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Effect of Electron-Electron Collisions on the Thermal Conductivity of a Dense Plasma

J. OSADA, J. R. M. BONILHA, A. C. FERREIRA and H. V. CAPELATO
Instituto de Física*, Universidade de *São Paulo*, São Paulo SP

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The effect of electron-electron collisions on thermal conductivity of a degenerate plasma is calculated by the perturbation method. Because of this effect, the conductivity is considerably reduced. In the case of a highly degenerate plasma, this effect becomes predominant and the usual formula, which is obtained by taking into account onlu the effect of ion-electron collisions, becomes useless.

O efeito das colisões elétron-elétron sobre a condutividade térmica de um plasma degenerado foi calculado pelo método de perturbações. Verificamos que por causa desse efeito, a condutividade é consideravelmente reduzida. No caso de um plasma altamente degenerado, esse efeito torna-se predominante e a fórmula usual, que é obtida levando-se em conta apenas o efeito das colisões elétron-ion, perde o significado.

Introduction

Many works on the thermal conductivity of degenerate plasmas have been presented until now[1], because it plays an important role in the study of the interna1 structures of white-dwarf stars. In a11 of them, however, only the effect of electron-ion collisions is taken into account, the effect of electron-electron collisions being neglected. On the other hand, in our recent work[2], we found that the mass-luminosity relation for white-dwarf stars depends very sensitively on the thermal conductivity of the stellar matter, so that it is interesting to examine the contribution of the latter effect to the conductivity.

In this preliminary report, we studied this effect in a simple way, that is, supposing that the electron-electron collision is a minor effect, we calculated the conductivity by applying the perturbation method. In conclusion, we'

*Postal Address: Caixa Postal 20516. 01000 - São Paulo SP.

obtained the surprising result, that such effect is predominant for highly degenerate matter, the usual formula being unapplicable in this case.

Calculations

Boltzmann's transport equation for the present probiem is written[3] as

$$\left(\frac{\partial f}{\partial t}\right)_{\text{ext}} + \left(\frac{\partial f}{\partial t}\right)_{\text{e-i}} + \left(\frac{\partial f}{\partial t}\right)_{\text{e-e}} = 0, \tag{1}$$

where

$$\left(\frac{\partial f}{\partial t}\right)_{\text{ext}} = p_z\left[gR + gSu\right]f_u', \quad gR = \frac{1}{m\mu}\left[eE + \left(\frac{\partial \mu}{\partial T} - \frac{\mu}{T}\right)\frac{\partial T}{\partial z}\right], \tag{2}$$

$$gS = \frac{1}{m}\cdot\frac{1}{T}\cdot\frac{\partial T}{\partial z}, \qquad u = \frac{\mathbf{p}^2}{2m\mu}, \ \psi = \frac{\mu}{kT}, \ f_u = \frac{1}{e^{\psi(u-1)} + 1}$$

and all other notations are the same of reference 3. The second term of Eq. (1) means the effect of electron-ion collisions and is given, according to Ref. 1, by

$$\left(\frac{\partial f}{\partial t}\right)_{\text{e-i}} = -p_z g u^{-3/2}\chi_u, \qquad g = \frac{4m}{(2mkT\psi)^{3/2}}\,\frac{Z^2 e^4}{16\pi I}, \tag{3}$$

where $I = (2V/N)\,[\ln\,(2/(1-\cos\theta_0))]^{-1}$; $p_z\chi_u = f(\mathbf{p}) - f_u$ and θ_0 is the minimum scattering angle. Here, we are using Heaviside units.

