Revista Brasileira de Física, Vol. 3, N.° 1, 1973

Field Observables in the Radiation Gauge for the Maxwell and Spin Two Fields

COLBER G. OLIVEIRA
Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro GB*

Recebido em 12 de Novembro de 1972

It is shown that, given the potentials for spin 1 massless fields, the process of producing field potentials which commute with the generator of gauge transformations is equivalent to replace the classical commutator (the Poisson bracket), which holds for these initial potentials, by the Dirac bracket which describes the commutation algebra for the gauge invariant field potentials in the radiation gauge This result is extended for spin 2 massless fields without self-interaction. This last case is taken as the weak field approximation of the full non-linear gravitational field equations of general relativity.

Mostra-se que, dados os potenciais para campos de spin 1 de massa nula, o processo de obter potenciais de campo que comutam com o gerador de transformações de *gauge* é equivalente a substituir o parêntesis de Poisson, que vale para os potenciais iniciais, pelo parêntesis de Dirac que descreve a álgebra de comutadores para os potenciais de campo, *gauge* invariantes no *gauge* de radiação. O resultado é também extendido a campos de spin 2 de massa nula. Esse último caso é tomado como uma aproximação de campo fraco das equações completas não lineares do campo de gravitação da relatividade geral.

Introduction

For the Maxwell field, it is well known that a process for obtaining field potentials which are gauge invariant, in the Hamiltonian formalism, is obtained by introducing the transverse field potentials, as those which are divergence free. Since a spin 1 gauge transformation adds to the potentials the gradient of a scalar function and thus a longitudinal vector, it is clear that any divergenceless potential will be invariant under such transformations, as long as the transverse character is conserved after the transformation. This may also be seen from the expression for the generator of gauge transformations,

$$C(x^0) = -\int \Lambda(\mathbf{x}, x^0)\, p_{r,r}(\mathbf{x}, x^0)\, d_3 x.$$

*Postal Address: Av. Wenceslau Braz, 71, 20000 - Rio de Janeiro GB.

In this formula, **A** is the gauge function and p_r is the canonical momentum density for the spin 1 field. Latin indices indicate degrees of freedom going from 1 to 3. Greek indices go from 1 to 4. The term commutator used in this paper refers to the classical commutator, that is, the Poisson bracket. The metric tensor is the Minkowski tensor $\eta_{\mu\nu}$ with signature + 2; thus all three-dimensional operations are done for the metric $\eta_{rs} = \delta_{rs}$, and no distinction is made between contravariant and covariant indices in three-dimensions.

Given arbitrary (gauge variant) potentials A,, we have

$$[A_i, C] = \Lambda_{,i}.$$

But the divergenceless field functional

$$A_i^T = A_i - \partial_i \frac{1}{\nabla^2} \partial_j A_j$$

has a null commutator with the gauge generator C. In this paper we show that the commutation algebra of the transverse potentials is identical to the algebra of the Dirac bracket. It is known that the algebra of the Dirac bracket for the case of the Hamiltonian formulation of general relativity is realized by the usual Poisson bracket algebra of the so called "starred field functionals"[1], which describe the field observables of the theory. Thus, we have shown that in the radiation gauge this conclusion may be extended for the spin 1 massless field in flat spacetime. Similarly, it is also proved that the same result applies to spin 2 massless fields without self interactions. This field is taken as the weak field approximation for the gravitational field equations of general relativity.

This result may also be recasted as a proof of equivalence between the A-D-M method for quantization[2] and the B-K method[3], in the radiation gauge and within the approximations presently considered.

With regard to notation, we indicate the partial derivatives by any one of the symbols, $\partial_i \phi$, $\partial\phi/\partial x^i$ or $\phi_{,i}$, for any quantity ϕ. Covariant derivatives are not used due to our approximation of a linearized gravitational field.

1. Gauge Invariant Canonical Variables for the Maxwell Field in Flat Space

The Hamiltonian theory for the Maxwell field contains one relation of constraint connecting the three components of the canonical momentum p,.

$$B \equiv p_{r,r} \approx 0. \tag{1}$$

The symbol $\approx$ means equal to zero in the weak sense in Dirac's notation[4]. We want to prove that for this case the construction of the "starred field components" of B-K formalism is equivalent to the construction of the T-type potentials of the A-D-M formalism. As was stated in the introduction, the P. B. (Poisson bracket) relations among the starred field components is the same as the Dirac bracket among the potentials themselves. Therefore, as long as we show that the T-type potentials have the same commutation algebra as the starred potentials, we have proved that the commutation algebra of the T-type potentials is the same as the Dirac bracket algebra among the potentials. This last algebra will be the commutation algebra for the field observables in the Hamiltonian formulation, now written entirely in terms of gauge invariant degrees of freedom.

