Revista Brasileira de Física, Vol. 2, N.2, 1972

Deep Inelastic Electron Scattering in a Nonrelativistic Bound State Model

J. BELLANDI FILHO*
Deutsches Elektronen-Synchrotron DESY, Hamburg

Recebido em 28 de Abril de 1972

Deep inelastic electron scattering is studied in a resonance model derived from a two spinless particle bound state, in the framework of nonrelativistic dynamics. The calculated structure function νW_2 does not obey approximate scaling for finite momentum and energy transfers but, in the Bjorken limit, approaches a scaling function, representing a quasi-elastic, scale invariant peak.

Estuda-se o espalhamento inelástico do elétron na região inelástica profunda, num modelo ressonante para um estado ligado de duas partículas sem spin dentro de uma dinâmica não relativística. A função de estrutura νW_2 não obedece a condição de ***scaling*** para transferências finitas de energia e momento. No entanto, no limite de Bjorken, aproxima-se de uma função delta, invariante por transformações de escala, representativa de um pico quase elástico.

1. Introduction

This work is motivated by the experiments on deep inelastic electron-proton scattering[1]. The large magnitude of the inelastic cross section leads one to consider the proton in terms of point-like constituents. In the parton model[2], the electron collides with a proton at high energies as if the proton were a gas of non-interacting point-like constituents (partons), the electron being scattered incoherently by these free partons. This model explains the scaling behaviour of the structure functions.

We are interested in the study of deep inelastic electron scattering within the framework of the quark model, where the hadronic target is considered as a state of strongly bound quarks[3]. The electron now is scattered by bound quarks and not by free particles as in the parton model. Because the quarks are strongly bound, they cannot be ejected, but they make transitions to excited states. We assume that the hadronic final state is completely expressible by a superposition of resonant states.

---

*On leave of absence from the ***Instituto de Física Teórica, S.*** Paulo, with a fellowship from FAPESP, S. Paulo.

Domokos *et al.*[4] discussed in a series of papers some aspects of a resonance model for inelastic electron-nucleon scattering. They argue that the excitation of many overlapping resonance states of an oscillator-like spectrum leads to structure functions like those experimentally observed. From an universality assumption on the transition form factors, they derive scaling in the Bjorken limit.

We are interested in the relation of deep inelastic electron scattering to quark dynamics. For this reason, we study as an exercise a resonance model derived from two spinless particle bound state in the framework of nonrelativistic dynamics. The explicitly calculated form factors of our oscillator model do not satisfy the universality assumption of Domokos. Therefore, we do not have approximate scaling for finite momentum and energy transfers, q^2 and ν. However, in the Bjorken limit, our structure function approaches a δ-function, representing a quasi-elastic, scale invariant peak.

In Section 2, we study the covariance properties of the current tensor $W_{\mu\nu}$ with respect to the Galilei group in nonrelativistic quantum mechanics. $W_{\mu\nu}$ is determined by two structure functions W_1 and W_2, which are calculated in a nonrelativistic oscillator model in Section 3.

2. The Current Tensor $W_{\mu\nu}$ in a Galilei Invariant Model

The invariance group of nonrelativistic quantum mechanics is the Galilei group. It consists of the translations in space and time, the rotations and the pure Galilei transformations. The canonical coordinates: position momentum p and time coordinate t of a single particle of mass m, transform like

$$\begin{aligned} \mathbf{x}' &= R\mathbf{x} + \mathbf{v}t + \mathbf{b}, \\ \mathbf{p}' &= R\mathbf{p} + m\mathbf{v}, \\ t' &= t + b_0, \end{aligned} \tag{1}$$

upon transformation from one inertial system to another. Here R denotes a three dimensional space rotation, $\mathbf{v}$ the relative velocity of the two referente systems, b a spacial and b_0 a time displacement.

The solutions of the Schrodinger equation transform, under the Galilei transformations, as[5]

$$\Psi'(\mathbf{x},t) = e^{-if(\mathbf{x}',t')}\,\Psi(\mathbf{x}',t'), \tag{2}$$

where $(\mathbf{x}',t')$ depend on $(\mathbf{x},t)$ according to (1) and

$$f(\mathbf{x}',t') = m\mathbf{v}\cdot\mathbf{x}' - \tfrac{1}{2}m\mathbf{v}^2 t'.$$

Unlike the relativistic case, the phase factor $f(\mathbf{x},t)$ cannot be eliminated. The solutions of the Schrodinger equation for a free spinless particle belongs to the physical representation of the Galilei group characterized by two real numbers, the particle mass and the internal energy.