The third term of Eq. (1) is the effect of electron-electron collisions in question. This can be written, using the Møller scattering cross section". as

$$\left(\frac{\partial f}{\partial t}\right)_{e-e} = \frac{me^4}{16\pi^5\hbar^3}\int d\mathbf{q}\,d\mathbf{s}\,\delta(\mathbf{s}\ (\mathbf{q}-\mathbf{p}))F(\mathbf{q},\mathbf{s})\times\left[\frac{1}{s^4} + \frac{1}{(\mathbf{q}-\mathbf{p})^4} - \frac{1}{s^2(\mathbf{q}-\mathbf{p})^2}\right]. \tag{4}$$

where

$$F(\mathbf{q},\mathbf{s}) = f(\mathbf{q})\,f(\mathbf{p}+\mathbf{s})\,[1-f(\mathbf{p})]\,[1-f(\mathbf{q}+\mathbf{s})]$$

$$\qquad - f(\mathbf{p})\,f(\mathbf{q}+\mathbf{s})\,[1-f(\mathbf{q})]\,[1-f(\mathbf{p}+\mathbf{s})],$$

and q and q + s are the initial (final) momenta of electrons and p and q + s are the final (initial) momenta. From the integrand of Eq. (4), we know that the main contribution to the integral comes from the region of small s^2 and $(q-\mathbf{p})^2$.

However, when both s^2 and $(q-\mathbf{p})^2$ are small at the same time, F is extremely small, so that the interference term $1/s^2(p-\mathbf{q})^2$ may be neglected compared to the terms $1/s^4$ and $1/(\mathbf{p}-\mathbf{q})^4$. Furthermore, observing that $F(\mathbf{q},-\mathbf{s}) = F(\mathbf{s}+p, q-\mathbf{p})$, Eq. (4) is written as

$$\left(\frac{\partial f}{\partial t}\right)_{e-e} = \frac{me^4}{8\pi^5\hbar^3}\int d\mathbf{q}\,d\mathbf{s}\,\delta(\mathbf{s}\cdot(\mathbf{q}-\mathbf{p}))\frac{1}{s^4}F(\mathbf{q},\mathbf{s}) \tag{5}$$

Now, we substitute $f(\mathbf{p}) = f_u + p_z\chi_u$, etc., into F, collect the terms of the first-order in χ, and expand the result with respect to s, retaining the terms up to the second-order in s. Then, it becomes possible to perform the integral of Eq. (5) analytically and, after some lengthy calculation, we have the following result:

$$\left(\frac{\partial f}{\partial t}\right)_{e-e} = p_z\frac{g'}{\psi}\int_0^\infty dt\sqrt{\frac{t}{u_>}}\left\{J_u f_t' + \frac{1}{3}\sqrt{\frac{t}{u}\cdot\frac{u_<}{u_>}}\times\right.$$

$$\times\left[J_u f_t' - 2f_u'J_t - \chi_u' f_t' - \chi_u f_t'' + f_u'\chi_t' + f_u''\chi_t\right.$$

$$\left.\left. + 2uJ_u'f_t' + 2uJ_u f_t''\right] + \frac{1}{5}\,\frac{t}{u}\,\frac{u_<}{u_>}\left[-2uf_u'J_t' - 2uf_u''J_t\right]\right\}, \tag{6}$$

where $t = \mathbf{q}^2/2m\mu$, $J_, = -\chi_u + \psi(2f_u - 1)\chi_u$, $u_>$ ($u_<$) is the larper (smaller) of u and t, $g' = (m_e{}^4/2\pi^3\hbar^3)\ln(s_2/s_1)$ and s_1 and s_2 are, respectively, the lower and upper limits of s, about which we shall discuss later.

Substituting Eqs. (2), (3) and (6) into Eq. (1) we have now a rather simple integro-differential equation for χ_u. When o r $g'/g \ll 1$, we can solve it by the perturbation method. The first-order solution is given by

$$\chi_u = R\,\{u^{3/2} f_u' + \sigma[(-1 + 3u^{1/2})f_u' + (-\tfrac{1}{3} - \tfrac{3}{5}u +$$

$$+ \tfrac{4}{3}u^{3/2})f_u'']\} + s\{u^{5/2} f_u' + \sigma[\tfrac{5}{3}(-1 + 4u^{3/2})f_u' +$$

$$+ (-\tfrac{1}{3} - u - 2u^{5/2})f_u'']\}, \tag{7}$$

where, assuming that ψ is sufficiently large, we have applied the following asymptotic expansions:

