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Dynamical Model of Regge Trajectories: Threshold Behavior and Linearity*

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A model for dominant Regge trajectories is discussed, both on theoretical and on experimental grounds. It is based on analyticity and unitarity and gives trajectories that are almost linear. The intercept is related to the asymptotic behavior and, as the trajectory approaches a straight-line, the intercept approaches 1/2.

Discute-se um modelo de trajetórias de Regge tanto do ponto de vista teórico quanto do experimental. As bases do modelo são a analiticidade da trajetória e as consequências da unitariedade. As trajetórias obtidas são aproximadamente lineares e o valor $\alpha(0)$ está relacionado com o comportamento assintótico de maneira que, ao aproximar-se a trajetória de uma reta, $\alpha(0)$ aproxima-se de 1/2.

1. Physicists from my generation grew under the impact of the beautiful ideas of Regge theory concerning the use of complex angular momentum in particle physics. Displaying a remarkable skill for survival in a time of so drastic changes, the ideas of Regge theory continue to provide a reasonably stable ground on which a theory of strong interactions may eventually be erected. Duality ideas recently gave new momentum to this line of research by introducing the infinitely rising Regge trajectories departing, in this way, from the traditional background of potential theory. The Veneziano model then summed up many of these features in an astonishingly simple meromorphic amplitude[1] and gave much status to straight-line trajectories. From the aesthetic viewpoint, the Veneziano picture of hadronic scattering is hardly to be surpassed.

Nature, alas, has been tough. All too often, experimental data have been ready to disprove the most beautiful theories, throwing out what, for a moment, seemed to us to be the perfect choice for the description of the world fabric. It is, however, a lesson from the past that, in the end,

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harmony imperates again, provided we look the things from a better vantage point.

In the meantime, we do that we can. We know that trajectories cannot be straight forever. What then would be the next simplest choice?

In this note, we look for a somewhat more realistic description of Regge trajectories while trying to keep aloof from the trees of uncorrelated details that prevent one from seeing the forest. We present a simple model of mesonic trajectories or, to be precise, revisit one which was introduced some time ago by Predazzi and Fleming[2], examined now under the light of some recent discoveries[3] which, in our opinion, contributed to make that model a reasonable candidate for describing "true-life" trajectories.

We base our discussions on analyticity and some bounds. As is traditional in talking about complex angular momentum, we omit many assumptions made: in fact, everyone knows that even the basic continuation of the partial-wave amplitudes to an analytic function of the (complex) angular momentum relies upon the existence of some as yet unproved dispersion relations. We assume the whole folklore of Regge pole theory, which owes its success to a reasonable description of two-body processes as well as to its flexibility, which allows it to accomodate in the same room such gruesome fiends as crossing and unitarity.

In Section 2, we review very briefly this folklore, just to fix notation, discussing, wherever it may seem useful, the present state of affairs. We will be mainly interested in justifying the analyticity we will use. Other category of assumptions to be made concerns the rate of growth of the trajectories with the energy along an arbitrary direction of the complex s plane. These are still more difiicult to argue for, and one must concede that it is the ultimate success of the results that justifies the assumptions. We draw, in this part, heavily upon potential theory. The diligent reader is referred to Ref. (4) which, to my taste, is by far the best exposition of Regge theory fundamentals, to be supplemented by Ref. (5).

In Section 3, the model is introduced and we discuss how plausible are our assumptions and to what degree are we really free to choose, in some choices we are compelled to make. We find a class of solutions of the model and dicuss their general properties. It is shown that the very few properties of the trajectories which have some universality are shared by our trajectories. This is obtained by comparing their threshold behavior to the general one predicted by Barut and Zwanziger[6] several years ago.

Finally, in Sectíon 4, we describe a different way to deal with threshold behaviors which gives further support to our model.

2. How should one introduce complex angular momentum in order to obtain physically interesting results? Let $F(s,t)$ be the scattering amplitude for some two-body process. The partial-wave expansion reads

$$F(s,t) = \sum_{l=0}^{\infty} (21+1) F_l(s) P_l(\cos\theta) \tag{1}$$

in an obvious notation. Our intention is defining a function $F(s,\lambda)$ which provides an interpolation for the (physical) partial-wave amplitudes, that is, such that $F(s,l) = F_l(s)$ for integer l. The interpolation $F(s,\lambda)$ must be unique, so as to characterize the physical system to which it refers. The mathematical tool that allows one to prove the uniqueness is a theorem due to Carlson[7] which says that, if a function $F(s,\lambda)$ exists such that

$$F(s,l) = F_l(s) \quad \text{for } 1 > \text{N},$$

analytic for Re $\lambda >$ N, with

$$F(s,\lambda) = O(\exp[a|\mathrm{Im}\,\lambda| + b\,\mathrm{Re}\,\lambda]), \tag{2}$$

where a $< n - \text{E}$, E > 0 and b are all real constants, then $F(s,\lambda)$ is uniquely determined.