In introducing the concept of the "starred field potentials", we have to introduce a gauge condition in terms of the dynamical components for the field. We choose the radiation gauge condition,

$$D \equiv A_{r,r} \approx 0. \tag{2}$$

The two constraints (1) and (2) form a set of two second-class constraints", since the **P.B.** of D with B is

$$[D, B'] = -\nabla^2 \delta(\mathbf{x} - \mathbf{x}'). \tag{3}$$

Therefore, in presence of second class constraints, two alternatives are possible, either we use the Dirac bracket directly instead of the **P.B.** or, equivalently, we still retain the **P.B.** but modify each component of the dynamical variables by adding to them a linear combination of the second-class constraints, such that it commutes with all second-class constraints. This last alternative defines the so called "starred dynamical variable". We use this process, by defining

$$A_i^*(\mathbf{x},x^0) = A_i(\mathbf{x},x^0) + \int \mu_i(\mathbf{x},\mathbf{x}')p'_{r,r}d_3x' + \int \alpha_i(\mathbf{x},\mathbf{x}')A'_{r,r}d_3\mathbf{x}', \tag{4}$$

where A_i is the vector potential, the configuration type variable in the Hamiltonian formalism for electrodynamics. Similarly, in place of the canonical momentum p_i, we write

$$p_i^*(\mathbf{x},x^0) = p_i(\mathbf{x},x^0) + \int \beta_i(\mathbf{x},\mathbf{x}')p'_{r,r}d_3x' + \int \gamma_i(\mathbf{x},\mathbf{x}')A'_{r,r}d_3x', \tag{5}$$

where the coefficients μ_i, α_i, β_i and γ_i are determined by the conditions

$$[A_i^*, B'] = [A_i^*, D'] = 0, \tag{6}$$

$$[p_i^*, B'] = [p_i^*, D'] = 0. \tag{7}$$

From the first two conditions, Eqs. (6) and (7), we see that the starred dynamical variables are gauge invariant, since the generator of gauge transformations is

$$C(x^0) = -\int \Lambda(\mathbf{x}, x^0)\, B\, d_3 x$$

and all starred dynamical functions commute with C, even if the gauge function A is also a function of the dynamical variables A_i and p_i (we have to put $p_{r,r}$ equal to zero after computing all commutators). For the case of spin 1 massless fields, the imposition of gauge invariance for the canonical momentum is not really necessary since p_i is gauge invariant by definition. However, for spin 2 massless fields, this imposition will be necessary and, since the method of definition of the "starred canonical variables" is general, we have maintained this condition here.

Conditions (6) and (7) imply

$$\nabla'^2 \alpha_i(\mathbf{x}, \mathbf{x}') = \delta_{,i'}(\mathbf{x} - \mathbf{x}'), \tag{8}$$

$$\nabla'^2 \mu_i(\mathbf{x}, \mathbf{x}') = 0, \tag{9}$$

$$\nabla'^2 \beta_i(\mathbf{x}, \mathbf{x}') = \delta_{,i'}(\mathbf{x} - \mathbf{x}'), \tag{10}$$

$$\nabla'^2 \gamma_i(\mathbf{x}, \mathbf{x}') = 0. \tag{11}$$

The solution of (8) and (10) is given by

$$\alpha_i = \beta_i = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right)_{,i},$$

so that the A_i^* and p_i^* have the form

$$A_i^* = A_i + \frac{1}{4\pi}\int \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right)_{,i} A'_{r,r}\, d_3 x' + \int \mu_i(\mathbf{x}, \mathbf{x}')\, p'_{r,r}\, d_3 x', \tag{12}$$

$$p_i^* = p_i + \frac{1}{4\pi}\int \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right)_{,i} p'_{r,r}\, d_3 x' + \int \gamma_i(\mathbf{x}, \mathbf{x}')\, A'_{r,r}\, d_3 x', \tag{13}$$

with μ_i and γ_i solutions of Laplace's equation. These formulas may be written as

$$A_i^* = A_i - \partial_i \frac{1}{\nabla^2}\partial_r A_r + \int \mu_i(\mathbf{x}, \mathbf{x}')\, p'_{r,r}\, d_3 x',$$

$$p_i^* = p_i - \partial_i \frac{1}{\nabla^2}\partial_r p_r + \int \gamma_i(\mathbf{x}, \mathbf{x}')\, A'_{r,r}\, d_3 x'.$$

Thus, up to the terms containing μ_i and γ_i, the A_i^* and p_i^* are just the transverse field variables of the A-D-M theory. It may be shown that the terms in μ_i and γ_i do not contribute to the commutation relations of the A_i^* and p_i^* (use (9) or (11)),

$$[A_i^*, p_k^*] = \delta_{ik}\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{4\pi}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{,ik} \equiv \delta_{ik}(\mathbf{x}-\mathbf{x}'), \tag{14}$$

where $\delta_{ik}(\mathbf{x}-\mathbf{x}')$ is the transverse delta function. Since all fields of interest, similarly to the case of the A-D-M theory, have to be free of singularities and vanish at spatial infinity, we can take as the solution of Laplace's equation

$$\mu_i = \gamma_i = 0.$$

In this case the A_i^* will commute with A_k^* (the same for the momentum p_i^*), and the identification of the "starred field potentials" with the T-type functionals of the A-D-M theory is completed:

$$A_i^* = A_i^T, \; p_i^* = p_i^T.$$

From the relations

$$[A_i^T, p_j'^T] = [A_i^*, p_j'^*] = [A_i, p_j']^*$$

(by $[f,g]^*$ we indicate the Dirac bracket of any given quantities f and g), we see that the commutation relations among the T-type functionals are just the commutations arising from the Dirac bracket of the initial gauge variant A_i and the original momentum p_i.