In the space of Schrodinger functions $\Psi(\mathbf{x},t)$, we introduce a basis $|\mathbf{p}\rangle$ of momentum eigenvalues normalized in a Galilei invariant way as

$$\langle \mathbf{p}|\mathbf{p}'\rangle = 2m\delta(\mathbf{p}-\mathbf{p}'). \tag{3}$$

The homogenous Galilei group is not semi-simple. There are reducible representations of this group which are not decomposable, and this makes the invariance considerations of the Galilei group rather involved.

For the description of the transformation properties of the energy p_g and the momentum $\mathbf{p}$, we form, together with the mass, a five-vector $p \equiv (m, \mathbf{p}, p_g)$, which transforms under pure Galilei transformations as

$$\begin{aligned} m' &= m, \\ \mathbf{p}' &= \mathbf{p} + m\mathbf{v}, \\ p_g' &= p_g + \mathbf{p}\cdot\mathbf{v} + \tfrac{1}{2}m\mathbf{v}^2. \end{aligned} \tag{4}$$

The following bilinear form of five-vectors,

$$p\cdot p' = mp_g' + m'p_g + mm' - \mathbf{p}\cdot\mathbf{p}', \tag{5}$$

is invariant under the Galilei transformations. The "five-momentum" squared,

$$p^2 = 2mp_g + m^2 - \mathbf{p}^2 = 2mE_g + m^2,$$

is related to the characteristic internal energy and the mass of the particle. For comparison with the relativistic four momentum squared $p^2 = p_0^2 - \mathbf{p}^2 = M_r^2$, we put $M_r = m + E$, and have approximately for $E \ll m$, $p^2 = M_r^2$.

We shall now consider some quantities which are important in a scattering process, when we have in the initial state two particles with five-momentum $k \equiv (m_1, \mathbf{k}, k_g)$ and $p \equiv (M, \mathbf{p}, p_g)$ and in the final state $k' \equiv (m_1, \mathbf{k}', k_g')$ and $p' \equiv (M, \mathbf{p}', p_g')$. Energy momentum conservation reads $p + k = p' + k'$. The center of mass energy $s = (p+k)^2$ is

$$s = (M+m)^2 + 2(M+m_1)E_s, \tag{6}$$

where E_s is the sum of the internal energy of the two particles plus a relativistic energy between the two particles:

$$E_s = E_k + E_p + \frac{1}{2(M + m_1)}\left(\sqrt{\frac{m_1}{M}}\,\mathbf{p} + \sqrt{\frac{M}{m_1}}\,\mathbf{k}\right)^2$$

The five-momentum transfer $q = (k - k')$ is $q = (0, \mathbf{k} - \mathbf{k}', k_g - k'_g)$, with the invariant five-momentum squared $q^2 = -|\mathbf{q}|^2$. The energy transfer in the laboratory system is

$$\nu = \frac{q \cdot p}{M_r} = \frac{M}{M + E_p}(k_g - k'_g).$$

For the discussion of electromagnetic processes, we have to discuss the transformation properties of the electromagnetic current under the pure Galilei transformations. It transforms like the m and p components of the five-momentum,

$$\begin{aligned} j_0(\mathbf{x}', t') &= j_0(\mathbf{x}, t), \\ \mathbf{j}(\mathbf{x}', t') &= \mathbf{j}(\mathbf{x}, t) + \mathbf{v} j_0(\mathbf{x}, t). \end{aligned} \tag{7}$$

We see, by expression (4), that this transformation carries a representation of the pure Galilei group.

For the one-particle expectation value of the current operator the normalization condition reads, with our convention on state normalization (equation (3)),

$$\langle \mathbf{p} | j_0^Q(0) | \mathbf{p} \rangle = \frac{2MQ}{(2\pi)^3}, \tag{8}$$

where Q is the charge of particle, and the phase space is $\frac{d^3p'}{2M}$. We see that this normalization is the limit of the relativistic case.

