

Diffusion of Heat in a Cylinder which is Generating Heat

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In the present paper, we have considered the problem of the diffusion of heat in a semiinfinite solid circular cylinder which is generating heat. The faces $z = 0$ and $r = a$ of a cylinder, of radius a , are maintained at prescribed temperatures and there is a initial distribution of temperature throughout the medium. The solution is obtained by an appeal to the finite Hankel transform and the Fourier sine transform. Some particular cases are given. The solution obtained is interpreted numerically by using an IBM 1620 computer. Figures are drawn to indicate the temperature distribution in the medium.

Considera-se o problema da difusão de calor em um sólido cilíndrico circular semi-infinito que gera calor. As faces $z = 0$ e $r = a$ ($a =$ raio do cilindro) são mantidas a temperaturas prefixadas e conhece-se a distribuição inicial de temperatura no meio. Obtem-se a solução utilizando-se a transformada de Hankel finita e a de Fourier e alguns casos particulares são discutidos. A solução obtida é interpretada numericamente fazendo-se uso de um computador IBM 1620. A distribuição de temperatura no meio é apresentada nas figuras.

1. Introdução

The cases in which heat is generated in the interior of the medium are important in physics and engineering. For example, the induction of heat in a solid undergoing radioactive decay or absorbing radiation. We have similar problems when there is generation or absorption of heat in the solid as a result of chemical reaction, the hydration of cement being an example of a process of this kind. Other processes of heat production are the passage of an electric current, dielectric or induction heating, mechanical generation in viscous or plastic flow¹.

The object of the present work is to consider the problem of diffusion of heat in a semiinfinite solid circular cylinder which is generating heat. The faces $z = 0$ and $r = a$ of the cylinder of radius a , are maintained at

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prescribed temperatures and there is an initial distribution of temperature throughout the medium. The solution is obtained by an appeal to finite Hankel transform and Fourier sine transform. The solution obtained is interpreted numerically and figures are drawn to indicate the temperature distribution.

2. Integral Transforms

We define the finite Hankel transform of a function $\theta(r, z, t)$ as

$$\bar{\theta}(\eta_i, z, t) = \int_0^a r \theta(r, z, t) J_0(r \eta_i) dr \quad (1)$$

where η_i is a root of the transcendental equation

$$J_0(a \eta_i) = 0 \quad (2)$$

and J_0 stands for the Bessel function of first kind of order zero.

If θ satisfies Dirichlet's conditions in the interval $(0, a)$ and if its finite Hankel transform in that range is defined as in (1), then at any point $(0, a)$ at which the function $\theta(r, z, t)$ is continuous (Ref. 3, p. 84), we have

$$\theta(r, z, t) = \frac{2}{a^2} \sum_i \bar{\theta}(\eta_i, z, t) \frac{J_0(r \eta_i)}{[J_1(a \eta_i)]^2}. \quad (3)$$

As usual we consider the Fourier sine transform of a function $\theta(r, z, t)$ as

$$\theta_s(r, \xi, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \theta(r, z, t) \sin(z \xi) dz \quad (4)$$

and consequently

$$\theta(r, z, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \theta_s(r, \xi, t) \sin(z \xi) d\xi. \quad (5)$$

3. The Solution of the Problem

Let us consider the problem of determining the distribution of temperature in a semiinfinite circular cylinder of radius a . whose faces $z = 0$ and $r = a$ are kept at prescribed temperatures $m(r, t)$ and $n(z, t)$ respectively and the initial distribution of temperature throughout the medium is $f(r, z)$. If we assume that the rate of generation of heat is independent of the tempe-

rature at the point, then the fundamental equation governing the conduction of heat in a solid medium takes the following form

$$\frac{1}{\kappa} \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} + A(r, z, t), \quad (6)$$

where the diffusivity $\kappa = k/\rho c$; k , ρ and c are the conductivity, density and the specific heat of the material of the body.