2. The Canonical Variables h_{ij}^* and p_{ij} for Spin 2 in the Linear Appmximation

Here we extend our previous conclusions for the weak field approximation of the general relativistic field equations of the gravitational field. The field obtained is a spin 2 massless field without self interactions. The corresponding Hamiltonian version contains four relations of constraint

$$\mathscr{H}_r \equiv 2p_{rs,s} \approx 0, \tag{15}$$

$$\mathscr{H}_L \equiv h_{rs,rs} - h_{rr,ss} \approx 0, \tag{16}$$

which correspond to the unique constraint (1) for electrodynamics. Here we have four constraints due to the fact that a spin 2 gauge transformation involves four arbitrary gauge functions instead of just one as was the case for the spin 1 massless field. In the Lagrangian formalism, which is a four-dimensional formalism, we have for the symmetric second rank Lorentz tensor $h_{\mu\nu}$ related to the metric $g_{\mu\nu}$ and to the Minkowski tensor $\eta_{\mu\nu}$ by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

the gauge transformation

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) - \Lambda_{\mu,\nu}(x) - \Lambda_{\nu,\mu}(x),$$

involving the four components of the gauge function Λ_μ. In the Hamiltonian theory we obtain a similar structure, but now divided in the gauge transformation of the configuration field variables, the h_{ij}, and the gauge transformation of the momentum variables p_{ij}. Before writing the expressions for the generators of these transformations, we give some formulas which will be needed. The weak field approximation (from now on it will be denoted by W.F.A.) is given by taking the previous metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where only $\eta_{\mu\nu}$ acts as the metric. The field equations are the spin 2 wave equation plus the Lorentz covariant gauge condition

$$\partial^\nu \gamma_{\mu\nu} = 0, \; \partial^\nu = \eta^{\nu\alpha}\partial_\alpha, \; \gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}. \tag{17}$$

We call attention to the fact that, in the W.F.A. approximation, the momentum p_{ik} is a first order quantity. The explicit expression for p_{ik} is obtained by linearization of the exact formula derived from the Dirac Lagrangian-density for general relativity:

$$p_{ik} = -\left(h_{ao,a} - \frac{1}{2}h_{aa,o}\right)\delta_{ik} + \frac{1}{2}(h_{io,k} + h_{ko,i} - h_{ik,o}). \tag{18}$$

Under a gauge transformation on the potentials $h_{\mu\nu}$, of the form written before, the p_{ik} change according to

$$p'_{ik} = p_{ik} - \delta_{ik}\nabla^2\Lambda^0 + \Lambda^0{}_{,ik}. \tag{19}$$

The generator for this type of gauge transformation, in the Hamiltonian theory, is (recall that $\eta^{00} = -1$)

$$G(x^0) = -\int \Lambda^0(x, x^0)\mathscr{H}_L(x, x^0)d_3x. \tag{20}$$

Since

$$[p_{ik}, G] = \Lambda^0{}_{,ik} - \delta_{ik}\nabla^2\Lambda^0,$$

the remaining part of the gauge transformations, that is, the part which acts on the configuration potentials h_{ik}, is generated by

$$J(x^0) = \int \Lambda_s(\mathbf{x}, x^0)\mathscr{H}_s(\mathbf{x}, x^0)\, d_3 x. \tag{21}$$

Thus, as before, the constraints are basically the generators for the invariance function group of the theory.

The Lorentz covariant gauge condition *(17)* is separated into the conditions giving the radiation gauge for spin 2,

$$h_{,,} = h_{,,} = 0, \tag{22-1}$$

$$A \equiv h_{ss} \approx 0,\ B_r \equiv h_{rs,s} \approx 0. \tag{22-2}$$

The set of eight constraints given by (15), (16) and (22-2) is of second class, since

$$[\mathscr{H}_L, \mathscr{H}'_r] = 0,\ [\mathscr{H}_L,\ A'] = 0,\ [\mathscr{H}_L,\ B'_s] = 0,\ [A, B'_s] = 0,$$

$$[\mathscr{H}_r,\ A'] = -2\delta_{,r}(\mathbf{x}-\mathbf{x}'),\ [\mathscr{H}_r,\ B'_s] = (\delta_{rs}\nabla^2 + \partial^2_{rs})\delta(\mathbf{x}-\mathbf{x}').$$

