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Cosmic Rotation and Mach's Principle in Three Special Cosmological Models

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A critica1 analysis is developed of the role played by Mach's principle m the formulation of the general theory of relativity. It is suggested that Whitrow's relation for closed spaces represents mathematically the principle considered Lineanzed solutions of the field equations as regards cosmic rotation are analyzed for the three possible static cosmological models in order to test the possibility of Mach's principle. The non-empty deSitter and Einstein models satisfy the two fundamental requirements of Mach's principle, whereas the asymptotic Euclidian models do not.

Desenvolve-se uma análise crítica do papel que representa o princípio de Mach na formulação da teoria da relatividade geral. Sugere-se que a relação de Whitrow para espaços fechados possa representar matematicamente o princípio de Mach. Soluções para as equações de campo linearizadas com respeito à rotação cósmica são analisadas para os três possíveis modelos cosmológicos estáticos a fim de se testar o princípio de Mach. Os modelos não-vazios de deSitter e Einstein satisfazem às duas exigências fundamentais do princípio de Mach, enquanto que não as satisfazem os modelos assintoticamente euclidianos.

1. Mach's Principle, the Equivalence Principle, the Generalized Principle of Relativity and Space-Time Curvature

The usual formulation of Mach's principle is actually due to Einstein's analysis' of the objection raised by Mach[2] against Newton's concept of absolute space. This will be explained in the following considerations of the basis of the general principle of relativity and the importance of Mach's principle.

The concept of an absolute reference system was exhibited by Newton[3] in his *Principia* as something that could explain the privileged character of non-inertial frames. The distinction between an accelerated frame and an inertial one could be accounted for through the existence of an absolute space independent of matter. Such a statement could bypass the difficulty exhibited by the principle of relativity which, in the domain of classical mechanics, is not sufficient by itself to make the existence of accelerated reference systems meaningful in a privileged sense. Newton's bizarre concepts of absolute space and absolute time independent of matter was at variance with the physical status proper to space and time, which cannot be conceived of as entirely independent realities where another reality, matter, is imbedded. It was the merit of Mach to have pointed out the bizarre status of these fundamental concepts of classical mechanics. The notion of reference system connot be disconnected from the idea of a physical object, and so an absolute frame disconnnected from matter is an empty concept, at least in the domain of physics. To overcome the difficulty of the privileged character of accelerated frames, Mach substituded the distribution of masses in the universe for Newton's absolute space.

With the idea of a cosmic inertial reference system to which the accelerated frames are referred, the cosmic distribution of masses would permit the possibility of distinguishing between inertial and accelerated frames.

Einstein, dealing with Mach's critique of Newton, added another objection against the notion of absolute space and time: accelerated frames, rotating objects, for instance, are the site of inertial forces. Since absolute empty space has the role of determining the difference between these frames and the inertial ones, it comes about that absoiute empty space is a kind of cause to inertial forces. But such a situation violates the principle of causality. In physics, a cause devoid of matter is not meaninful. Therefore inertial forces should be viewed as an effect of an interaction between local accelerated objects and the distribution of cosmic distant masses.

Einstein's insight into the problem is essentially centered on the idea of causality, which was not the case with Mach's critique. The usual formulation of Mach's principle is in effect derived from Einstein's analysis and it is only indirectly Mach's. Hence, two fundamental assumptions constitute Mach's principle as interpreted by Einstein:

a. Accelerated frames are distinguished from inertial ones through a cosmic reference system defined by the distribution of the distant masses in the universe;
b. Inertial forces appear in accelerated objects due to interaction of these with the distant cosmic masses. Inertial effects are connected to gravitational interaction.

A straightforward conclusion of Mach's principle is the non-existence of privileged reference systems. The systems of reference are defined through the existence of ponderable bodies and so no absolute reference frame independent of matter should be postulated. Since this must be the case in order to save the validity of the principle of causality, mechanical laws should be universally valid, i.e., valid for any reference system. This situation does not appear in classical mechanics, where Newton's laws are applicable only with respect to inertial frames. The argument developed by Einstein poses the necessity for a generalization of the principle of relativity. namely: the laws of mechanics must be invariant for arbitrary coordinate transformations.

It is worthwhile to note that Mach's principle in its original presentation has a philosophical status and no mathematical formulation. This should be kept in mind. For, as it will be discussed later, it is one of the difficulties of Mach's principle in the domain of the general theory of relativity.