Thus our problem is to solve the differential equation (6) with the following initial and boundary conditions:

$$\theta(r, z, 0) = f(r, z), \quad t = 0, \quad (7)$$

$$\theta(r, 0, t) = m(r, t), \quad z = 0, \quad 0 \leq r \leq a, \quad t > 0, \quad (8)$$

$$\theta(a, z, t) = n(z, t), \quad z > 0, \quad r = a, \quad t > 0. \quad (9)$$

Now by an appeal to the Fourier sine transform and condition (8), we get

$$\frac{1}{\kappa} \frac{\partial \theta_s}{\partial t} = \frac{\partial^2 \theta_s}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_s}{\partial r} + \sqrt{\frac{2}{\pi}} \xi m(r, t) - \xi^2 \theta_s + A_s(r, \xi, t). \quad (10)$$

Multiplying both sides of this equation by $r J_0(r \eta_i)$ and then integrating over the section of the cylinder, we obtain

$$\frac{1}{\kappa} \frac{d \bar{\theta}_s}{dt} + (\xi^2 + \eta_i^2) \bar{\theta}_s = a \eta_i J_1(a \eta_i) n_s(\xi, t) + \sqrt{\frac{2}{\pi}} \xi \bar{m}(\eta_i, t) + \bar{A}_s(\eta_i, \xi, t), \quad (11)$$

which on using the initial condition leads to the solution

$$\bar{\theta}_s(\eta_i, \xi, t) = \bar{f}_s(\eta_i, \xi) \exp[-\kappa(\xi^2 + \eta_i^2)t] + G(\eta_i, \xi, t), \quad (12)$$

where

$$G(\eta_i, \xi, t) = \kappa \int_0^t \left[a \eta_i J_1(a \eta_i) n_s(\xi, x) + \sqrt{\frac{2}{\pi}} \xi \bar{m}(\eta_i, x) + \bar{A}_s(\eta_i, \xi, x) \right] \cdot \exp[\kappa(\xi^2 + \eta_i^2)(x-t)] dx. \quad (13)$$

Applying the inversion theorem for finite Hankel transforms (3) and the Fourier sine formula (5), in succession, to (12), we set the temperature function as

$$\theta(r, z, t) = \frac{2}{a^2} \sqrt{\frac{2}{\pi}} \sum_i \frac{J_0(\eta_i r)}{[J_1(\eta_i a)]^2} \int_0^\infty [\bar{f}_s(\eta_i, \xi) \exp\{-\kappa(\xi^2 + \eta_i^2)t\} + G(\eta_i, \xi, t)] \sin(\xi, z) d\xi. \quad (14)$$

4. Particular Cases

Let us assume that

$$f(r, z) = f(r) \cdot g(z), \quad (15)$$

where $f(r) = a^2 - r^2$ and $g(z) = \lambda e^{-z}$; then [Ref. 3, p. 525, 31]

$$f_s(\eta_i, \xi) = \sqrt{\frac{2}{\pi}} \frac{4a}{\eta_i^3} \frac{\lambda \xi}{\Lambda + \xi^2} J_1(a\eta_i). \quad (16)$$

Further if we suppose that $m(r, t) = \beta$ (constant),

$$n(z, t) = \gamma z \cdot e^{-\frac{1}{2}z^2} \quad \text{and} \quad A(r, z, t) = 6 \cdot (a^2 - r^2) z \cdot e^{-\frac{1}{2}z^2}$$

then³ we have the temperature function as

$$\theta(r, z, t) = \frac{2}{a^2} \sum_i \frac{f_s(\eta_i, \xi)}{[J_1(\eta_i a)]^2} \frac{1}{\pi \eta_i^2} \frac{\lambda \xi}{1 + \xi^2} J_1(a\eta_i) \cdot \exp\{-\kappa(\xi^2 + \eta_i^2)t\} + G'(\eta_i, \xi, t) \sin(\xi, z) d\xi, \quad (17)$$

where

$$G'(\eta_i, \xi, t) = a\eta_i J_1(a\eta_i) \gamma \xi e^{-\frac{1}{2}\xi^2} \left[\frac{1}{\xi^2 + \eta_i^2} - \frac{\exp\{-\kappa(\xi^2 + \eta_i^2)t\}}{\xi^2 + \eta_i^2} \right] + \sqrt{\frac{2}{\pi}} \xi \frac{a\beta}{\eta_i} J_1(a\eta_i) \left[\frac{1}{\xi^2 + \eta_i^2} - \frac{\exp\{-\kappa(\xi^2 + \eta_i^2)t\}}{\xi^2 + \eta_i^2} \right] + \frac{4a\delta}{\eta_i^3} J_1(a\eta_i) \xi e^{-\frac{1}{2}\xi^2} \left[\frac{1}{\xi^2 + \eta_i^2} - \frac{\exp\{-\kappa(\xi^2 + \eta_i^2)t\}}{\xi^2 + \eta_i^2} \right]$$