Thus, as before, we define the quantities

$$h^*_{ij} = h_{,,} + \int \mu_{ijr}(x,\mathbf{x}')\mathscr{H}_r(x,x^0)d_3x' + \int \alpha_{ij}(\mathbf{x},\mathbf{x}')\mathscr{H}_L(x',x^0)d_3x'$$

$$+ \int \beta_{ij}(\mathbf{x},\mathbf{x}')A(\mathbf{x}',x^0)d_3x' + \int \gamma_{ijs}(\mathbf{x},\mathbf{x}')B_s(\mathbf{x}',x^0)d^3x', \tag{23}$$

$$p^*_{ij} = p_{ij} + \int \lambda_{ijr}(\mathbf{x},\mathbf{x}')\mathscr{H}_r(\mathbf{x}',x^0)d_3x' + \int \phi_{ij}(\mathbf{x},\mathbf{x}')\mathscr{H}_L(\mathbf{x}',x^0)d_3x'$$

$$+ \int \psi_{ij}(\mathbf{x},\mathbf{x}')A(\mathbf{x}',x^0)d_3x' + \int \tau_{ijs}(\mathbf{x},\mathbf{x}')B_s(\mathbf{x}',x^0)d_,x', \tag{24}$$

the coefficients being determined by the conditíons

$$[h^*_{ij},\ \mathscr{G}'] = 0, \quad [p^*_{ij},\ \mathscr{G}'] = 0,$$

where by 9 we indicate the set of all eight constraints

$$\mathscr{G} = \{A,\ B_s,\ \mathscr{H}_L,\ \mathscr{H}_s\}.$$

We determine first the coefficients standing on Eq. (23). From the condition that h^*_{ij} commutes with $\mathscr{H}_r$, A and B_r, we get the equations:

$$\delta_{ir}\delta_{,j}(\mathbf{x}-\mathbf{x}') + \delta_{jr}\delta_{,i}(\mathbf{x}-\mathbf{x}') = -2\int \beta_{ij}(\mathbf{x},\mathbf{x}'')\delta_{,r''}(\mathbf{x}''-\mathbf{x}')\,d_3x''$$

$$-\int \gamma_{ijs}(\mathbf{x},\mathbf{x}'')(\delta_{sr}\nabla''^2 + \partial''^2_{rs})\delta(\mathbf{x}''-\mathbf{x}')\,d_3x'', \quad (25)$$

$$\frac{\partial}{\partial x'^r}\mu_{ijr}(\mathbf{x},\mathbf{x}') = 0, \quad (26)$$

$$\nabla'^2\mu_{ijr}(\mathbf{x},\mathbf{x}') + \frac{\partial^2\mu_{ijk}(\mathbf{x},\mathbf{x}')}{\partial x'^k\partial x'^r} = 0. \quad (27)$$

From the commutator of h^*_{ij} with $\mathscr{H}_L$, we get no information since h^*_{ij} automatically commutes with $\mathscr{H}_L$ within the gauge conditions presently used.

We can write (25) in the form

$$\frac{\partial}{\partial x'^s}\left(-2\delta_{sr}\beta_{ij} + \frac{\partial\gamma_{ijr}}{\partial x'^s} + \frac{\partial\gamma_{ijs}}{\partial x'^r}\right) = \frac{\partial}{\partial x'^s}(\delta_{ir}\delta_{sj} + \delta_{jr}\delta_{is})\delta(\mathbf{x}-\mathbf{x}').$$

From this equation we can write

$$-2\delta_{sr}\beta_{ij} + \gamma_{ijr,s'} + \gamma_{ijs,r'} = (\delta_{ir}\delta_{sj} + \delta_{jr}\delta_{is})\delta(\mathbf{x}-\mathbf{x}'). \quad (28)$$

In obtaining (28) we have neglected a divergenceless term $\phi_s(\mathbf{x}')$ (such term may be added to the left-, or to the right-side of (28) but, as we shall see, we can obtain the desired solution without using this new term). Solving (28) for the β_{ij}, we obtain

$$\beta_{ij} = -\frac{1}{3}\left[\delta_{ij}\delta(\mathbf{x}-\mathbf{x}') - \gamma_{ijr,r'}(\mathbf{x},\mathbf{x}')\right]. \quad (29)$$

Now, from (26) and (27), we have

$$\nabla'^2\mu_{ijr} = 0,$$

and since we look for fields which are free of singularities and which tend to zero at spatial infinity, we may take

$$\mu_{ijr} = 0. \quad (30)$$

Eqs. (29) and (30) allow us to write the h^*_{ij} of (23) as

$$h^*_{ij} = \tilde{h}_{ij} + \int \alpha_{ij}(\mathbf{x},\mathbf{x}')\left(\frac{\partial B'_s}{\partial x'^s} - \nabla'^2 A'\right)d_3x' +$$

$$+\frac{1}{3}\int \frac{\partial \gamma_{ijr}(\mathbf{x},\mathbf{x}')}{\partial x'^r} A' d_3 x' + \int \gamma_{ijs}(\mathbf{x},\mathbf{x}') B'_s d_3 x', \qquad (31)$$

where $\tilde{h}_{ij}$ is the trace free combination

$$\tilde{h}_{ij} = h_{ij} - \frac{1}{3}\delta_{ij} A.$$

In order that the h^*_{ij} be of the type *TT* of the A-D-M theory, *it is first* of **all** necessary that the trace of h^*_{ij} vanishes. In the formula (23), or equivalently in (31), the coefficients a, and γ_{ijr} are symmetric in i, j (the other coefficients having the same symmetry). Taking the trace in (31), it is simple to verify that h^*_{ii} is zero only if

$$\alpha_{ss}(\mathbf{x},\mathbf{x}') = \gamma_{ssr}(\mathbf{x},\mathbf{x}') = 0.$$

