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Application of the Statistical Perturbation Method to the Heisenberg Ferromagnet

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Extension of the statistical pertubation method to a non-Hermitian Hamiltonian is made and applied to the Dyson Hamiltonian. We obtained a low-temperature magnetization which is equal to that obtained by the Green's function method and is essentialy the Dyson result. This means that the kinematical interactions are properly taken into account in that temperature range. Finally, the renormalized energy of the spin-waves in the symmetric aproximation is obtained and the result coincides with that obtained by Callen using the Green's function method. These results show that the statistical perturbation and Green's function methods are equivalent.

Estendemos o método da perturbação estatística a um Hamiltoniano não hermitiano para aplicá-lo ao Hamiltoniano de Dyson. Na região de baixas temperaturas, obtemos um resultado para a magnetização que é idêntico ao obtido com o método das funções de Green e que é, essencialmente, o resultado de Dyson. Isso significa que as interações cinemáticas são bem tratadas naquela região de temperatura. Finalmente, as energias renormalizadas das ondas de spin são determinadas na aproximação simétrica, o resultado sendo idêntico ao obtido por Callen com o método das funções de Green. Esses resultados mostram que os métodos da perturbação estatística e das funções de Green são equivalentes.

1. Introduction

As pointed out by Wortis[1], the Heisenberg ferromagnet thermodynamics must be calculated without violating the following properties of the Heisenberg model: (i) spin kinematics must always be obeyed and (ii) the low-lying states have a propagational, particle-like behavior. The violation of these properties by any formalism introduces spurious terms in the thermodynamics.

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The purpose of this paper is to show that these two properties are not violated by the application of the statistical perturbation method[2] to the Dyson spin-waves. As pointed out by Wallace[2], the application of this method to the Bloch spin-waves obscures the local spin kinematics. This can be understood by the fact that the Bloch spin-waves are obtained by a Fourier transform of the spin operators and this transformation defines new operators which do not obey the boson commutators. This implies that the local spin kinematics is violated and kinematical interactions are overestimated when a careless approximation is introduced in the commutator of these new operators. This means that the Bloch spin-waves must be carefully treated.

On the other hand, the Dyson spin-waves are obtained by a transformation from the spin to boson operators. Hence, we expect that the local spin kinematics will not be violated by the application of the statistical perturbation method to the Dyson spin-waves. In Section 2, we show that the statistical perturbation method can be extended to a non-Hermitian Hamiltonian and it is applied to the Dyson spin-waves. The obtained magnetization is equal to that obtained by Tahir-Kheli and ter Haar[4] using the Green's function method and it is essentially the Dyson result[3]. Then we may conclude that the local spin kinematics and the particle-like behavior are not violated by the application of the statistical perturbation method to the Dyson spin-waves. From these results, we also may conclude that the statistical perturbation and Green's function methods are equivalent.

Wallace[2] has shown that there is a T^3 spurious term in the magnetization when the statistical perturbation method is applied to the Bloch spin-waves for $S = 1/2$. In addition, we must observe that in the Wallace results there is no T^4 term. This result is equal to that obtained by Callen[5] for $S = 1/2$, using the Green's function method with a symmetric decoupling. To obtain at, low temperatures, a result for the correlation function $\langle S_j^- S_j^z S_f^+ \rangle$, better than that obtained by Callen[5], it has been shown by Dembinski[6] that the basic equation of the $S = 1/2$ Callen's decoupling must be modified. The present authors have shown that the $S = 1/2$ Callen's decoupling violates the local spin kinematics. Then we may conclude that this property is strongly violated by the careless aplication of the statistical perturbation method to the Bloch spin-waves.

Finally, in Section 3, we obtain the renormalized energy of the spin-waves in the symmetric approximation[5]. The result for this renormalized energy is equal to that obtained by Callen[5]. The result obtained here is a generalization of the Wallace result[2]. The selfconsistent equation for the $S = 1/2$ magnetization is equal to that obtained by Callen. We must remember that the magnetization obtained from this selfconsistent equation also presents spurious terms. Again we may conclude that the statistical perturbation and Green's function methods are equivalent.

