Revista Brasileira de Física, Vol. 2, N.° 3, 1972

Dissipation of Charge Layers in Dielectrics*

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Recebido em 28 de Julho de 1972

It is studied the dissipation of uniformly charged layers (box distribution) in solids with an intrinsic conductivity. It is shown that an exact mathematical solution can be found, which is particularly simple in the short circuited case.

Estuda-se a dissipação de camadas uniformemente carregadas (distribuição tipo caixa) em sólidos dotados de condutividade intrínseca. É mostrado que, no presente caso, é possível se achar uma solução matemática exata, a qual é particularmente simples quando os eletródios estão em curto.

1. Introduction

In the present article, we study the motion of free space charge in dielectric solids, with prescribed initial conditions (box distribution at zero time). The subject has already been partially treated in the literature by Wintle[1], but we think that the method of solution proposed here is simpler even in those points covered by Ref. 1. Plane symmetry and a medium with intrinsic conductivity are assumed. Although reference is made to free space charge, it is known that the results we will obtain can also be applied to solids with a system of single shallow traps[2].

2. Excess Charge Touching One of the Electrodes

Suppose we have initially a uniformly distributed excess of charge ρ_0,

*Work supported by BNDE/FUNTEC, CNPq, FAPESP and CNEN.

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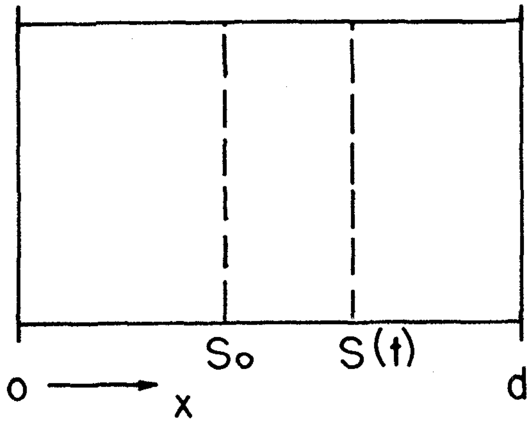

Fig. 1.

occupying a distance s_0, $s_0 < d$, d being the thickness of the dielectric sample (Fig. 1). The dieletric can be said to have an intrinsic conductivity a, in such a way that the total conduction current is given by

$$i(x,t) = \mu\rho(x,t)\,E(x,t) + \sigma E(x,t), \tag{1}$$

$\rho(x,t)$ and $E(x,t)$ being the density of charge and electric field at point x and time t; μ is the mobility of the excess charge. From the assumed initial conditions, we have

$$\rho(x,0) = \rho_0, \qquad 0 < x < s_0,$$
$$\rho(x,0) = 0, \qquad s_0 < x < d.$$

We suppose also a constant applied voltage V:

$$\int_0^d E(x,t)\,dx = \mathrm{V}. \tag{2}$$

We want to show that, with the given initial and boundary conditíons, the exact solution of the Poisson and continuity equations is given by the box charge distribution, that is, a uniform time-varying density of charge $\rho(t)$ (independent of x), spreading from $x = 0$, till $x = s(t)$.

The solution of the pertinent equations should give p and s as functions of time.

The Poisson and continuity equations are:

$$\varepsilon \frac{\partial E}{\partial x} = \rho, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial i}{\partial x} = 0,$$

with $i(x, t)$ given by (1), ε being the permissivity.

Assume a solution of the form p = $\rho(t)$, for $0 < x < s(t)$. We have then

$$\frac{d}{dt}\rho + \frac{\mu}{\varepsilon}\rho^2 + \frac{\sigma}{\varepsilon}\rho = 0. \tag{3}$$

From this, $\rho(t)$ could be found as a function of $\rho(0) = \rho_0$. Integrating the continuity equation from x = 0 to x = d, we get, in general,

$$\frac{d}{dt} q = i(0, t) - i(d, t) = [\mu\rho(0, t) + \sigma]\, E(0, t) - \sigma E(d, t), \tag{4}$$

with $q(t) = \int_0^d \rho\, dx = \int_0^{s(t)} \rho\, dx$, taking into account that, at $x = d$, the excess of charge is zero, that is, the space charge has not reached this point yet.