The analytic continuation that satisfies (2) was discovered by Froissart and Gribov[8] and is obtained through the use of a fixed s dispersion relation for $F(s,$ t). One has

$$F(s,t) = \sum_{n=0}^{N-1} C_n(s) t^n + \frac{t^N}{\pi} \int_{t_0}^{\infty} dt' \frac{A_t(s,t')}{t'^N (t'-t)} +$$

$$+ \frac{u^N}{\pi} \int_{u_0}^{\infty} du' \frac{A_u(s,u')}{u'^N (u'-u)}, \tag{3}$$

where A, and A, are the absorptive parts of $F(s,$ t) in tbe t and u channels respectively, including eventually some poles disguised as delta functions. The amplitude for the l-th partial wave is defined by

$$F_l(s) = \frac{1}{2} \int_{-1}^{+1} d(\cos\theta) P_l(\cos\theta) F(s, t(\cos\theta)). \tag{4}$$

Putting (3) into (4), it is easily seen that, for $l > N$, the polynomial in t, as well as the terms that come from the subtractions made, do not contribute to the final expression which reads:

$$F_l(s) = \frac{1}{\pi} \int_{v_0}^{\infty} dv \frac{1}{2q^2(s)} Q_l \left\{1 + \frac{v}{2q^2(s)}\right\} \{A_t(s,v) + (-1)^l A_u(s,v)\}, \tag{5}$$

where use was made of

$$Q_l(z) = \frac{1}{2} \int_{-1}^{+1} dx \frac{P_l(x)}{(z-x)},$$

$$Q_l(-z) = -(-1)^l Q_l(z)$$

and $v_0 < \min(t_0, u_0)$.

Expression (5) is convergent for $l > N$ but fails to meet the requirements about behavior at infinity of Carlson's theorem, on account of the factor $(-1)^l$. The way out is the following: two functions are introduced,

$$F_{\pm}(s,\lambda) = \frac{1}{\pi} \int_{v_0}^{\infty} dv \frac{1}{2q^2(s)} Q_\lambda \left(1 + \frac{v}{2q^2}\right) A_{\pm}(s,v), \tag{6}$$

where

$$A_{\pm}(s,v) = A_t(s,v) \pm A_u(s,v). \tag{7}$$

For $l > N$, then, $F_{\pm}(s,\lambda = l) = F_l(s)$ according to l being even or odd. It is possible to show that the existence of a unique $F(s,\lambda)$ that interpolates both even and odd partial waves implies that $A_u = 0$, namely, that no exchange forces exist. This situation is known as exchange degeneracy and seems to be, in fact, favored by nature[9]. It is our special interest to study the singularities of $F_{\pm}(s,\lambda)$ in the s-plane. A careful study of expression (6) shows that $F_{\pm}(s,\lambda)$, like $F_l(s)$, has two cuts, one for $s > 4u^2$, the other for $s < -8u^2$, u being the mass of the external particles. They come from the cuts of $A_{\pm}(s,v)$ at

$$s > 4u^2 + \frac{4u^4}{v - 4u^2} \quad \text{and } s < -v - \frac{4u^4}{v - 4u^2} \tag{8}$$

Let us assume $F_{\pm}(s,\lambda)$ to have a pole for some value of λ, say $\lambda = \alpha(s)$. If λ is not a negative integer, it is called a Regge pole and the function