2. Statistical Perturbation Method

2.1. Summary of the Wallace Results

Let us assume that for a given Hamiltonian it is possible to find approximate creation operators $\theta_i^\dagger$, which satisfy the following equation of motion:

$$[H, \theta_i^\dagger]_- = \omega_i \theta_i^\dagger + R_i^\dagger, \tag{2-1}$$

where the ω_i are real positive numbers and the $R_i^\dagger$ operators may be considered as a perturbation in the sense that it gives small contributions to the statistical averages[2].

In the zeroth-order approximation, we neglect the $R_i^\dagger$ operators, and the basic equations for this approximation are[2]

$$\langle \theta_i^\dagger \Omega \rangle = \phi_i^{(\pm)} \langle [R, \theta_i^\dagger]_\pm \rangle \tag{2-2a}$$

and

$$\langle \Omega \theta_i \rangle = \phi_i^{(\pm)} \langle [\theta_i, \Omega]_+ \rangle, \tag{2-2b}$$

where

$$\phi_i^{(\pm)} = (\exp(\beta\omega_i) \pm 1)^{-1}, \tag{2-3}$$

$\beta = (k_B T)^{-1}$ and R is an operator. The statistical average is defined in the usual manner.

In the special case $R = \theta_i$, and taking into account the $R_i^\dagger$ operators, we can obtain the statistical averages in the first-order approximation, that is,

$$\langle \theta_i^\dagger \theta_i \rangle = \phi_i^{(\pm)} \langle [\theta_i, \theta_i^\dagger]_\pm \rangle \pm \beta \{\phi_i^{(\pm)} \pm 1\} \langle R_i^\dagger \theta_i \rangle + 0(R_i^\dagger R_i). \tag{2-4}$$

The renormalized energies, in the first-order approximation, are given by $\omega_i + o,,$ where[2]

$$\omega_{1i} = \langle R_i^\dagger \theta_i \rangle / \langle \theta_i^\dagger \theta_i \rangle. \tag{2-5}$$

Thus, with the ω_{1i} given by Eq. (2-5), the first-order basic equation for $\langle\theta_i^\dagger\theta_i\rangle$ reduces to a zeroth-order basic equation with the renormalized energies

$$(e;\ \theta_i\rangle = \phi_{1i}^{(\pm)}\langle[\theta_i,\ e;],), \tag{2-6}$$

where

$$\phi_{1i}^{(\pm)} = (\exp\beta(\omega_i - \omega_{1i}) - 1)^{-1}. \tag{2-7}$$

In principle, the statistical average $\langle\theta_i^\dagger\theta_i\rangle$ must be obtained selfconsistently from the coupled equations (2-5) and (2-6). However, in practice ω_{1i} is evaluated in the zeroth-order approximation and this result is used in **Eq.** (2-6).

Actually, the Wallace procedure is to treat the effect of the $R_i^\dagger$ as a perturbation and then make an a *posteriori* check to see if it does indeed give small contributions to the statistical average.

In general, the usefulness of Eq. (2-6) is that the commutator or the anticommutator are much simpler than the product of $\theta_i^\dagger$ with θ_i. In particular, in the case that the $\theta_i^\dagger$ operators are bosons or fermions, from Eq. (2-6) we obtain the occupation numbers of renormalized quasi-particles in the first-order.

2.2. Extension of the Statistical Perturbation Method to a Non-Hermitian Hamiltonian

We assume that we are dealing with a non-Hermitian *H*,, Hamiltonian and that it is possible to find approximate creation operators $\Gamma_i^\dagger$, which satisfy the commutator equation

$$[H,,\ ,\ r;]_- = \omega_i\Gamma_i^\dagger + \bar{R}_i^\dagger, \tag{2-8}$$

where the ω_i are real positive numbers.