Similarly to the well-known relativistic case[6], the cross section of an electromagnetic process is described by the current tensor $W_{\mu\nu}$

$$W_{\mu\nu} = (2\pi)^4 \sum_n \delta(p + q - p') \langle \mathbf{p} | j_\mu(0) | \mathbf{p}' n \rangle \langle \mathbf{p}' n | j_\nu(0) | \mathbf{p} \rangle . \tag{9}$$

The transformation properties of the current implies that the current tensor transforms, under the pure Galilei transformations, like

$$\begin{aligned} W'_{00} &= W_{00}, \\ W'_{0i} &= W_{0i} + v_i W_{00}, \\ W'_{ij} &= W_{ij} + 2v_i W_{0j} + v_i v_j W_{00}. \end{aligned} \tag{10}$$

In order to write the general form of the tensor $W_{\mu\nu}$, we observe that there are only two five-vectors on which this tensor can depend, p and $\mathbf{q}$, and only two independent scalars which can be formed from these five vectors, $\mathbf{q}^2$ and $\mathbf{q}\cdot p$. Thus, the most general form for the components of $W_{\mu\nu}$ is

$$\begin{aligned} W_{00} &= A(q^2,\ p\cdot q), \\ W_{0i} &= B(q^2,\ p\cdot q)q_i + A(q^2,\ p\cdot q)p_i/M, \\ W_{ij} &= C(q^2, p\cdot q)q_i q_j + A(q^2, p\cdot q)p_i p_j/M^2 + B(q^2, p\cdot q)(q_i p_j/M + p_i q_j/M) \\ &\quad + E(q^2, p\cdot q)\delta_{ij}. \end{aligned} \tag{11}$$

Current conservation implies that $q_g W_{00} - q_i W_{i0} = q_g W_{i0} - q_i W_{ij} = 0$. These relations are sufficient to eliminate two invariant functions and, as in the relativistic case, $W_{\mu\nu}$ is completely expressible by two invariant functions, $W_1(\mathbf{q}^2, p\cdot\mathbf{q})$ and $W_2(\mathbf{q}^2, p\cdot\mathbf{q})$:

$$\begin{aligned} W_{00} &= W_2, \\ W_{0i} &= W_2\left(p_i - \frac{p\cdot q}{q^2}q_i\right)/M, \\ W_{ij} &= W_1(\delta_{ij} + q_i q_j/q^2) + W_2\left(p_i - \frac{p\cdot q}{q^2}q_i\right)\left(p_j - \frac{p\cdot q}{q^2}q_j\right)/M^2. \end{aligned} \tag{12}$$

3. Deep Inelastic Electron Scattering in a Nonrelativistic Bound State Model

We shall consider now the deep inelastic electron scattering in a nonrelativistic spinless two-particle bound-state model. The electron is scattered by a bound state of two particles with charge Q_1 and Q_2 and the same mass m. We assume that the interaction between the two particles is of the harmonic oscillator type.

The time independent Schrodinger equation is

$$-\frac{1}{2m}(\nabla_1^2 + \nabla_2^2)\Psi(\mathbf{r}_1, \mathbf{r}_2) + V(\mathbf{r}_1 - \mathbf{r}_2)\Psi(\mathbf{r}_1, \mathbf{r}_2) = E\Psi(\mathbf{r}_1, \mathbf{r}), \tag{13}$$

where $\mathbf{r}_i$ is the position operator for the ith-particle and

$$V(\mathbf{r}_1 - \mathbf{r}_2) = \tfrac{1}{2}k(\mathbf{r}_1 - \mathbf{r}_2)^2. \tag{14}$$

If we introduce the center of mass and relative coordinates $\mathbf{R} = (1/2)(\mathbf{r}_1 + \mathbf{r}_2)$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, we have two independent equations, one for the center of mass wave function $\psi_{CM}(\mathbf{R})$ and the other for the relative wave function $\psi_{rel}(\mathbf{r})$

$$
\begin{aligned}
-\frac{1}{2M}\nabla_R^2\psi_{CM}(\mathbf{R}) &= E_{CM}\psi_{CM}(\mathbf{R}),\\
-\frac{1}{2\mu}\nabla_{\mathbf{r}}^2\psi_{rel}(\mathbf{r}) + \frac{1}{2}k\mathbf{r}^2\psi_{rel}(\mathbf{r}) &= E_{rel}\psi_{rel}(\mathbf{r}),
\end{aligned}
\tag{15}
$$

where $\Psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_{CM}(\mathbf{R})\psi_{rel}(\mathbf{r})$, $M = 2m$, $\mu = m/2$ and $E_{CM} + E_{rel} = E$.

