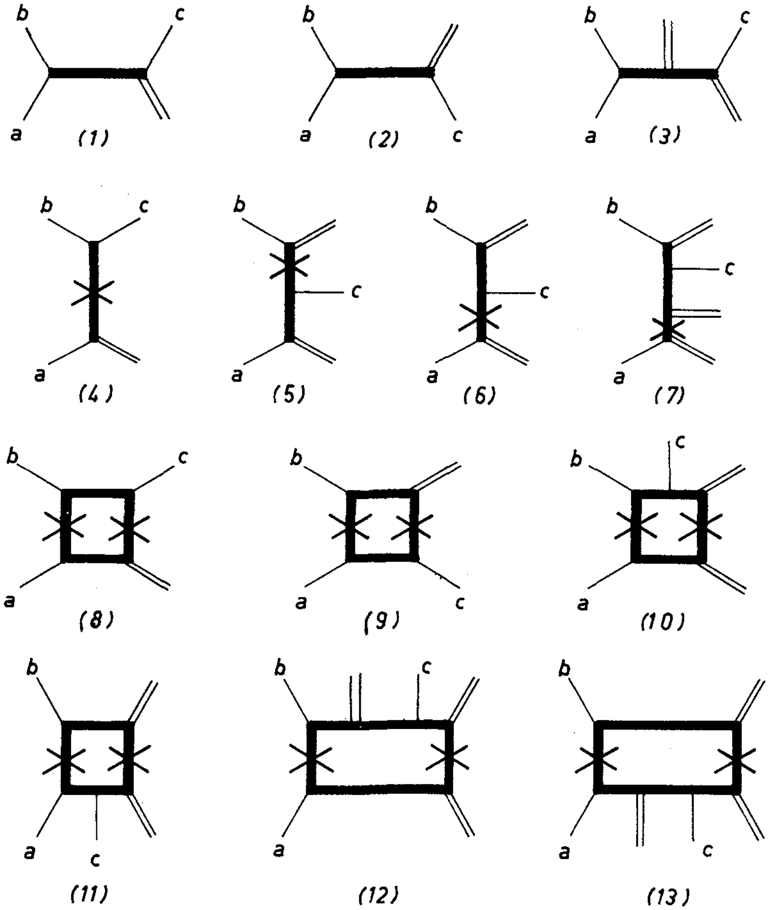


Fig. 23 - The 13 components of the Tye-Veneziano model for one-particle inclusive distributions.

butions. We shall not review here this analysis, but quote only a general result. The energy sum rule, Eq. (2-46) with $\mu = 0$, written in the C.M. frame in terms of the scaling variable x_c (2-14), becomes in the present case:

$$\frac{a}{2} \sum_{i=1}^{9} \int F_i^{(1)}(s, x_c, p_c)\, p_c\, dp_c\, dx_c = \sigma_T(ab), \tag{3-89}$$

where we have assumed to deal with identical particles. For the present purposes, it is sufficient to decompose the inclusive density, similarly to Eq. (3-67), as

$$F_i^{(1)}(s, x_c, p_c) \approx g_i(x_c, p_c) + \tilde{g}_i(x_c, p_c)\, s^{-1/2}, \tag{3-90}$$

so that, if the system ab is exotic and $\sigma_T(ab)$ is practically a constant, Eq. (3-89) gives for the non-leading terms

$$\sum_{i=1}^{9} \int \tilde{g}_i(x_i, p_i)\, p_c\, dp_c\, dx_c = 0. \tag{3-91}$$

This shows that, since some of the $\tilde{g}_i$ are shown to be positive, others must be negative: then it is possible to have inclusive cross sections approaching the scaling limit from below[53], in contrast to the case of total cross section which, according to duality, reach their constant limit from above.

Analytic expression for the distribution functions have been given for few components of the above model. The contribution from the components 1, 2 and 4 of Fig. 23 have been computed applying Mueller's approach to the 6-point tree amplitude[26] indicated in Fig. 24. (For the analytical expression of dual 6-point amplitudes we refer to the review paper by

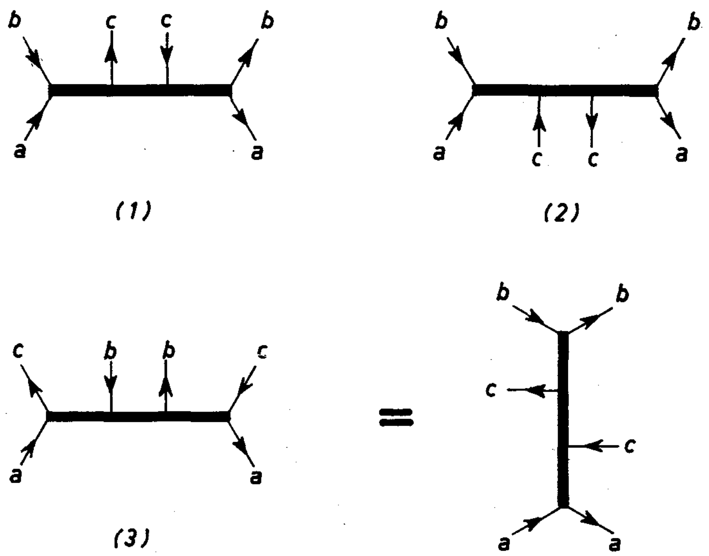


Fig. 24 - Tree graphs analysed for one-particle inclusive distributions.