Transforming the *H*,, operator, by a non-unitary matrix, into a H^h Hermitian operator, Eq. (2-8) becomes

$$[H_h,\ \theta_i^\dagger]_- = \omega_i\theta_i^\dagger - R_i^\dagger, \tag{2-9}$$

where

$$H_h = T^{-1} H_{nh} T, \tag{2-10a}$$
$$\theta_i^\dagger = T^{-1} \Gamma_i^\dagger T, \tag{2-10b}$$
$$R_i^\dagger = T^{-1} \bar{R}_i^\dagger T, \tag{2-10c}$$

and

$$T^{-1} \mathrm{T} = 1. \tag{2-11}$$

The Wallace formalism may be applied to **Eq.** (2-9). The zeroth- and first--order basic equations are Eqs. (2-2) and (2-6), respectively. However, Eq. (2-2) may be rewritten as follows:

$$\langle T^{-1} \Gamma_i^\dagger T T^{-1} \varphi T \rangle = \phi_i^{(\pm)} \langle [T^{-1} \varphi T,\ T^{-1} \Gamma_i^\dagger T]_\pm \rangle \tag{2-12a}$$

and

$$\langle T^{-1} \varphi T T^{-1} \Gamma_i T \rangle = \phi_i^{(\pm)} \langle [T^{-1} \Gamma_i T,\ T^{-1} \varphi T]_\pm \rangle, \tag{2-12b}$$

where $\varphi = T^{-1}\Omega T.$ Of course, the density matrix is now given by $\rho = \exp(-\beta H_{nh})$.

But

$$[T^{-1} \varphi T,\ T^{-1} \Gamma_i^\dagger T]_+ = T^{-1} [\varphi,\ \Gamma_i^\dagger]_+ \mathrm{T}. \tag{2-13}$$

Hence, substituting Eq. (2-13) into Eq. (2-12) and using the cyclic permutation theorem for the traces, we obtain

$$\langle \Gamma_i^\dagger \varphi \rangle = \phi_i^{(\pm)} \langle [\phi,\ \Gamma_i^\dagger]_\pm \rangle \tag{2-14a}$$

and

$$\langle \varphi \Gamma_i \rangle = \phi_i^{(\pm)} \langle [\Gamma_i,\ \varphi]_\pm \rangle. \tag{2-14b}$$

We also may easily obtain

$$\langle \Gamma_i^\dagger \Gamma_i \rangle = \phi_i^{(\pm)} \langle [\Gamma_i, \Gamma_i^\dagger]_\pm \rangle \pm \beta \{\phi_i^{(\pm)} \mp 1\} \langle \bar{R}_i^\dagger \Gamma_i \rangle + 0(\bar{R}_i^\dagger \bar{R}_i). \tag{2-15}$$

Finally, we may obtain the zeroth-order basic equation with renormalized energies:

$$\langle \Gamma_i^\dagger \Gamma_i \rangle = \phi_{1i}^{(\pm)} \langle [\Gamma_i,\ \Gamma_i^\dagger]_\pm \rangle \tag{2-16}$$

where

$$\omega_{1i} = \langle \bar{R}_i^\dagger \Gamma_i \rangle / \langle \Gamma_i^\dagger \Gamma_i \rangle. \tag{2-17}$$

Thus, similar equations for the operators $\theta_i^\dagger$ and θ_i are valid for the operators $\Gamma_i^\dagger$ and Γ_i, which are related to the non-Hermitian Hamiltonian.

We want to emphasize that equations (2-14) to *(2-17)* will be useful in calculating the Dyson spin-wave thermodynamics by the statistical perturbation method.

2.3. Dyson Spin-Waves

Let us consider a ferromagnetic material described by N localized spins in a crystal lattice and coupled through the isotropic Heisenberg Hamiltonian

$$H = -\sum_{ij}' J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j - \mu h \sum_j S_j^z, \quad J_{ij} > 0, \tag{2-18}$$

where $\mathbf{S}_j$ is the spin operator ($\hbar = 1$) of the j lattice site, J_{ij} is the exchange integral between spins at i and j lattice sites, h is the externa1 magnetic field and $\mu = g\mu_B$ (g = spectroscopic factor and μ_B = Bohr magneton).