$$\int_0^\infty dt\, G(t) f'_t = -G(1) - \frac{\pi^2}{6\psi^2} G''(1) - \ldots\ldots, \tag{8}$$

$$\int_0^u dt\, G(t) f'_t = G(u) f_u - G(1) + \frac{\ln 2}{\psi} G'(1) - \frac{\pi^2}{12\psi^2} \times$$

$$\times [G''(1)] - \frac{1}{2} G'(1) \frac{u-1}{\psi} - \left[\frac{1}{4} G''(1) - \frac{\psi G'(1)}{8}\right] \frac{(u-1)^2}{\psi^2} + \ldots,$$

retaining only the higher order terms.

Now, the thermal conductivity λ is given by

$$\lambda = -\frac{Q}{\partial T/\partial z} = -\frac{1}{GmTS} \int d\mathbf{p} \frac{2}{(2\pi\hbar)^3} \frac{\mathbf{p}^2}{2m} \frac{p_z}{m} f(\mathbf{p})$$

$$= -\frac{1}{GmTS} \frac{2\pi}{3} \frac{(2mkt\psi)^{7/2}}{m^2(2\pi\hbar)^3} \int_0^\infty du\, u^{5/2} \chi_u, \tag{9}$$

where Q is the thermal current. Substituting Eq. (7) into Eq. (9) and eliminating R by using the condition that the electric current J is equal to zero, namely,

$$J = \int d\mathbf{p} \frac{2}{(2\pi\hbar)^3} \left(-\frac{e}{c} \frac{p_z}{m}\right) f(\mathbf{p})$$

$$= -\frac{1}{3} \frac{e}{c} \frac{4\pi}{3(2\pi\hbar)^3} (2mkT\psi)^{5/2} \int_0^\infty du\, u^{3/2} \chi_u = 0, \tag{10}$$

we obtain the following thermal conductivity:

$$\lambda = \lambda_0 (1 - \frac{4\sigma}{5n^2} *), \tag{11}$$

where we have used Eq. (8) again and

$$\lambda_0 = \frac{\pi^2}{3} k^2 T \frac{16\pi\mu m}{3(2\pi\hbar)^3} \frac{\mu^2 16\pi I}{Z^2 e^4}$$

is the usual thermal conductivity given in Ref. 1.

Conclusions

When ψ is sufficiently large, the expansion parameter o may be written as

$$\sigma \simeq \frac{6}{Z} \frac{\ln(s_2/s_1)}{\ln(2/\theta_0)} .$$

The determination of the cut-off parameters s_1 and s_2 would require a more thorough examination of the collision problem but, since only their logarithms appear in the above expression for σ, a rough estimation will be sufficient. Here we use $s_1 = \sqrt{2m\mu}\,\theta_0$ for the minimum momentum transfer and $s_2 = 2\sqrt{2m\mu}$ for the maximum momentum transfer for the sake of simplicity. (This choice of s_2 is beyond the limit under which the expansion of F with respect to s is valid. However, if we choose this upper limit as s_2, the integrand in Eq. (5) must be very small, so that, extrapolating the expanded result, we can choose the above value for s,). Then, we may put o $\simeq 6/Z$. Substituting this value into Eq. (11), we have

$$\lambda \simeq \lambda_0 \left(1 - \frac{\psi}{2Z} \right). \tag{12}$$

For example, for $\psi = 6$ and Z = 12, the thermal conductivity is reduced by 25%. Perhaps, the temperature gradient inside the white-dwarf stars may be larger than that considered until now.

Since the above result was obtained by using the perturbation method, our result is valid only when $\psi/2Z < 1$. This means that, when the electron gas is highly degenerate, the effect of electron-electron collisions becomes predominant and the usual formula, which is obtained by taking into account only the effect of electron-ion collisions, becomes useless. In such a case, we must solve the integro-differential equation obtained above in a completely opposite way, taking the electron-ion collision as a perturbation.

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