Einstein reinforced his argument for a general principle of relativity appealing to the principle of equivalence: inertial and gravitational forces are interchangeable through suitable coordinate transformations.

This principle has an experimental basis in Roland von Eotvos work[4]. where it is verified, with an accuracy of the order of 5 x 10^{-9}, that inertial and gravitational effects are independent of the nature of ponderable bodies. Most recently, Robert Dicke[5] repeated von Eötvös experiment. with a higher accuracy (of the order of 10^{-11}), confirming the previous results. Dicke has pointed out that the principle of equivalence based on the highly accurate Eotvos experiment can be viewed as the fundamental argument for geometnzation of inertia and gravitation, i.e., as the basis för the'general theory of relativity. This is indeed good for the theory, since the classical experimental tests are very poor due to their low accuracy. Furthermore, the classical tests verify consequences which can be interpreted in terms of theories other than Einstein's. Following Dicke, we put forward an argument in modified terms as follows.

According to Eötvös' experiment, inertial and gravitational effects are independent of nuclear, electromagnetic and weak coupling, i.e., independent of the nature of the bodies. This fact suggests a generalization of the principle of inertia: the path of a body under inertia and gravitation is uniquely deterrnined by the space-time geometry. The principle of equivalence gives to inertia and gravitation a sweeping universality of the same level as that of space-time, since both orders of physical facts are independent of the nature of the bodies. Therefore, space-time and gravitation should be connected, and the path of a body *solely* under inertia and gravitation should be uniquely determined by the metrical field of space-time. This conclusion cannot be formalized in terms of Euclidean geometry, which can afford universal laws only for inertial reference frames. A Riemannian manifold, in which the metric field $g_{\mu\nu}$ takes on inertial and gravitational meanings, seems to be the most reasonable consequence. The generalized principle of inertia could be expressed in terms of geodesics in a Riemannian 4-space.

The generalized principle of relativity, i.e., the equivalence of all inertial and accelerated frames, is expressed through the invariance of Riemann's space-time metric form,

$$ds^2 = g_{\mu\nu} dx^p dx^\nu . \tag{1}$$

The generalized principle of inertia is given by

$$\delta \int g_{\mu\nu} dx^\mu dx^\nu = 0, \tag{2}$$

which gives the geodesic equations

$$\frac{d}{ds}(g_{\mu\nu} u^\nu) - \frac{1}{2} g_{\nu\alpha,\mu} u^\nu u^\alpha = 0. \tag{3}$$

The geodesic equations (3) present a generalized equivalence between inertial and gravitational forces.

The connection of space-time with gravitation, that is, of geometry with matter, is made explicit through Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = -\kappa T_{\mu\nu}, \tag{4}$$

where $T_{\mu\nu}$ is the matter energy-tensor and $R_{\mu\nu}$ the space-time Ricci tensor. It is interesting to point out that the Einsteinian outlook on space-time restates ancient philosophical concepts of the Greeks: for Plato[6] and Aristotle[7], space and time are inseparable from matter and a pure space void is unthinkable.

Mach's principle, important as it is for the origin of Einstein's general theory of relativity, is nonetheless not completely manifested in the solutions of Einstein's field equations. The Gadel cosmological model[a], for instance, exhibits a constant rotation everywhere and no inertial frame to which is related such a rotation. Furthermore, the energy-momentum tensor for this model is the same as that for Einsteins's model universe, which means that the field equations do not determine uniquely the inertial properties of a test body[9]. Another similar example is that of de Sitter's empty static universe. An inertial field can be obtained for this model, notwithstanding the fact that no cosmic inertial frame can be defined, since there is no matter. Certain ideas have been put forward to clear up this weak point of the theory, for example, Brans and Dicke's modified field equations[10] and Wheeler's boundary conditions[11].

In Wheeler's theory, the boundary conditions restrict the solutions of the field equations to those which define spaces with positive curvature, that is, closed universes. This result reinforces Einstein's suspicion[12] that only closed spaces satisfy Mach's principle. We may arrive at this result perhaps by a more direct reasoning as follows.