Alessandrini *et al.*[51]). The distribution function can be written in the form

$$F^{(1)}(P_c) \approx s^{\alpha(0)-1}\, g(x_r, p_c) \tag{3-92}$$

and, since in the sanie approximation one gets a, $\sim s^{\alpha(0)-1}$ (first component of Fig. 22), one generally uses the inclusive density and speaks still of scaling behaviour. With a convenient choice of the Regge trajectory intercepts, finite limiting distributions are obtained both in the fragmentation and in the central regions. A reasonable fit of the experimental data is obtained with the distribution evaluated from the third diagram of Fig. 24 (Ref. 54). In the central region, only this diagram contributes; it gives

$$g(x_r, p_c) \propto \mu_c^{-5} \exp[-4(p_r^2 + m_c^2)]. \tag{3-93}$$

We note the remarkable result that the inclusive distributions show an exponential cut-off in the square transverse momentum.

In order to use a consistent descnption of the Pomeranchuk singularity in the duality scheme, one-particle inclusive distributions have been evaluated also from the 6-point loop amplitudes represented in Fig. 25 (Ref. 55); they correspond to the components 5 and 6 of Fig. 23. The results confirm also in this case an exponential cut-off in the square transverse moinentum.

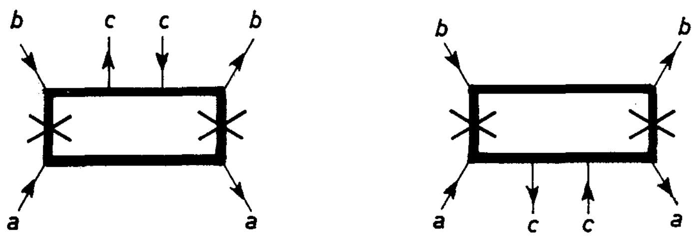


Fig. 25 - One-loop graphs analysed for one-particle inclusive distributions.

4. Conclusions

Referring to the main features of inclusive reactions outlined in Sec. 2.4, we summarize here the most important predictions of the multiperipheral and dual models.

The property of scaling and limiting fragmentation is a general output of these models. This property is clearly exhibited by the CERN-ISR inclusive cross sections, which show, at higher energies, the presence of a plateau in the central region of the rapidity plot (see Fig. 4). The data indicate also the presence of a short-range correlation[56] of the type predicted by Eq. (3-77); even if long-range correlations are certainly present, the main contribution can be interpreted in terms of a correlation length $L \approx [\alpha(0) - \alpha_M(0)]^{-1} \approx 2$, corresponding to Pomeranchuk and meson-trajectory exchanges.

Scaling is predicted by most of the popular models, so that it is belived to be based on very general grounds. The rate of approach to scaling is very important for differentiating the various models; several interesting predictions have been given[53] and they await for experimental tests.

The logs dependence of the average multiplicity (n), which is a very general prediction of the multiperipheral scheme, seems to be confirmed by all high energy data; deviation of this simple law at lower energies is probably due to the fact that the logs regime is fully attained when the plateau in the central (pionization) region is already populated.

Besides the average multiplicity (n), it is very important to analyze the correlation coefficients, such as, for the case of two-particle inclusive reactions, the dispersion $D^{(2)}$ (Eq. (2-40)).

The energy dependence of these coefficients provides a useful test to discriminate among different models. While the diffractive model predicts the behaviour $D^{(2)} \sim s^{1/2}$, the multiperipheral model with a leading pole with factorized residue (or, equivalently, the Mueller-Regge model) gives $D^{(2)} \sim \log s$. Modified multiperipheral models, in which one takes into account unitarity corrections[61] (diffractive contributions), generating long-range correlations, give $D^{(2)} \sim (\log s)^2$ and a multiplicity distribution broader than Poisson's.

The experimental data[S9] seem to favour a $(\log s)^2$ dependence, indicating the presence also of long-range correlation effects. Then modified multiperipheral models would be in better shape than diffractive ones.

The same behaviour $D^{(2)} \sim (\log s)^2$ is exhibited by the two-component models[60], in which the distribution functions consist of two terms: one, corresponding to short-range correlation, is the main responsible for high multiplicities, while the other, corresponding to diffraction, is dominant at low multiplicities.

The strong transverse momentum cut-off in the high energy multi-particle reactions is perhaps one of the most fundamental features of hadron dynamics. This property is introduced as a main ingredient into the multiperipheral scheme, through a momentum transfer cut-off. One of the successes of dual models is that they predict a strong cut-off, of the type $\exp[-a(p^2 + m^2)]$. We note that the dependence $\exp(-a\sqrt{p^2 + m^2})$, shown by the experimental distributions, follows rather directly from the statistical model; the essential point is a bootstrap postulate[24] which leads to density of hadronic levels increasing exponentially with energy. The cut-off prediction of dual models is probably related to the analogous energy dependence of the multiplicity of levels[58].

Multiperipheral ideas, implemented with duality, seem to provide an adequate description of multlparticle phenomena; the formulation of more specific models in this general scheme will probably require more stringent tests from both inclusive and exclusive analyses.

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