By using the results (Ref. 2, p. 65, 76), (17) can be rewritten in a simplified form as

$$\theta(r, z, t) = \frac{2}{a^2} \sqrt{\frac{2}{\pi}} \sum_i \frac{J_0(\eta_i r)}{[J_1(\eta_i a)]^2} \left[\sqrt{\frac{2}{\pi}} \frac{4a\lambda}{\eta_i^3} J_1(\eta_i a) \frac{\pi}{4} \exp\{\kappa t - \kappa \eta_i^2 t\} \cdot \left\{ e^{-z} \operatorname{Erfc}\left(\sqrt{\kappa t} - \frac{1}{2} \frac{z}{\sqrt{\kappa t}}\right) - e^z \operatorname{Erfc}\left(\sqrt{\kappa t} + \frac{1}{2} \frac{z}{\sqrt{\kappa t}}\right) \right\} + \left(a\gamma \eta_i J_1(\eta_i a) + \frac{4a\delta}{\eta_i^3} J_1(\eta_i a) \right) \right]$$

$$\begin{aligned}
& \cdot \left\{ \frac{\pi}{4} e^{\frac{1}{2}\eta_i^2} \left(e^{-\eta_i z} \operatorname{Erfc} \left(\frac{\eta_i - z}{\sqrt{2}} \right) - e^{\eta_i z} \operatorname{Erfc} \left(\frac{\eta_i + z}{\sqrt{2}} \right) \right) \right. \\
& - \frac{\pi}{4} e^{\frac{1}{2}\eta_i^2} \left(e^{-\eta_i z} \operatorname{Erfc} \left(\sqrt{\kappa t + 1/2} \eta_i - \frac{1}{2} \frac{z}{\sqrt{\kappa t + 1/2}} \right) \right. \\
& \left. \left. - e^{\eta_i z} \operatorname{Erfc} \left(\sqrt{\kappa t + 1/2} \eta_i + \frac{1}{2} \frac{z}{\sqrt{\kappa t + 1/2}} \right) \right) \right\} \\
& + \sqrt{\frac{2}{\pi}} \frac{a\beta}{\eta_i} J_1(\eta_i a) \left\{ \frac{\pi}{2} e^{-\eta_i z} - \frac{\pi}{4} \left(e^{-\eta_i z} \operatorname{Erf} \left(\sqrt{\kappa t} \eta_i - \frac{1}{2} \frac{z}{\sqrt{\kappa t}} \right) \right. \right. \\
& \left. \left. - e^{\eta_i z} \operatorname{Erfc} \left(\sqrt{\kappa t} \eta_i + \frac{1}{2} \frac{z}{\sqrt{\kappa t}} \right) \right) \right\}. \quad (18)
\end{aligned}$$

Let us consider a mild steel solid circular cylinder of unit radius ($a = 1$). If we set that the surfaces $z = 0$ and $r = a$ are kept at zero temperature, i.e., $\beta = \gamma = 0$ and the initial distribution of temperature in the medium is also zero ($\lambda = 0$), then (18) reduces to

$$\begin{aligned}
\theta(r, z, t) &= \frac{2}{a^2} \sqrt{\frac{2}{\pi}} \sum_i \frac{J_0(\eta_i r)}{[J_1(\eta_i a)]^2} \left[\frac{4a\delta}{\eta_i^3} J_1(a\eta_i) \cdot \right. \\
& \cdot \left\{ \frac{\pi}{4} e^{\frac{1}{2}\eta_i^2} \left(e^{-\eta_i z} \operatorname{Erfc} \left(\frac{\eta_i - z}{\sqrt{2}} \right) - e^{\eta_i z} \operatorname{Erfc} \left(\frac{\eta_i + z}{\sqrt{2}} \right) \right) \right. \\
& - \frac{\pi}{4} e^{\frac{1}{2}\eta_i^2} \left(e^{-\eta_i z} \operatorname{Erfc} \left(\sqrt{\kappa t + 1/2} \eta_i - \frac{1}{2} \frac{z}{\sqrt{\kappa t + 1/2}} \right) \right. \\
& \left. \left. - e^{\eta_i z} \operatorname{Erfc} \left(\sqrt{\kappa t + 1/2} \eta_i + \frac{1}{2} \frac{z}{\sqrt{\kappa t + 1/2}} \right) \right) \right\} \right]. \quad (19)
\end{aligned}$$

By an appeal to formula (19) with $\delta = 200$, $a = 1$, the following tables have been prepared. They show the temperature distribution at various points in the cylinder at different intervals of time. To further illustrate the nature of the temperature distribution, Figs. 1 to 3 have been drawn; they are self-explanatory.