Whitrow had suggested that a relation of the form

$$(GM/c^2 r) \sim 1 \qquad (5)$$

defines Mach's principle. In this relation, M is the mass contained within the observable radius of the universe, r. This relation was deduced in a straightforward way by Whitrow from the equivalence of inertia with gravitation for cosmic masses:

$$Mc^2 \sim GM^2/r. \qquad (6)$$

An objection against equation (5) is that variations in M due to clumpiness in the distribution of galaxies, for instance, may violate the relation. To overcome this difficulty, Dicke[10] has proposed a variable G. Another objection may be raised, however, as follows: relation *(5)* is not a determined law, for it is based on the concept of a fiat space, i.e., an infinite universe, in which boundary conditions are not well defined. Relation *(5)* is established only by disregarding the non-observable infinite mass. This objection can be bypassed through a finite world model, i.e., a closed universe. If R is the "radius" of curvature of the closed space and M the total mass of the universe, the total gravitational energy of the universe may be defined as

$$GM^2/aR, \qquad (7)$$

where a is a constant which depends on the model.

Using Whitrow's argument, Mach's principle may then be stated as follows:

$$GM/c^2 R = \alpha \sim \mathrm{i}. \qquad (8)$$

This relation is verified in two cosmological models, the Einstein model and the de Sitter non-empty static universe. The non-empty de Sitter model has the particular feature of exhibiting a negative pressure. This non-classical concept, which led de Sitter to reject the non-empty model, is nonetheless capable of an interpretation in the domain of general relativity, according to McCrea's analysis[14] of the energy-momentum tensor in his reformulation of Hoyle's model universe.

Note that in the aforementioned world models, relation **(8)** comes from the field equations. With the usual expanding models this is not so. Thus, if Mach's principle is to be built into the field equations of the expanding models, new physical concepts must be introduced. For example, Jordan[15], in order to keep relation (8) valid for expanding models, made M and G functions of the cosmic time.

Another possibility would be M only as a function of cosmic time, a hypothesis which we have considered recently in a finite expanding model universe with matter injection[16].

2. Cosmic Rotation for the Three Possible Static Homogeneous Models. The Solutions Compaaible with Mach's Principle

There are three possible homogeneous static universes: Euclidean, Einstein's and de Sitter's''. The last two obey Mach's principle as presented in relation (8). In this section we analyse the behavior of solutions of cosmic rotation for the three models. The procedure is equivalent to that developed by Lausberg[18], i.e., it is assumed that the rotation is sufficiently small, such that the diagonal Einstein equations for the three models are not perturbed, that is, they are kept linear, giving the known solutions of the three non perturbed models. Thus, the perturbed metric should be:

$$ds^2 = -\left[e^{\nu} dr^2 + r^2 \sin^2\theta (d\phi - \Omega\, dt)^2 + r^2 d\theta^2\right] + e^{\lambda} c^2 dt^2, \qquad (9)$$

where $\Omega(\theta, r)$ is the rotational perturbation.

For the Euclidean, Einstein and de Sitter universes we have, respectively,

$$e^{\nu} = e^{\lambda} = 1, \qquad (10)$$

$$e^{\nu} = [1-(r/R)^2]^{-1}, \; e^{\lambda} = 1, \qquad (11)$$

$$e^{\nu} = e^{-\lambda} = [1-(r/R)^2]^{-1}. \qquad (12)$$

2.1. Rotation and Mach's Principle for the Euclidean Model

The covariant and contravariant components of the metric tensor for this case are:

$$
\begin{aligned}
g_{11} &= -1, & g^{11} &= -1, \\
g_{22} &= -r^2, & g^{22} &= -r^{-2}, \\
g_{33} &= -r^2 \sin^2\theta, & g^{33} &= -\frac{c^2 - r^2\Omega^2\sin^2\theta}{r^2c^2\sin^2\theta}, \\
g_{03} &= g_{30} = r^2\Omega\sin^2\theta, & g^{03} &= \Omega c^{-2}, \\
g_{00} &= c^2 - r^2\Omega^2\sin^2\theta, & g^{00} &= c^{-2}.
\end{aligned}
\tag{13}
$$

The surviving Christoffel symbols for the three models are:

$$
\begin{aligned}
&\left\{\begin{matrix}1\\03\end{matrix}\right\}, \quad \left\{\begin{matrix}1\\00\end{matrix}\right\}, \quad \left\{\begin{matrix}1\\33\end{matrix}\right\}, \quad \left\{\begin{matrix}1\\22\end{matrix}\right\} \\
&\left\{\begin{matrix}2\\00\end{matrix}\right\}, \quad \left\{\begin{matrix}2\\03\end{matrix}\right\}, \quad \left\{\begin{matrix}2\\12\end{matrix}\right\}, \quad \left\{\begin{matrix}2\\33\end{matrix}\right\} \\
&\left\{\begin{matrix}3\\23\end{matrix}\right\}, \quad \left\{\begin{matrix}3\\13\end{matrix}\right\}, \quad \left\{\begin{matrix}3\\20\end{matrix}\right\}, \quad \left\{\begin{matrix}0\\13\end{matrix}\right\} \\
&\left\{\begin{matrix}0\\13\end{matrix}\right\}, \quad \left\{\begin{matrix}0\\01\end{matrix}\right\}.
\end{aligned}
\tag{14}
$$