$\alpha(s)$ is the Regge trajectory[10]. Our goal is the determination of the singularities of the function $\alpha(s)$. Writing (6) in the form

$$F_{\pm}(s,\lambda) = E_{\pm}(s,\lambda) + D_{\pm}(s,\lambda), \tag{9}$$

with

$$D_{\pm}(s,\lambda) = \frac{1}{\pi}\int_{a^2}^{\infty} dv \frac{1}{2q^2} Q_{\lambda}\left(1 + \frac{v}{2q^2}\right) A_{\pm}(s,v) \tag{10}$$

and E, defined by the corresponding integral from v, to a^2, it is easily seen that E, (s, A) is meromorphic in the I-plane, its poles being located at negative integral values of **I. All** other singularities are singularities of D,. Since a^2 is arbitrary, we may take it as large as we wish, so that the singularities of $D_{\pm}$ depend only on the asymptotic behavior of A, (s, v) as $v \to$ x , that is, on the high energy behavior of the t- and u-channel absorptive parts. The position of the poles may be obtained by solving the equation

$$D^{-1}(s,\alpha(s)) = 0 \tag{11}$$

(from now on we will omit the $\pm$ sign). If $D^{-1}(s,\lambda)$ is regular in a neighborhood of (s, **I** $= \alpha(s)$) and

$$\left[\frac{\partial D^{-1}(s,\lambda)}{\partial\lambda}\right]_{\lambda=\alpha(s)} \neq 0, \tag{12}$$

the implicit function theorem tells us that (11) defines a function $\alpha(s)$ which is regular in the same neighborhood. *So,* singularities of $\alpha(s)$ are expected to appear at those points where either $D^{-1}(s,1)$ is singular or (12) is not true. Using (10) and (8) it is apparent that $D(s,\lambda)$ has a cut for $s >$ 4u^2 and that the nearest left-hand cuts start at

$$s = -a^2 - \frac{4u^2}{a^2 - 4u^2}$$

or

$$s = 4u^2 - a^2.$$

As a^2 may be taken arbitrarily large, the branch points may be sent to – **x**, so that we may as well ignore them. Therefore, the Regge trajectories do not inherit the left-hand branch points and have, as only singularity, a branch point at 4u^2.

There is, however, another mechanism for generating singularities: the derivative at (12) may vanish somewhere. Now, when (11) is true and (12) is not, a zero of higher multiplicity of D^{-1} is present. This means that two trajectories intersect at that point. It became usual to call this phenomenon a "collision of singularities". It may give rise to new branch points for both "colliding" trajectories. Such trajectories are called "complex" and have recently deserved some attention''. In this paper we deal with trajectories that are not complex. They are, therefore, analytic functions of s in the plane cut from threshold to $\pm \infty$ along the real axis. According to the reflection principle, $\alpha(s^*) = \alpha^*(s)$, the so-called reality condition.

As for bounds for $\alpha(s)$, we assume that there is some number k such that

$$\alpha(s)/s^k = O\,(\exp r^{\beta}), \tag{13}$$

with $\beta < 1$ uniformiy in the upper half s-plane and that

$$\lim_{s\to\pm\infty} \alpha(s)/s^k = C_{\pm}, \tag{14}$$

the limits being taken along the real axis. The notation used in (13) is that of Titchmarsh (see Ref. (7)). We can then apply the Phragmén-Lindelof theorem[12] to the function $\alpha(s)/s^k$ in the upper half-plane to conclude that $C_+ = C_-$.

We could be more precise and correlate these assumptions to the behavior of $\alpha(s)$ below threshold. We think, however, to meet the reader's interest by refraining to do that and sending him instead to Ref. (13), where all these questions are dealt with exhaustively. Sufiice it to say that the same results would obtain for a function like $\alpha(s)/s^k\,(\ln s)^l\,(\ln\ln s)^m$.

3. We may start to prepare the ground for our model. First, a further consequence of the Phragmén-Lindelof's theorem is that $\alpha(s)/s^k$ is bounded along any direction, so that it is possible to write a dispersion relation for $\alpha(s)$, provided enough subtractions are made. We are interested in infinitely rising trajectories that approach a linear function of s, so we will subtract the dispersion relation once, namely,

$$\alpha(s) = \alpha(s_0) + \frac{s-s_0}{\pi}\int_{s_0}^{\infty} ds' \frac{\operatorname{Im}\alpha(s')}{(s'-s_0)(s'-s)} \tag{15}$$

The essential input will be the width function. When the real part of the trajectory is rising and, for a real s above threshold, takes an integral value of correct parity, the partial-wave amplitude has a Breit-Wigner form with a width given by

$$\Gamma(s) = \frac{\mathrm{Im}\,\alpha(s)}{\sqrt{s}\,\mathrm{Re}\,\alpha'(s)} \tag{16}$$

It is assumed that this function interpolates smoothly the widths of the several resonances that lie on the trajectory. By studying the variation of these widths with s one can then, in principle, determine $\Gamma(s)$. In the present situation, there is no definite experimental knowledge about $\Gamma(s)$, though many functional dependences are clearly ruled out[14]. We can, however, show that, under the hypotheses made, some conclusions about the behavior of $\Gamma(s)$ for large s follow.