R \ T	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
.20	.58	.56	.52	.46	.39	.32	.24	.15	.07	0.00
.40	1.06	1.02	.94	.84	.72	.58	.44	.28	.14	0.00
.60	1.45	1.39	1.29	1.15	.98	.80	.60	.39	.19	0.00
.80	1.77	1.69	1.57	1.40	1.20	.97	.73	.48	.23	0.00
1.00	2.03	1.94	1.80	1.61	1.38	1.12	.84	.55	.26	0.00
1.20	2.25	2.15	1.99	1.78	1.52	1.24	.93	.61	.29	0.00
1.40	2.42	2.32	2.15	1.92	1.64	1.33	1.00	.65	.32	0.00
1.60	2.57	2.46	2.28	2.03	1.74	1.41	1.06	.69	.33	0.00
1.80	2.69	2.57	2.38	2.13	1.83	1.48	1.11	.73	.35	0.00
2.00	2.79	2.67	2.47	2.21	1.89	1.53	1.15	.75	.36	0.00

Table 1: $z = .10$

R \ T	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
.20	1.16	1.37	1.65	1.90	2.04	1.98	1.71	1.24	.64	0.00
.40	2.62	2.89	3.23	3.52	3.61	3.41	2.89	2.07	1.06	0.00
.60	3.96	4.25	4.60	4.86	4.88	4.53	3.79	2.69	1.37	0.00
.80	5.40	5.63	5.88	6.00	5.85	5.32	4.38	3.08	1.56	0.00
1.00	6.58	6.76	6.93	6.94	6.66	5.98	4.87	3.40	1.72	0.00
1.20	7.56	7.69	7.79	7.71	7.32	6.51	5.27	3.67	1.84	0.00
1.40	8.36	8.46	8.50	8.35	7.87	6.96	5.61	3.89	1.95	0.00
1.60	9.03	9.10	9.10	8.88	8.32	7.32	5.88	4.07	2.04	0.00
1.80	9.58	9.63	9.58	9.32	8.70	7.63	6.11	4.22	2.11	0.00
2.00	10.04	10.06	9.99	9.68	9.01	7.88	6.30	4.34	2.17	0.00

Table 2: $z = .50$

R \ T	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
.20	2.54	2.62	2.71	2.75	2.66	2.40	1.97	1.38	.69	0.00
.40	5.03	5.10	5.15	5.08	4.80	4.26	3.44	2.39	1.20	0.00
.60	7.28	7.30	7.25	7.03	6.54	5.73	4.58	3.16	1.58	0.00
.80	9.25	9.21	9.05	8.67	7.98	6.92	5.49	3.77	1.88	0.00
1.00	10.93	10.83	10.56	10.04	9.17	7.90	6.24	4.26	2.12	0.00
1.20	12.33	12.18	11.83	11.19	10.17	8.72	6.86	4.67	2.32	0.00
1.40	13.55	13.33	12.90	12.15	10.99	9.39	7.36	5.00	2.48	0.00
1.60	14.56	14.30	13.80	12.95	11.68	9.94	7.77	5.28	2.61	0.00
1.80	15.40	15.11	14.55	13.62	12.25	10.41	8.12	5.51	2.73	0.00
2.00	16.11	15.79	15.18	14.18	12.73	10.80	8.42	5.70	2.82	0.00

Table 3: $z = 1.00$

R \ T	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
.20	1.38	1.37	1.35	1.29	1.18	1.03	.81	.56	.27	0.00
.40	2.72	2.68	2.60	2.45	2.22	1.90	1.49	1.01	.50	0.00
.60	3.96	3.88	3.72	3.48	3.12	2.65	2.06	1.40	.69	0.00
.80	5.07	4.95	4.73	4.39	3.91	3.30	2.55	1.72	.85	0.00
1.00	6.07	5.91	5.63	5.20	4.61	3.87	2.99	2.01	.99	0.00
1.20	6.95	6.75	6.41	5.90	5.21	4.36	3.36	2.25	1.11	0.00
1.40	7.72	7.49	7.09	6.51	5.74	4.79	3.68	2.46	1.21	0.00
1.60	8.39	8.13	7.69	7.04	6.20	5.16	3.96	2.65	1.30	0.00
1.80	8.96	8.68	8.19	7.50	6.59	5.47	4.19	2.80	1.37	0.00
2.00	9.46	9.16	8.64	7.90	6.93	5.75	4.40	2.94	1.44	0.00