On the other hand, assuming the box distribution, we can find $\frac{dq}{dt}$:

$$\frac{d}{dt} q = \frac{d}{dt}\rho s = \rho \frac{ds}{dt} + s\frac{d\rho}{dt} \quad .$$

Using the fact that $\frac{ds}{dt}$ is equal to $\mu E(s, t) = \mu E(d, t)$, since from $s < x < d$ the dielectric is free from charge, and using (3) for $\frac{d}{dt}\rho(t)$, we get:

$$\frac{d}{dt} q = \mu\rho E(d, t) - [\mu\rho + \sigma]\frac{q}{\varepsilon} \quad . \tag{5}$$

Now, if we substitute on **Eq.** 4 the explicit field values, namcly,

$$\begin{aligned} E_1(x, t) &= \frac{\rho x}{\varepsilon} + \frac{q}{\varepsilon d}\left[\frac{s}{2} - d\right] + \frac{V}{d}, \qquad 0 < x < s, \\ E_2(x, t) &= \frac{\rho s^2}{2\varepsilon d} + \frac{V}{d}, \qquad s < x < d, \end{aligned} \tag{6}$$

we see that Eqs. (4) and (5) agree with each other. Actually, we have supposed that the field at $x = 0$ is negative and that the charge is dissipated at the plate therein. However, it is shown in detail in Section 5, that even here the box distribution is the exact solution of the space charge problem we are dealing with.

Formally, we could also make the proof as follows. We have

$$\rho(x,t) = \theta[s(t) - x]\,\rho(t),$$
$$i(x,t) = \mu E(x,t)\,\theta[s(t) - x]\,\rho(t) + \sigma E(x,t),$$

with

$$\theta[s(t) - x] = 1, \qquad 0 < x < s(t),$$
$$\theta[s(t) - x] = 0, \qquad s(t) < x < d.$$

We then show that these functions satisfy the continuity equation. The time derivatives of ρ and i are given by

$$\frac{\partial \rho}{\partial t} = \frac{\partial \theta}{\partial s}\frac{ds}{dt}\rho + \theta\frac{d\rho}{dt} = \delta[s(t) - x]\frac{ds}{dt}\rho + \theta\frac{d\rho}{dt},$$
$$\frac{\partial i}{\partial x} = \mu\frac{\partial E}{\partial x}\theta\rho - \mu\rho E\delta[s(t) - x] + \sigma\frac{\partial E}{\partial x}\cdot$$

For $0 < x < s(t)$, we have our previous Eq. **(3).** To get a meaningful relation around $x = s$, we integrate in x in a small range 2Δ, centered at $x = s$. The finite contributions come from the terms containing delta functions:

$$\lim_{\Delta \to 0}\int_{s-\Delta}^{s+\Delta}\frac{\partial \rho}{\partial t}\,dx = \rho\frac{ds}{dt},$$

$$\lim_{\Delta \to 0}\int_{s-\Delta}^{s+\Delta}\frac{\partial i}{\partial x}\,dx = -\mu E(s,t).$$

Because $\dfrac{ds(t)}{dt}$ was assumed to be $\mu E(s,t)$, we see that the continuity equation is satisfied.

It can be shown along the same lines that $\rho(t)$ given as a solution of **(3)** is the correct one after the charge reaches the plate at d.

The integration of (3) furnishes $\rho(t)$, which can be used in the equation

$$\frac{ds}{dt} = \mu E(d, t) = \frac{\mu}{d}\left[\frac{\rho s^2}{2\varepsilon} + V\right] \tag{7}$$

to give s as a function of time.