We assume $r\Omega/c \ll 1$ such that the diagonal equations are kept unaltered, as it was pointed out before: in other terms, we are linearizing the problem. To find solutions for $\Omega(\theta, r)$ in the present context is to solve the non-diagonal equation:

$$\kappa\left(T_{03} - \frac{1}{2}g_{03}T\right) = g_{03}\Lambda - R_{03}. \tag{15}$$

Since in the present analysis the diagonal equations are maintained unaltered through the assumption of the very small perturbation considered before, then for the Euclidean model the following relations are yet valid:

$$\rho + (p/c^2) = 0, \qquad \Lambda = \kappa\rho/c^2 \tag{16}$$

and it follows:

$$T_{03} = -(p/c^2)g_{03}. \tag{17}$$

Therefore,

$$R_{03} = 0 \tag{18}$$

Bearing in mind that

$$\frac{\partial}{\partial x^0}\left\{ {\tau \atop 3\quad\tau} \right\} = \frac{\partial}{\partial t}\left\{ {\tau \atop 3\quad\tau} \right\} = 0,$$

we have:

$$R_{03} = -\frac{\partial}{\partial x^\sigma}\left\{ {\sigma \atop 0\quad 3} \right\} + \left\{ {\sigma \atop 0\quad\tau} \right\}\left\{ {\tau \atop 3\quad\sigma} \right\} - \left\{ {\sigma \atop 0\quad 3} \right\}\left\{ {\tau \atop \sigma\quad\tau} \right\}. \tag{19}$$

The first term on the right hand side of (19) is

$$\frac{\partial}{\partial x^\sigma}\left\{ {\sigma \atop 0\quad 3} \right\} = \frac{\partial}{\partial r}\left\{ {1 \atop 0\quad 3} \right\} + \frac{\partial}{\partial \theta}\left\{ {2 \atop 0\quad 3} \right\}, \tag{20}$$

while the two last terms are

$$\left\{ {\sigma \atop 0\quad\tau} \right\}\left\{ {\tau \atop 3\quad\sigma} \right\} - \left\{ {\sigma \atop 0\quad 3} \right\}\left\{ {\tau \atop \sigma\quad\tau} \right\} = \left\{ {3 \atop 0\quad 1} \right\}\left\{ {1 \atop 3\quad 3} \right\} + \left\{ {3 \atop 0\quad 2} \right\}\left\{ {2 \atop 3\quad 3} \right\}$$

$$+ \left\{ {1 \atop 0\quad 0} \right\}\left\{ {0 \atop 3\quad 1} \right\} + \left\{ {2 \atop 0\quad 0} \right\}\left\{ {0 \atop 3\quad 2} \right\} - \left\{ {1 \atop 0\quad 3} \right\}\left\{ {2 \atop 1\quad 2} \right\}. \tag{21}$$

Developing (20) and (21) according to (14), the following differential equation is obtained:

$$R_{03} = \frac{\sin^2\theta}{2}\left[r^2\frac{\partial^2\Omega}{\partial r^2} + 4r\frac{\partial\Omega}{\partial r}\right] + \frac{1}{2}\left[\sin^2\theta\frac{\partial^2\Omega}{\partial\theta^2} + 3\sin\theta\cos\theta\frac{\partial\Omega}{\partial\theta}\right] = 0. \tag{22}$$

Putting

$$\Omega = \phi(r)\psi(\theta), \tag{23}$$

we get two differential equations with Γ as the constant of separation:

$$\begin{aligned} r^2\frac{d^2\phi}{dr^2} + 4r\frac{d\phi}{dr} + \Gamma\phi &= 0, \\ \frac{d^2\psi}{d\theta^2} + 3\cot\theta\frac{d\psi}{d\theta} - \Gamma\psi &= 0. \end{aligned} \tag{24}$$

The second equation is an associated Legendre differential equation if we put

$$\Gamma = 2 - l(l+1), \qquad l = 1, 2, 3, \dots . \tag{25}$$

Relation (25) is highly convenient since it affords solutions with no singularities in the whole domain of values for θ. Solutions for this equation are of the type

$$\psi(y) = \frac{d}{dy} P_l(y), \qquad y = \cos\theta. \tag{26}$$

It is important to note that we have arrived at equations (24) by neglecting non linear terms.