Table 4: $\alpha = 1.50$

R \ T	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
.20	2.31	2.33	2.32	2.26	2.12	1.86	1.49	1.03	.51	0.00
.40	4.53	4.50	4.42	4.23	3.89	3.37	2.67	1.83	.91	0.00
.60	6.53	6.44	6.26	5.92	5.38	4.61	3.62	2.47	1.22	0.00
.80	8.28	8.14	7.85	7.36	6.63	5.65	4.41	2.99	1.48	0.00
1.00	9.80	9.60	9.21	8.59	7.70	6.52	5.07	3.43	1.70	0.00
1.20	11.11	10.85	10.38	9.64	8.60	7.25	5.63	3.80	1.87	0.00
1.40	12.23	11.92	11.37	10.53	9.37	7.88	6.10	4.11	2.02	0.00
1.60	13.18	12.83	12.22	11.29	10.02	8.41	6.49	4.37	2.15	0.00
1.80	13.99	13.61	12.94	11.94	10.58	8.86	6.83	4.59	2.26	0.00
2.00	14.69	14.28	13.55	12.49	11.05	9.24	7.12	4.78	2.35	0.00

Table 5: $\alpha = 2.00$

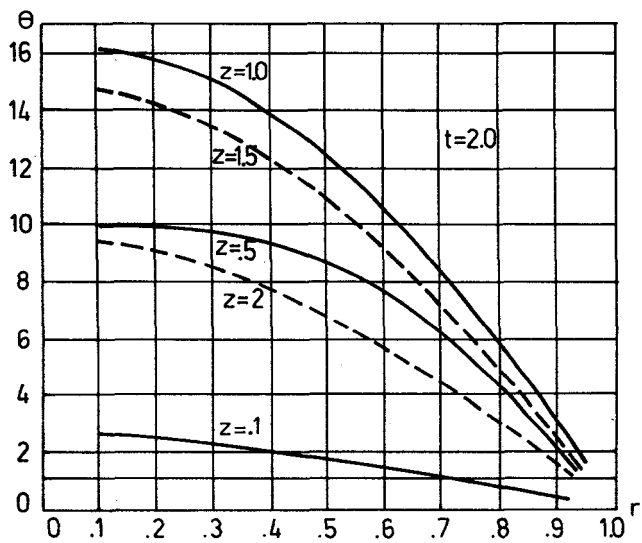


Figure 1

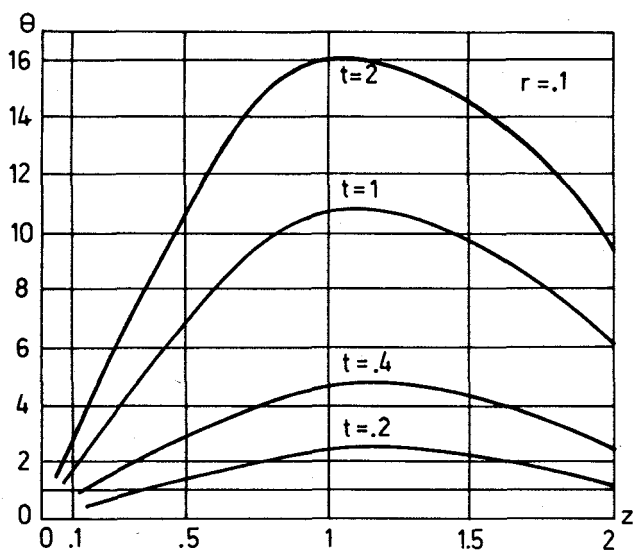


Figure 2

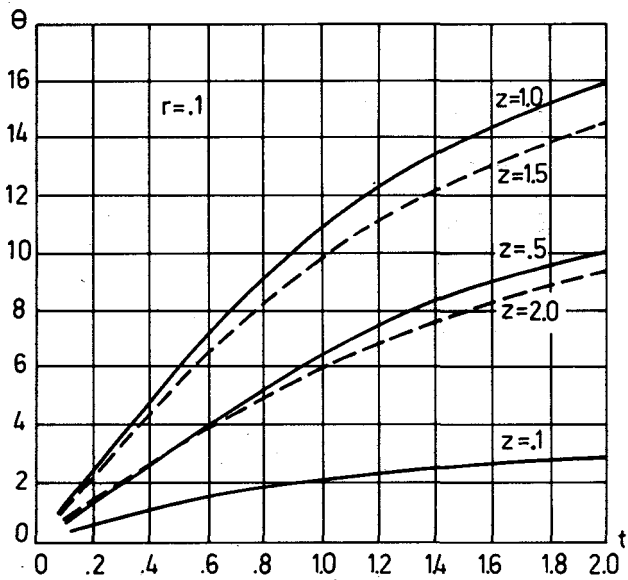


Figure 3

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