3. Short Circuited Electrodes

The solution of Eq. (3) is independent of the boundary Eq. (2). It gives, with $\rho'(t) = \frac{\rho(t)}{\rho(0)}$ and $a = \frac{o}{\mu\rho_0}$:

$$\rho'(t) = \frac{a}{(1 + a)\exp\left(\frac{\sigma t}{\varepsilon}\right) - 1}\,. \tag{8}$$

On the other hand, the solution of (7) is facilitated if V is set equal to zero. Therefore, $\frac{ds}{dt} = \frac{\mu\rho s^2}{2\varepsilon d}$ (9) and $s(t)$ can be found in terms of $\rho(t)$, substituting dt from *(3).* We have

$$\frac{ds}{s^2} = \frac{-1}{2d}\cdot\frac{\mu d\rho}{\sigma + \mu\rho},$$

which gives, with $s(0) = s_0$,

$$\frac{1}{s} - \frac{1}{s_0} = \frac{1}{2d}\log\frac{\sigma + \mu\rho}{o + \mu\rho_0}\,. \tag{10}$$

For large enough intrinsic conductivity, the density of charge can drop to zero before the front reaches the plate at d. The distance s', when the density is zero, can be found from (9), setting p' = 0. We find

$$\frac{2d}{s'} = \frac{2d}{s} + \log\frac{a}{a + 1}\,. \tag{11}$$

However, Eq. (8) tells us that s' is reached only asymptoticaly since the required time is infinite. This is not surprising because, according to Eq. (9), $\frac{ds}{dt}$ is proportional to p.

For low conductivity, the charge reaches the plate at d in a time t_d that can be calculated from (10) and (8) putting $s = d$. We get

$$t_d = \frac{-\varepsilon}{\rho} \log\left\{1 + a\left[1 - \exp\left(\frac{2d}{s_0} - 2\right)\right]\right\}. \tag{12}$$

The external current $j(t)$, can be found using the conditions prevailing at $x = d$, that is, no conduction current from the space charge

$$j(t) = \varepsilon \frac{dE(d,t)}{dt} + \sigma E(d,t) = -\frac{\rho^2 s^2}{2\varepsilon d}\left(1 - \frac{s}{d}\right).$$

The same result is found if use is made of the expression deduced by Lindmayer, Gross and Perlman[3],

$$j(t) = -\rho(x_0, t)\frac{dx_0}{dt} = -\rho(t)\frac{dx_0}{dt},$$

where x_0 gives the position of the zero field plane.

The total charge on the external circuit is

$$Q(t) = \int_0^t j(t)\,dt = \varepsilon[E(d,t) - E(0,t)] + \frac{\sigma}{\mu}(s - s_0).$$

4. A Solution with an Applied Voltage

Here, as in the short circuited problem, the differential equation for ρ is Eq. (8). With it, we could solve Eq. (7). However, we must consider the sign of the field at $x = 0$. If it is positive, the charge will be detached from the nearby plate, drifting to the right: so, two velocities should be calculated, those of the front and of the back of the space cloud. This will be discussed more thoroughly in the next section. But we should add that our results do not agree with those reported in the literature[2]. On the other hand, if the field is negative, we see from Eq. (6) that eventually the field at $x = 0$ reaches positive values (if the conductivity is low enough), when detachment will take place. Supposing that the applied voltage is such that we do not have detachment from the plate, and low conductivity, a situation is reached in which the whole charge spreads uniformily from $x = 0$ to $x = d$. Without applied voltage this time would equal the time t_d 'calculated in Eq. (12).

Integrating the equation $j(t) = i + \varepsilon \frac{\partial E}{\partial t}$, we get the well known relation $j(t) = \frac{1}{d}\int_0^d i dx$; in our case, this gives

$$j(t) = \frac{\sigma V}{d} + \mu\rho(t)V. \tag{13}$$

Eq. (13) shows that the conductivity only adds an ohmic term to the external current. This solution is valid till the detachment of the space charge from the plate at $x = 0$.

5. Floating Space Charge

Now we consider the situation when initially a space charge of density ρ is uniformly distributed from $s_1(0)$ to $s_2(0)$, its initial thickness being $s(0) = s_2(0) - s_1(0)$; Fig. 2. The dielectric thickness is d and an applied voltage V is present.