Therefore, the two-point functions α_{ij}, γ_{ijr} have to be symmetric in i, j and traceless for all **x** and x'. Besides, they cannot depend on the dynamical variables since this would generate higher order terms which are neglected in the W.F.A.. Since in (31) we have no further information on the explicit form of α_{ij} and γ_{ijr}, we can make use of this arbitrariness in the form of these functions in rewriting (31) with coefficients

$$\tilde{\alpha}_{ij} = \alpha_{ij} - \frac{1}{3}\delta_{ij}\alpha_{ss},$$

$$\tilde{\gamma}_{ijr} = \gamma_{ijr} - \frac{1}{3}\delta_{ij}\gamma_{ssr},$$

in place of the α_{ij} and γ_{ijr}. These last two-point functions are symmetric in i, j (if the original two-point functions α_{ij} and γ_{ijr} are so) and are trace free. This recalibration in (31) implies that h^*_{ij} is trace free but, if we compute its divergence in x^j, we find

$$h^*_{ij,j} = h_{ij,j} - \frac{1}{3}h_{,i} + \int \tilde{\alpha}_{ij}(\mathbf{x},\mathbf{x}')(B'_{m,m} - \nabla'^2 A') d_3 x'$$

$$+\frac{1}{3}\int \frac{\partial^2 \tilde{\gamma}_{ijr}(\mathbf{x},\mathbf{x}')}{\partial x^j \partial x'^r} A' d_3 x' + \int \tilde{\gamma}_{ijs,j}(\mathbf{x},\mathbf{x}') B'_s d_3 x'$$

Imposing that the divergence of h^*_{ij} vanishes, we get a relation between the $\tilde{\alpha}_{ij}$ and $\tilde{\gamma}_{ijr}$. This relation after partial integration may be presented in the form

$$\int \left(\frac{1}{3}\partial_i' A' - B_i'\right)\delta(\mathbf{x}-\mathbf{x}')\,d_3x' = -\int \tilde{\alpha}_{ij,\,jm'}(\mathbf{x},\mathbf{x}')B_m'\,d_3x'$$

$$+\int \tilde{\alpha}_{ij,\,im'}(\mathbf{x},\mathbf{x}')\partial_m' A'\,d_3x' - \frac{1}{3}\int \tilde{\gamma}_{ijr,\,i}(\mathbf{x},\mathbf{x}')\partial_r' A'\,d_3x'$$

$$+\int \tilde{\gamma}_{iis,\,j}(\mathbf{x},\mathbf{x}')B_s'\,d_3x'$$

We separate this equation into two relations, one containing only $\partial_m' A'$, the other involving only B_m' (this is possible since A and B, represent independent combinations):

$$\frac{1}{3}\int \delta_{im}\partial_m' A'\cdot\delta(\mathbf{x}-\mathbf{x}')\,d_3x' = \int \tilde{\alpha}_{ij,\,jm'}(\mathbf{x},\mathbf{x}')\partial_m' A'\,d_3x'$$

$$-\frac{1}{3}\int \tilde{\gamma}_{ijm,\,j}(\mathbf{x},\mathbf{x}')\partial_m' A'\,d_3x',$$

$$-\int \delta_{is}B_s'\delta(\mathbf{x}-\mathbf{x}')\,d_3x' = -\int \tilde{\alpha}_{ij,js'}(\mathbf{x},\mathbf{x}')B_s'\,d_3x' + \int \tilde{\gamma}_{ijs,\,j}(\mathbf{x},\mathbf{x}')B_s'\,d_3x'$$

Consequently,

$$\frac{1}{3}\delta_{im}\delta(\mathbf{x}-\mathbf{x}') = \tilde{\alpha}_{ij,\,jm'}(\mathbf{x},\mathbf{x}') - \frac{1}{3}\tilde{\gamma}_{ijm,\,j}(\mathbf{x},\mathbf{x}'),$$

$$-\delta_{is}\delta(\mathbf{x}-\mathbf{x}') = -\tilde{\alpha}_{ij,\,js'}(\mathbf{x},\mathbf{x}') + \tilde{\gamma}_{ijs,\,j}(\mathbf{x},\mathbf{x}').$$

These equations are compatible for $\tilde{\alpha}_{ij} = 0$, since then we get just one independent equation:

$$\tilde{\gamma}_{ijs,\,j}(\mathbf{x},\mathbf{x}') = -\delta_{is}\delta(\mathbf{x}-\mathbf{x}'), \tag{32}$$

which is a condition fixing the value of $\tilde{\gamma}_{ijs}$. Its solution is

$$\tilde{\gamma}_{ijs}(\mathbf{x},\mathbf{x}') = -\delta_{is}D_{,j}(\mathbf{x}-\mathbf{x}') - \delta_{js}D_{,i}(\mathbf{x}-\mathbf{x}') + \frac{2}{3}\delta_{ij}D_{,s}(\mathbf{x}-\mathbf{x}')$$

$$+ \frac{1}{2}\tilde{\Delta}_{ij,\,s}(\mathbf{x}-\mathbf{x}'), \tag{33}$$

where $D(\mathbf{x}-\mathbf{x}')$ is the Green function of the Poisson equation and

$$\tilde{\Delta}_{ij}(\mathbf{x}-\mathbf{x}') = \frac{1}{\nabla^2} D_{,ij}(\mathbf{x}-\mathbf{x}') - \frac{1}{3}\delta_{ij} D(\mathbf{x}-\mathbf{x}'). \tag{34}$$