The first equation is of the Cauchy type, which gives the general solution

$$\phi(r) = Ar^{l-1} + Br^{-(l+2)}. \tag{27}$$

The light postulate demands the condition

$$\lim_{r\to\infty} \Omega r < c. \tag{28}$$

Comparing (28) with (25), it follows that

$$A = 0, \qquad \phi(r) = Br^{-(l+2)}. \tag{29}$$

Inspection of (29) shows that for small values of r, the linear approximation breaks down. Non linear equations involving exp v, exp λ and R are required. The solution can no longer be a flat space-time. We infer from this that cosmic rotation is possible in so far as the model universe is asymptotically Euclidean. It is clear that, for $r \to \infty$, $\Omega \to 0$ and this. means that the Machian reference frame is given by masses at infinity. Notwithstanding the validity of the first item for Mach's principle, the second, gravitational interaction, cannot be verified for this universe. On the other hand, if a universe everywhere Euclidean is assumed, only the following trivial solution could be possible:

$$\Omega(r, \theta) \equiv 0, \tag{30}$$

which means that for a completely flat space-time, inertial forces could not exist. Such a universe cannot be Machian either.

2.2. Rotation and Mach's Principle for the Non-Empty de Sitter Static Model

Making

$$r/R = \sin\chi, \tag{31}$$

the perturbed metric for this model is

$$ds^2 = -R^2[d\chi^2 + \sin^2\chi\sin^2\theta(d\phi - \Omega dt)^2 + \sin^2\chi d\theta^2] + c^2 e^{\lambda} dt^2,$$
$$e^{\lambda} = \cos\chi. \tag{32}$$

The covariant and contravariant components of the metric tensor are

$$\begin{aligned}
g_{11} &= -R^2, & g^{11} &= -R^{-2},\\
g_{22} &= -R^2\sin^2\chi, & g^{22} &= -(R\sin\chi)^{-2}\\
g_{33} &= -R^2\sin^2\chi\sin^2\theta & g^{33} &= -\frac{c^2 e^{\lambda} - R^2\Omega^2\sin^2\chi\sin^2\theta}{R^2\sin^2\chi\sin^2\theta e^{\lambda}c^2},\\
g_{03} &= g_{30} = R^2\Omega\sin^2\chi\sin^2\theta, & g^{03} &= \Omega e^{-\lambda}c^{-2},\\
g_{00} &= [c^2 e^{\lambda} - R^2\Omega^2\sin^2\chi\sin^2\theta], & g^{00} &= e^{-\lambda}c^{-2}.
\end{aligned} \tag{33}$$

Developing the surviving symbols by use of (33) and neglecting non-linear terms, we obtain

$$\begin{aligned}
R_{03} = &-\frac{1}{2}(\sin\theta\sin\chi)^2\frac{\partial^2\Omega}{\partial x^2} - \frac{1}{2}\sin^2\theta\frac{\partial^2\Omega}{\partial\theta^2}\\
&-\left[2\sin\chi\cos\chi\sin^2\theta + \frac{1}{2}\sin^2\chi\tan\chi\sin^2\theta\right]\frac{\partial\Omega}{\partial\chi}\\
&-\frac{3}{2}\cos\theta\sin\theta\frac{\partial\Omega}{\partial\theta} + [2\sin^2\chi\sin^2\theta - \cos^2\chi\sin^2\theta]\Omega + \Omega\sin^2\theta.
\end{aligned} \tag{34}$$

For the de Sitter model,

$$1/R^2 = \frac{1}{3}(\Lambda + \kappa\rho), \qquad \rho + (p/c^2) = 0. \tag{35}$$

Combining (35) with (34) and using separation of variables as in (23), we get the two differential equations

$$\begin{aligned}
&\sin^2\chi\frac{d^2\phi}{d\chi^2} + (4\sin\chi\cos\chi + \tan\chi\sin^2\chi)\frac{d\phi}{d\chi} + \Gamma\phi = 0,\\
&\frac{d^2\psi}{d\theta^2} + 3\cot\theta\frac{\partial\psi}{\partial\theta} - \Gamma\psi = 0.
\end{aligned} \tag{36}$$

The last equation is the same as *(24).* Again we assume relation *(25).*

The first equation can be put in the form

$$x^2(1-x^2)\frac{d^2\phi}{dx^2} + 4x(1-x^2)\frac{d\phi}{dx} + \Gamma\phi = 0, \tag{37}$$

with $x = \sin\chi$.