Our proof that the box distribution is the solution of the problem is

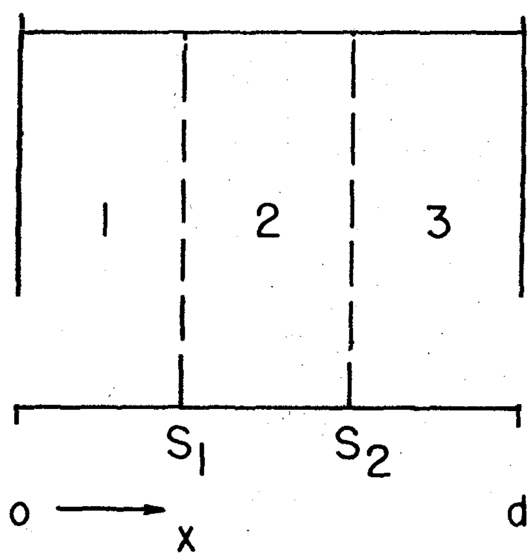

Fig. 2.

essentially the same as before. The variation of the total charge is given by Eq. (4):

$$\frac{d}{dt}q(t) = i(0,t) - i(d,t) = \sigma[E(0,t) - E(d,t)]. \tag{14}$$

On the other hand, assuming the box distribution, we can write again Eq. (3), namcly,

$$\frac{d\rho}{dt} + \frac{\mu\rho^2}{\varepsilon} + \frac{\sigma}{\varepsilon}\rho = 0. \tag{3}$$

We have the differential equations for s_1(t) and $s_2(t)$:

$$\frac{ds_2}{dt} = \mu E(d,t), \qquad \frac{ds_1}{dt} = \mu E(0,t). \tag{15}$$

The fields in the regions 1, 2, 3 (Fig. 2) are, for the box distribution, given by

$$E_1 = \frac{\rho}{2\varepsilon d}(s_2^2 - s_1^2 - 2s_2 d + 2ds_1) + \frac{V}{d}, \qquad 0 < x < s_1,$$

$$E_2 = \frac{\rho x}{\varepsilon} + \frac{\rho}{2\varepsilon d}(s_2^2 - s_1^2 - 2s_2 d) + \frac{V}{d}, \qquad s_1 < x < s_2,$$

$$E_3 = \frac{\rho}{2\varepsilon d}(s_2^2 - s_1^2) + \frac{V}{d}, \qquad s_2 < x < d.$$

Substituting the appropriate values in Eqs. (15), it follows that

$$\frac{\mathrm{ds}}{\mathrm{dt}} = \frac{ds_2}{\mathrm{dt}} - \frac{\mathrm{ds},}{\mathrm{dt}} = \frac{\mu\rho s}{\mathrm{E}}. \tag{16}$$

The time derivative of $q(t) = \rho(t)s(t) = q_0$ is

$$\frac{dq}{dt} = \rho\frac{ds}{dt} + s\frac{d\rho}{dt} = 0. \tag{17}$$

With (3) and (16) substituted here, and the field values fed into Eq. (14), it is seen that the two expressions are the same. From (16) and (3) it is possible to calculate $s(t)$. It gives

$$s - s_0 = \frac{q_0\mu}{\sigma}\left[1 - \exp\left(-\frac{\sigma t}{\varepsilon}\right)\right]$$

For vanishing o, we see that the charge spreads uniformly with time, its velocity being $\frac{\mu}{E} q_0$. It is interesting to note that this result does not depend on the presence of an applied voltage: the spread comes from the electrostatic repulsion in the cloud itself; its velocity is proportional to the total charge and does not depend on the thickness of the sample.

Calling $v(t) = (d - s_2) - s_1$, which measures the asymmetry of the charge distribution with respect to the plates, it is easy to find it as function of time for the case when o = 0. It gives, using the expressions for $\frac{ds_1}{dt}$ and $\frac{ds_2}{dt}$,

$$\left| v(t) - \frac{2V}{q_0} \right| = \left| v(0) - \frac{2V}{q_0} \right| \exp\left(\frac{\mu q_0 t}{\varepsilon d}\right).$$

So, the asymmetry tends to increase with time, but if at the beginning is zero, it remains so for all times. The same result has already been found[4] in the open circuit condition.

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[NOTE ADDED IN PROOF: It came to the knowledge of one of the authors that Prof. J. van Turnhout arrived independently to about the same results reported here].