The relation (33) represents a c-number two-point function, symmetric in the first pair of indices and traceless over this pair of indices. Note that the divergence of $\tilde{\gamma}_{ijs}$ of (33) in x'^s gives

$$\tilde{\gamma}_{ijs,s'}(\mathbf{x},\mathbf{x}') = \frac{3}{2} D_{,ij}(\mathbf{x}-\mathbf{x}') - \frac{1}{2}\delta_{ij}\delta(\mathbf{x}-\mathbf{x}') \tag{35}$$

which is different from its divergence in x^j which is given by (32).

With the choice $\tilde{\alpha}_{ij} = 0$ and $\tilde{\gamma}_{ijs}$ given by (33), the h^*_{ij} is identical to the h^{TT}_{ij} of the A-D-M theory for spin 2, in the so called N-decomposition for a second rank symmetric tensor[4]. From (33), (35) and $\tilde{\alpha}_{ij} = 0$, we have for h^*_{ij},

$$h^*_{ij} = h_{ij} - \frac{1}{2}\delta_{ij} h_{ss} + \frac{1}{2}\partial^2_{ij}\frac{1}{\nabla^2} h_{ss} - \partial_j \frac{1}{\nabla^2}\partial_k h_{ki}$$
$$- \partial_i \frac{1}{\nabla^2}\partial_k h_{kj} + \frac{1}{2}\delta_{ij}\partial_s \frac{1}{\nabla^2}\partial_m h_{ms} + \frac{1}{2}\partial_s \frac{1}{\nabla^2}\partial^2_{ij}\frac{1}{\nabla^2}\partial_k h_{ks}.$$

An inspection on this formula shows that indeed the trace and divergence of h^*_{ij} are zero. This relation coincides with the usual form for presenting a TT part of a given tensor h_{ij} in the N-decomposition.

The gauge invariance of h^*_{ij} is made clear, even before the identification with h^{TT}_{ij}, since it commutes with the gauge generators (20) and (21) even for q-number gauge transformations (when the gauge functions depend on the dynamical variables).

The A-D-M method may be looked at as a process for producing field functionals such that, from given initial arbitrary canonical fields, we obtain new fields which satisfy the gauge conditions A = B, = 0. The method for obtaining the starred field variables is similar in this point and this is made clear from the fact that we used only the left hand side of the radiation gauge conditions for the definition of h^*_{ij} and did not take directly $A = B_s = 0$, but rather showed that the final h^*_{ij} may be chosen so as to satisfy these requirements.

For the momentum p^*_{ij} of (24), we cannot use the gauge conditions under form (22-2) since this leads to a contradiction. Indeed, taking the commutator of p^*_{ij} of **(24)** with $\mathscr{H}_L$, se get

$$[p_{ij},\ \mathscr{H}'_L] = 0,$$

which cannot be true. To avoid this difficulty, we rewrite the radiation gauge conditions for spin 2 in a form slighly different, but mathematically equivalent[5], the quantity A being replaced by

$$Q \equiv \nabla^2 p_{ss} - p_{rs,rs} \approx 0, \tag{36}$$

while the remaining conditions $B_s = 0$ are retained. Then, we can write

$$p^*_{ij} = p_{ij} + \int \lambda_{ijr}(\mathbf{x},\mathbf{x}')\mathscr{H}'_r d_3 x' + \int \phi_{ij}(\mathbf{x},\mathbf{x}')\mathscr{H}'_L d_3 x'$$
$$+ \int \psi_{ij}(\mathbf{x},\mathbf{x}')Q' d_3 x' + \int \tau_{ijs}(\mathbf{x},\mathbf{x}')B'_s d_3 x'. \tag{37}$$

The imposition that p^*_{ij} commutes with **all** constraints leads to the following equations:

$$\nabla'^2 \tau_{ijr}(\mathbf{x},\mathbf{x}') + \partial'^2_{rs}\tau_{ijs}(\mathbf{x},\mathbf{x}') = \mathbf{0}, \tag{38}$$

$$\nabla'^2\nabla'^2 \psi_{ij}(\mathbf{x},\mathbf{x}') + \frac{1}{2}(\delta_{ij}\nabla'^2 - \partial'^2{}_{ij})\delta(\mathbf{x}-\mathbf{x}') = 0, \tag{39}$$

$$\nabla'^2\nabla'^2 \phi_{ij}(\mathbf{x},\mathbf{x}') = 0, \tag{40}$$

$$\nabla'^2 \lambda_{ijs}(\mathbf{x},\mathbf{x}') + \partial'^2{}_{rs}\lambda_{ijr}(\mathbf{x},\mathbf{x}') = -\frac{1}{2}(\delta_{js}\partial_i + \delta_{is}\partial_j)\delta(\mathbf{x}-\mathbf{x}'). \tag{41}$$