This equatioa can be solved readily for $l = .1$, which gives, as we shall see, a solution that obeys Mach's principle. The solution considered is:

$$\phi = A + Bx^{-3}. \tag{38}$$

In this case,

$$\psi = \text{const} = 1.$$

For $x \to 0$, the linear approximation is no longer valid and it is necessary to treat the problem in terms of non linear equations.

Using the boundary condition

$$A = -B, \tag{39}$$

it can be seen that for the event horizon $(x = 1)$ we have $R = 0$ and this defines an inertial region. Such regions represent Mach's cosmic inertial reference system, to which accelerations are referred.

Since

$$\rho = M(2\pi^2 R^3)^{-1}, \tag{40}$$

making $A = 0$ in (35) yields

$$R = A[1-(\kappa M/6\pi^2)^3 r^{-3}]. \tag{41}$$

The constant A depends on the cosmic angular momentum L and cosmic mass M, as

$$L = \int (r\sin\theta)^2\, T^{03}\sqrt{-\bar{g}}\,dV$$

$$= R^5\rho\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\pi}\Omega(\chi)\exp(3\nu/2)\sin^4\chi\sin^3\theta d(\sin\chi)d\theta d\phi.$$

The integral is finite and it can be shown that

$$A = (18\pi^4 L)/(\kappa^2 M^3).$$

It is apparent from the above relation that cosmic inertial forces depend on the total mass of the universe through the gravitational constant κ. This consequence expresses the second item of Mach's principle, namely, that inertial forces arise as a result of the gravitational interaction. This is not the case when an asymptotic Euclidean universe is considered, as can be seen by the preceding analysis. Therefore, it seems that curvature plays an essential role for the possibility of Mach's principle, and Whitrow's relation (8) should be considered seriously.

23. Rotation and Mach's Principle for the Einstein Model

For the Einstein model, we have recourse to relations (31), (32) and (33) with the condition $\exp\lambda = 1$.

Following a similar development, the next differential equations obtained are

$$\sin^2\chi \frac{d^2\phi}{d\chi^2} + 4\sin\chi\cos\chi\frac{d\phi}{d\chi} - 4\left(\sin^2\chi - \frac{\Gamma}{4}\right)\phi = 0,$$

$$\frac{d^2\psi}{d\theta^2} + 3\cot\theta\frac{d\psi}{d\theta} - \Gamma\psi = 0. \tag{42}$$

The second equation is again equal to those of the two former cases. Equations (42) were already given by Lausberg[18] in dealing with the Lense-Thirring effect in the Einstein universe.

For 1 = 1, the general solution for the first equation is

$$\phi(\chi) = A\frac{\cot\chi}{\sin^2\chi} + B\frac{1-\chi\cot\chi}{\sin^2\chi}. \tag{43}$$

For the solution of the Lense-Thirring problem, Lausberg[18] gave the boundary condition

$$A = \pi B. \tag{44}$$

Using (44), the Machian inertial region would be given by the relation

$$\cot\chi = \frac{1}{\chi-\pi}, \quad \frac{\pi}{2} < \chi < \pi. \tag{45}$$

For the Einstein model, the following relations hold:

$$1/R^2 = \frac{\kappa}{2}[\rho + (p/c^2)], \tag{46}$$

$$\Lambda = \frac{\kappa}{2}[\rho + 3(p/c^2)].$$

Making the pressure zero and using relation (40), it follows that

$$R = \kappa M/4\pi^2. \tag{47}$$

This relation can be substituted into *(43)* and, *as* in the former case, it is clear that the inertial effect is intimately related to the total mass of the universe. The two items which define Mach's principle appears to be verified for the Einstein model also.

3. Conclusion

If we are to accept Mach's principle as formulated in Whitrow's modified relation (8), the two possible static closed homogeneous world models realize its fundamental properties, at least in the domain of the special solutions for very small cosmic rotations. The Euclidean static model and the asymptotic Euclidean model are not Machian universes in this sense.

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