The normalized bound state wave functions are[7]

$$
|\mathbf{p},\ n\rangle = \frac{1}{(2\pi)^{3/2}}\sqrt{2M}\, e^{i\mathbf{p}\cdot\mathbf{R}} f_n(\mathbf{r}),
\tag{16}
$$

where

$$
f_n(\mathbf{r}) = \left[\frac{\alpha}{\sqrt{\pi}}\right]^{3/2} e^{-(1/2)\alpha^2 r^2} \prod_{i=1}^{3}\left(\frac{1}{2^{n_i} n_i!}\right)^{1/2} H_{n_i}(\alpha r_i)
\tag{17}
$$

is the normalized harmonic oscillator wave function with $a^2 = (\mu k)^{1/2}$.

The electromagnetic current of point-like particles in the Schrodinger picture, at $t = 0$, is

$$
\begin{aligned}
j_0^Q(\mathbf{x}) &= \sum_{i=1}^{2} Q_i\delta(\mathbf{x}-\mathbf{r}_i),\\
\mathbf{j}^Q(\mathbf{x}) &= \sum_{i=1}^{2} Q_i[\dot{\mathbf{r}}_i\delta(\mathbf{x}-\mathbf{r}_i)]_{sym},
\end{aligned}
\tag{18}
$$

where $\mathbf{r}_i$ and $\dot{\mathbf{r}}_i$ are the position and velocity of each i-particle. As a consequence of the time independent Schrodinger equations, this current is conserved. The current matrix elements between the bound states of our model are

$$
\begin{aligned}
\langle\mathbf{p}', n|j_0^Q(0)|\mathbf{p}\rangle &= \frac{2MQ}{(2\pi)^3}F_n(|\mathbf{q}|^2),\\
\langle\mathbf{p}', n|\mathbf{j}(0)|\mathbf{p}\rangle &= \frac{2Q}{(2\pi)^3}[\mathbf{p}'F_n(|\mathbf{q}|^2) + 4\alpha^2\nabla_{\mathbf{q}}F_n(|\mathbf{q}|^2)],
\end{aligned}
\tag{19}
$$

where $\mathbf{q} = \mathbf{p}'-\mathbf{p}$, $Q = Q_1 + Q_2$ and

$$
\begin{aligned}
F_n(|\mathbf{q}|^2) &= \int d^3r e^{i(1/2)\mathbf{q}\cdot\mathbf{r}} f_n^*(\mathbf{r}) f_0(\mathbf{r})\\
&= (i)^n\frac{1}{(2\pi)^{3/2}}\prod_{i=1}^{3}\left(\frac{1}{2^{n_i}n_i!}\right)^{1/2}\left(\frac{q_i}{2\alpha}\right)^{n_i}\exp\left[-\frac{1}{2}\left(\frac{q_i^2}{8\alpha^2}\right)\right].
\end{aligned}
\tag{20}
$$

In order to calculate the structure functions, we saturate the intermediate states with the resonant states and take the sum over all resonant states. By expression (12), W_2 is given by W_{00}:

$$W_2 = W_{00} = \frac{4M^2 Q^2}{(2\pi)^2} \sum_{n_i=0}^{\infty} \theta(p_g + q_g)\delta[(p + q)^2 - M(2E_{p'} + M)]\,|F_n(|\mathbf{q}|^2)|^2. \tag{21}$$

In the laboratory system, we take q in the z-direction and the sum reduces to one only over a single index:

$$W_2 - \frac{4M^2 Q^2}{(2\pi)} \exp\left(-\frac{|\mathbf{q}|^2}{8\alpha^2}\right) \sum_{n=0}^{\infty} \theta(p_g + q_g)\frac{1}{n!}\left(\frac{|\mathbf{q}|^2}{\ldots}\right)^n \delta(2p\cdot q + q^2 - 8(n + 3/2)\alpha^2). \tag{22}$$