From **(40)** and **(39)** we conclude that

$$\phi_{ij} = 0, \tag{42}$$

and

$$\psi_{ij} = \frac{1}{\nabla'^4}\,\frac{1}{2}\,[\phi_{,ij}(\mathbf{x}-\mathbf{x}') - \delta_{ij}\nabla'^2\delta(\mathbf{x}-\mathbf{x}')]. \tag{43}$$

Now we note that **all** avaílable two-point functions have to depend on δ**(x – x')** or on the Green function D**(x – x'),** since they have to be c-numbers. Thus, any double differentiation with respect to **x" is** equivalent to differentiate on x^r and we may rewrite **(41)** as

$$\nabla^2 \lambda_{ijs} + \partial^2_{rs}\lambda_{ijr} = -\frac{1}{2}(\delta_{js}\partial_i + \delta_{is}\partial_j)\delta(\mathbf{x}-\mathbf{x}').$$

Differentiation in x^s gives

$$\nabla^2 \lambda_{ijs,s}(\mathbf{x},\mathbf{x}') = -\frac{1}{2}\delta_{,ij}(\mathbf{x}-\mathbf{x}'),$$

which has as solution

$$\lambda_{ijs,s}(\mathbf{x},\mathbf{x}') = -\frac{1}{2}\int D(\mathbf{x}-\mathbf{x}'')\delta_{,i''j''}(\mathbf{x}''-\mathbf{x}')d_3\mathbf{x}''. \tag{44}$$

Differentiation with respect to x'^r in Eq. (38) gives

$$\nabla'^2 \tau_{ijs,s'}(\mathbf{x},\mathbf{x}') = 0,$$

which implies that

$$\tau_{ijs,s'}(\mathbf{x},\mathbf{x}') = 0$$

and thus, τ_{ijs} is constant. We take this constant as zero since we know that p^*_{ij} cannot depend on the h_{ij}, as it would depend if τ_{ijs} did not vanish. Eq. (44) is integrated over the delta function to give

$$\lambda_{ijs,s}(\mathbf{x},\mathbf{x}') = -\frac{1}{2}D_{,ij}(\mathbf{x}-\mathbf{x}'). \tag{45}$$

The solution of this equation is

$$\lambda_{ijs}(\mathbf{x},\mathbf{x}') = -\frac{1}{2}\partial^2_{ij}\partial_s\frac{1}{\nabla^2}D(\mathbf{x}-\mathbf{x}') + \Theta_{ijs}(\mathbf{x}-\mathbf{x}'), \tag{46}$$

for $\Theta_{ijs}(\mathbf{x}-\mathbf{x}')$ a c-number two-point function, symmetric over i,j and divergenceless over the last index:

$$\Theta_{ijs,s}(\mathbf{x}-\mathbf{x}') = 0. \tag{47}$$

The value for Θ_{ijs} is obtained by imposing consistency of the solution (46) with the original equation (41). With this end, we compute the Laplacian of (46):

$$\nabla^2 \lambda_{ijs} = -\frac{1}{2}\partial^2_{ij}\partial_s D(\mathbf{x}-\mathbf{x}') + \nabla^2\Theta_{ijs}(\mathbf{x}-\mathbf{x}')$$

and, also from (46),

$$\partial^2_{rs}\lambda_{ijr}(\mathbf{x},\mathbf{x}') = -\frac{1}{2}\partial^2_{ij}\partial_r\partial^2_{rs}\frac{1}{\nabla^2}D(\mathbf{x}-\mathbf{x}')$$

(condition (47) was used). Therefore,

$$\partial^2_{rs}\lambda_{ijr}(\mathbf{x},\mathbf{x}') = -\frac{1}{2}\partial^2_{ij}\partial_s D(\mathbf{x}-\mathbf{x}')$$

and the left hand side of the original Eq. *(41)* is

$$\nabla^2 \lambda_{ijs} + a_{,,}^2 \lambda_{ijr} = -\partial_{ij}^2 \partial_s D(\mathbf{x}-\mathbf{x}') + \nabla^2 \Theta_{ijs}(\mathbf{x}-\mathbf{x}').$$

By consistency, we should have

$$-\partial_{ij}^2 \partial_s D(\mathbf{x}-\mathbf{x}') + \nabla^2 \Theta_{ijs}(\mathbf{x}-\mathbf{x}') = -\frac{1}{2}\left[\delta_{js}\delta_{,i}(\mathbf{x}-\mathbf{x}') + \delta_{is}\delta_{,j}(\mathbf{x}-\mathbf{x}')\right].$$

This is a differential equation in Θ_{ijs}. Its solution is

$$\Theta_{ijs} = -\frac{1}{2}\left(\delta_{js}\frac{1}{\nabla^2}\delta_{,i}(\mathbf{x}-\mathbf{x}') + \delta_{is}\frac{1}{\nabla^2}\delta_{,j}(\mathbf{x}-\mathbf{x}')\right) + \frac{1}{\nabla^2}\partial_{ij}^2\partial_s D(\mathbf{x}-\mathbf{x}'),$$

which may be written as,

$$\Theta_{ijs} = -\frac{1}{2}\left(\delta_{js} D_{,i}(\mathbf{x}-\mathbf{x}') + \delta_{is} D_{,j}(\mathbf{x}-\mathbf{x}')\right) + \frac{1}{\nabla^2}\partial_{ij}^2\partial_s D(\mathbf{x}-\mathbf{x}'). \quad (48)$$