As we are considering infinitely rising trajectories, we write

$$\lim_{s \to -\infty} \frac{\alpha(s)}{(-s)^k} = -A. \tag{17}$$

The Phragmén-Lindelofs theorem asserts that the same value is the limit for $s \to + x$. It follows. then, that

$$\alpha(s) \underset{s \to +\infty}{\sim} -A e^{-i\pi k} s^k, \tag{18}$$

that is,

$$\mathrm{Re}\,\alpha(s) \underset{s \to + w}{\sim} -A \cos(\mathrm{nk}) s^k,$$

$$\mathrm{Im}\,\alpha(s) \underset{s \to +\infty}{\sim} A \sin(\pi k) s^k. \tag{19}$$

The requirement that $\mathrm{Im}\,\alpha(s) > 0$ which, in potential theory, is a consequence of unitarity, gives

$$0 < k < 1, \tag{20}$$

or other intervals which are not of interest to our model.

A prediction for the asymptotic behavior of the width function follows at once from the use of (16) and (19). A simple computation gives

$$\Gamma(s) \underset{s \to +\infty}{\sim} -\mathrm{tg}(\pi k) \sqrt{s}, \tag{21}$$

which is the asymptotic behavior we looked for. Observe that the requirement that $\Gamma(s) > 0$ gives us a further restriction on k, namely,

$$1/2 < k < 1. \tag{22}$$

We reached therefore the conclusion that, whatever the functional dependente of $\Gamma(s)$ is, it must behave, for large s, as a square root. This is a very useful result, as these asymptotic regions are, obviously, outside of the range of the experiments, so that (21) is needed at least to supplement experimental data. Remark that (21) is true independently of the positivity requirements which gave origin to (20) and (22) as well as from the particular assumption (14), in the sense that logarithmic factors may be introduced in the asymptotic behavior[15] of $\alpha(s)$.

Now, once we know $\Gamma(s)$ we can use (16) to transform (15) into an integro-differential equation. Let us rewrite (15) as follows:

$$\mathrm{Re}\,\alpha(s) = \alpha(s_0) + \frac{s-s_0}{\pi} \int_{s_0}^{\infty} ds' \frac{\mathrm{Im}\,\alpha(s')}{(s'-s_0)(s'-s)}\,. \tag{23}$$

where the integral is now a principal value.

It is convenient to introduce the function

$$g(s) = \sqrt{s}\,\Gamma(s), \tag{24}$$

which allows us to get, from (23),

$$\mathrm{Re}\,\alpha(s) = \alpha(s_0) + \frac{s-s_0}{\pi} \int_{s_0}^{\infty} ds' \frac{g(s')\,\mathrm{Re}\,\alpha'(s)}{(s'-s_0)(s'-s)} \tag{25}$$

An equivalent equation, obtained by first differentiating (23) and then using (24) is

$$\mathrm{Im}\,\alpha(s) = \frac{g(s)}{\pi} \int_{s_0}^{\infty} \frac{ds'}{s'-s} \frac{d}{ds'} [\mathrm{Im}\,\alpha(s')]. \tag{26}$$

We make now an explicit assumption as to the form of $g(s)$, namely,

$$g(s) = \gamma(s-s_0), \tag{27}$$

where γ is a constant. This has, of course, the correct asymptotic behavior and is compatible with the experiments in the accessible region[16]. Using a Hilbert transform table[17] it is not difficult to find the solution

$$\operatorname{Im} \alpha(s) = \frac{\gamma A}{\varepsilon} (s - s_0)^{\varepsilon}, \tag{28}$$

for $s > s_0$, giving

$$\operatorname{Re} \alpha(s) = \frac{A}{\varepsilon^2} (s - s_0)^{\varepsilon} + \alpha(s_0). \tag{29}$$

For $s < s_0$, eq. (26) gives

$$\alpha(s) = \frac{A}{\varepsilon^2 \cos \pi\varepsilon} (s_0 - s)^{\varepsilon} + \alpha(s_0). \tag{30}$$

Consistency of (28) with (26) requires that

$$-\varepsilon \cot(\pi\varepsilon) = \frac{1}{\delta} \cdot \tag{31}$$

As the width must be positive, a further condition obtains:

$$\varepsilon \cot(\pi\varepsilon) < 0, \tag{32}$$

that is,

$$1/2 < \varepsilon < 1, \tag{33}$$

which is precisely condition (22).