In view of the fact that the intermediate resonant states have overlapping width for large *n*, we may substitute the sum by an integration. This procedure was discussed by Domokos. For this we assume that $\gamma = \dfrac{8\alpha^2}{2M\nu} \ll 1$ and the summation over *n* can be replaced by an integration on the variable $x = \dfrac{8\alpha^2}{2M\nu}$ n. The n ! can be approximated by a gamma function on the x variable. We therefore write:

$$W_2 \approx \frac{4M^2 Q^2}{(2\pi)^5} \frac{1}{8\alpha^2} \exp\left(-\frac{|\mathbf{q}|^2}{8\alpha^2}\right) \int_0^{\infty} dx\,\delta\left(1 - \frac{1}{\omega} - x\right) \frac{1}{\Gamma\left(\frac{x}{\gamma} + 1\right)} \left(\frac{|\mathbf{q}|^2}{8\alpha^2}\right)^{x/\gamma},$$

where $\omega = \dfrac{2M\nu}{|\mathbf{q}|^2}$ is the scaling variable. We obtain

$$W_2 \approx \frac{4M^2 Q^2}{(2\pi)^5} \frac{1}{8\alpha^2} \frac{1}{\Gamma\left(\frac{1 - 1/\omega}{\gamma} + 1\right)} \exp(-|q|^2/8\alpha^2) \left(\frac{|q|^2}{8\alpha^2}\right)^{\frac{1-1/\omega}{\gamma}}. \tag{23}$$

The behaviour of W_2 is like a continuous Poisson distribution and if $\dfrac{|\mathbf{q}|^2}{8a^2} \gg 1$, this expression can be approximated by a normal distribution of half width $\left(\dfrac{|\mathbf{q}|^2}{8\alpha^2}\right)^{1/2}$:

$$\nu W_2 \approx \frac{2MQ^2}{(2\pi)^5} \frac{2M\nu}{8\alpha^2} \frac{1}{\sqrt{2\pi}} \frac{1}{\left(\frac{|\mathbf{q}|^2}{8\alpha^2}\right)^{1/2}} \exp\left[-\frac{1}{2} \frac{\left\{\frac{1 - 1/\omega}{\gamma} - \frac{|\mathbf{q}|^2}{8\alpha^2}\right\}^2}{\frac{|\mathbf{q}|^2}{8\alpha^2}}\right]. \tag{24}$$

In terrns of the scaling variable,

$$\nu W_2 \approx \frac{2MQ^2}{(2\pi)^5} \sqrt{\frac{\omega}{2\pi\gamma}} \exp\left[-\frac{1}{2} \frac{\omega(1-2/\omega)^2}{\gamma} \right] \tag{25}$$

It is not difficult to see that $W_{,,} = 0$ and, consequently, that $W_{,} = 0$. This is consistent with the parton model where, for spinless partons, W_1 must be zero. νW_2 does not satisfy scaling for finite q^2 and ν. In the Bjorken limit, the width of the distribution (25) goes to zero and νW_2 is strongly peaked around $\omega = 2$,

$$\nu W_2 \approx \frac{2MQ^2}{(2\pi)^5} \delta(1 - 2/\omega). \tag{26}$$

This distribution corresponds to elastic scattering on quasi-free constituents with the mass $m = M/2$.

The author wishes to thank Prof. H. Joos for suggesting this theme and for many helpful discussions, as well as for the hospitality extended to him at DESY.

References

1. E. Bloom *et al.*, Phys. Rev. Letters 23,930,935 (1969). H. W. Kendall - *Deep inelastic elertron scattering in the continuum region* - Proceedings of the International Symposium on Electron--Photon Interactions at High Energies (Cornell University, August 1971).

2. R. P. Feynman, Procetdings of 3rd High Energy Collision Conference at Stony Brook (Gordon and Breach 1970) and Phys. Rev. Letters **23,** 1415 (1969).

3. M. Bohm, H. Joos and M. Krammer, N. Cimento **7A,** 21 (1971) and DESY Report 72/11 (1972).

4. G. Domokos, S. Kovesi Domokos, N. Cimento **2A,** 270 (1971), G. Domokos, S. Kovesi Domokos and E. Schonberg, preprints, The Johns Hopkins University NYO 4076 - 11, 12 and 13.

5. J. M. Levy-Leblond, J. Math. Phys. 4, 776 (1963).

6. S. D. Drell and J. D. Walecka, Annal of Physics **28,** 18 (1964).

7. L. I. Schiff, *Quantum Mechanics* (Third Edition, McGraw Hill Book Company) chap. 13.