The Θ_{ijs} of *(48)* satisfies condition *(47)*. Therefore, from *(46)* we have

$$3_{ijs} = -\frac{1}{2}\partial_{ij}^2\partial_s\frac{1}{\nabla^2} D(\mathbf{x}-\mathbf{x}') + \frac{1}{\nabla^2}\partial_{ij}^2\partial_s D(\mathbf{x}-\mathbf{x}')$$
$$-\frac{1}{2}\left[\delta_{js} D_{,i}(\mathbf{x}-\mathbf{x}') + \delta_{is} D_{,j}(\mathbf{x}-\mathbf{x}')\right]. \quad (49)$$

We note that by an argument similar to the one used for the A-equation, we can write *(43)* in the form

$$\psi_{ij} = \frac{1}{\nabla^4}\,\frac{1}{2}\left[\delta_{,ij}(\mathbf{x}-\mathbf{x}') - \delta_{ij}\nabla^2\delta(\mathbf{x}-\mathbf{x}')\right]. \quad (50)$$

This formula may be simplified to

$$\psi_{ij} = \frac{1}{2}\,\frac{1}{\nabla^2} D_{,ij}(\mathbf{x}-\mathbf{x}') - \frac{1}{2}\delta_{ij} D(\mathbf{x}-\mathbf{x}'). \quad (51)$$

Using the value for the several coefficients, we can finally write down the formula for the p_{ij}^*:

$$p_{ij}^* = p_{ij} + \int\left[\frac{1}{\nabla^2}\partial_{ij}^2 D_{,s}(\mathbf{x}-\mathbf{x}') - \frac{1}{2}\partial_{ij}^2\partial_s\frac{1}{\nabla^2} D(\mathbf{x}-\mathbf{x}')\right.$$
$$\left. -\frac{1}{2}\left\{\delta_{js} D_{,i}(\mathbf{x}-\mathbf{x}') + \delta_{is} D_{,j}(\mathbf{x}-\mathbf{x}')\right\}\right]\mathscr{H}_s(\mathbf{x}')\,d_3x' +$$
$$+\frac{1}{2}\int\left[\frac{1}{\nabla^2} D_{,ij}(\mathbf{x}-\mathbf{x}') - \delta_{ij} D(\mathbf{x}-\mathbf{x}')\right] Q(\mathbf{x}')\,d_3x' \quad (52)$$

which is a functional of the original momentum p_{ij}. As for the configuration variables, we also have here that $p^*_{ij} = p^{TT}_{ij}$.

3. Conclusion

In the usual formalism, involving the gauge variant canonical variables h_{ij} and p_{ij} for the linearized gravitational spin 2 field, the gauge functions cannot be arbitrarily chosen in the radiation gauge. Under a gauge transformation, the h_{ij} change as

$$h'_{ij} = h_{ij} - \Lambda_{i,j} - \Lambda_{j,i}.$$

Then, the gauge conditions

$$A = h_{ss} = 0, \quad B_s = h_{sr,r} = 0,$$

are valid in a new gauge frame only if

$$A_{,,} = 0, \quad \nabla^2 A_{,} = 0.$$

These last relations are equivalent to impose that the left hand side of the above gauge conditions commute with the generators of the gauge transformations:

$$[A, J] = 0, \quad [B_s, J] = 0$$

(if we use in place of the gauge condition $A = h_{ss} = 0$, the condition $Q = \nabla^2 p_{ss} - p_{rs,rs} = 0$, a similar situation holds). However, in the case where we work with the functionals h^*_{ij} and p^*_{ij}, no condition need to be imposed on the gauge functions Λ_0 and $A_{,}$, since under gauge transformations

$$h^{*\prime}_{ij} = h^*_{ij}, \quad p^{*\prime}_{ij} = p^*_{ij},$$

and thus h^*_{ij} and p^*_{ij} automatically commute with the generators of the gauge transformations:

$$[A^*, J] = 0, \quad [B^*_s, J] = 0$$

and also trivially we have,

$$[A^*, G] = 0, \quad [B^*_s, G] = 0.$$

Since $[A^*, J] = [A, J]^*$, with the same property for the other commutators, we have that in the Dirac bracket algebra all eight constraints $\mathscr{H}_r$, $\mathscr{H}_L$, A and B, become of the first class. This indeed was the basic idea underlining this new commutation algebra. What was proven is that this new commutation algebra is just the commutation algebra of the transverse-transverse field functionals of the type used in the so called N-decomposition of the **A-D-M** formalism.

Since in the **A-D-M** theory a process is suggested for generalizing this for the full non-linear gravitational field equations of general relativity, in the so called C-decomposition, we may hope that similarly it may also apply for the starred field functionals underlined in the Dirac commutation algebra. Since no closed and simple form is known for g^*_{ij} and p^{*ij} in general relativity, it may happen that this analogy turns out to be useful in this case.

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