We would like to have resonances with a well defined spin. This means that, at an energy in which the partial-wave amplitude resonates, the trajectory has a value

$$\alpha(s) = l + i\gamma, \tag{34}$$

the imaginary part γ being small compared to the real part. It is easy to see that this is equivalent to requiring ε to be near 1. This is, of course, what is strongly suggested by experiments: trajectories that are very nearly straight lines.

Another free parameter we have is the quantity $\alpha(s_0)$. To get some information about it, let us compare our trajectory, near threshold, to the threshold behavior found, on general grounds, by Barut and Zwanziger[18],

$$\operatorname{Im} \alpha(s) \simeq (s - s_0)^{\alpha(s_0) + 1/2} \sin \pi[\alpha(s_0) + 1/2]. \tag{35}$$

It is extremely interesting that the requirement that ε be nearly one has here the consequence that $\alpha(s_0)$ be nearly 1/2. If we have a trajectory with a low threshold (for instance, the ρ trajectory) we may take $\alpha(0) \simeq \alpha(s_0)$, getting the result

$$\alpha(0) \simeq 1/2, \tag{36}$$

in good agreement with experiments. In fact, the last determinations of the ρ-trajectory (the effective Regge trajectory as measured in $\pi^- p \to \pi^\circ n$) give[19] an intercept

$$\alpha(0) \simeq 0.55. \tag{37}$$

4. The threshold behavior may be used in a somewhat more general way to obtain information about the width function. The idea is using, in equation (26), an asymptotic theorem due to Tricomi[20] which allows one to compute the asymptotic behavior of a Cauchy integral at the integration limits, provided we know the behavior of the integrand at these limits. Using Eq. (35) we can then compute the asymptotic behavior of the integral at Eq. (26). Eq. (35) is used again to write $\operatorname{Im} \alpha(s)$ in the left-hand side of the equation. Consistency determines, therefore, the threshold behavior of $g(s)$. The asymptotic theorem essentially asserts that, if in the Cauchy integral

$$f(t) = \frac{1}{\pi} \int_{x_0}^{\infty} \frac{\phi(x)}{x-t}\, dx \tag{38}$$

whe have

$$\phi(x) \sim A(x-x_0)^{-\alpha} \tag{39}$$

with $0 < \alpha < 1$, then

$$f(x) \underset{x \to x_0}{\sim} A \cot(\alpha\pi)(x-x_0)^{-\alpha}. \tag{40}$$

Applied to our case, it gives, after a very simple computation,

$$g(s) \underset{s \to s_0}{\sim} -\frac{\tan \pi[\alpha(s_0) + 1/2]}{\alpha(s_0) + 1/2}(s-s_0). \tag{41}$$

Since $g(s)$ must be positive, it is necessary that

$$0 < \alpha(s_0) < 1/2. \tag{42}$$

We see, in this way, that our choice of $g(s)$ is favored by theoretical arguments, also near threshold. The quantity a(~,) is restricted to an interval that makes sense, on the light of experimental data, and, of course, tends to 1/2 if the trajectory is assumed to be approximately linear.

Let us sum up everything. In the search of the "next simplest choice" for a Regge trajectory to replace the unrealistic straight-lines suggested by dual bootstraps and the Veneziano model we, after insisting in keeping the relevant analyticity and unitarity, ended up with a trajectory which can approximate arbitrarily well the linear ones, yet interpolate finite-width resonances. As an extra prize we got a correlation between the asymptotic behavior of the trajectory and its intercept: as the trajectory gets closer and closer to a straight-line, the intercept approaches the value 1/2, characteristic of the dominant non-strange mesonic trajectones. In face of these features we think, and propose, that a trajectory of thís kind be given a try in the construction of dual amplitudes with non-linear trajectories, such as those by Baker and Coon[21].

We wish to acknowledge the very strong influence Enrico Predazzi had upon this work, as well as upon this whole line of research started and developed by him Once more it is my pleasure to thank my friend and teacher.

References and Notes

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