The spin operators are defined by the commutators

$$[S_i^+, S_j^-]_- = 2S_i^z \delta_{ij}, \tag{2-19a}$$

$$[S_i^z, S_j^+]_- = S_i^+ \delta_{ij}, \; [S_i^z, S_j^-]_- = -S_i^- \delta_{ij}, \tag{2-19b}$$

$$[S_i^+, S_j^+]_- = [S_i^-, S_j^-]_- = 0. \tag{2-19c}$$

The S^- and S^+ operators satisfy the following subsidiary condition:

$$(S_j^+)^{2S+1} = (S_j^-)^{2S+1} = 0 \tag{2-20}$$

which is responsible for the kinematical interactions.

Let us transform the spin operators to boson operators by the Maleev transformation[8]

$$S_j^+ = (2S)^{1/2}(b_j + b_j^\dagger b_j b_j/2S),$$
$$S_j^- = (2S)^{1/2} b_j^\dagger, \; S_j^z = S - b_j^\dagger b_j, \tag{2-21}$$

where the $b_j^\dagger$ and b_j are creation and annihilation boson operators, respectively.

Dyson spin-wave creation and annihilation operators are boson operators and are defined by the Fourier transform of the $b_j^\dagger$ and b_j operators, respectively:

$$b_j^{\dagger} = N^{-1/2} \sum_k \exp(-i\mathbf{k}\cdot\mathbf{j}) a_k^{\dagger},$$
$$b_j = N^{-1/2} \sum_k \exp(i k\,.\,\mathbf{j}) a_k. \tag{2-22}$$

Dealing with the transformations *(2-21)* and (2-22), the isotropic Heisenberg Hamiltonian *(2-18)* will read

$$H_{nh} = -\mu h N S - J_0 S^2 + \sum_k \omega_k a_k^{\dagger} a_k$$
$$+ N^{-1} \sum_{kk'k''} (J_k - J_{k-k''}) a_k^{\dagger} a_{k'}^{\dagger} a_{k''} a_{k+k'-k''}, \tag{2-23}$$

where

$$\omega_k = \mu h + 2S(J_0 - J_k), \tag{2-24}$$
$$J_k = \sum_{m-j} J_{mj} \exp[ik\,.\,(\mathrm{m} - \mathbf{j})]. \tag{2-25}$$

The Heisenberg Hamiltonian in terms of the $a_k^{\dagger}$ and a_k operators is known as the Dyson spin-wave non-Hermitian Hamiltonian.

The commutator $[H_{,,}\ \ a_k^{\dagger}]_-$ is given by

$$[H_{,,}\,,\ a_k^{\dagger}]_- = \omega_k a_k^{\dagger} + (2/N) \sum_{k'k''} (J_{k'} - J_{k'-k}) a_{k'}^{\dagger} a_{k''}^{\dagger} a_{k'+k''-k}. \tag{2-26}$$

Comparing Eqs. *(2-8)* and (2-26), we see that

$$\bar{R}_k^{\dagger} = (2/N) \sum_{k'k''} (J_{k'} - J_{k'-k})\ a_{k'}^{\dagger} a_{k''}^{\dagger} a_{k'+k''-k} \tag{2-27}$$

At low temperatures, taking the leading term in Eq. *(2-26)* and using the zeroth-order approximation basic equation (2-14), we obtain

$$\langle a_k^{\dagger} a_k \rangle = \phi_k, \tag{2-28}$$

where ϕ_k is defined by Eq. *(2-3)*.

From Eqs. *(2-21)* and (2-22), we easily obtain

$$N^{-1} \sum_k \langle a_k^{\dagger} a_k \rangle = S - (S^z). \tag{2-29}$$

Thus, the zeroth-order magnetization is given by

$$\langle S^z \rangle = S - \phi, \tag{2-30}$$

where $4 = N^{-1} \sum_k \phi_k$.

Considering only nearest-neighbor exchange and primitive cubic lattices, has the usual temperature power series[2]

$$\phi = Z_{3/2}\theta^{3/2} + (3/4)\pi\nu Z_{5/2}\theta^{5/2} + \pi^2\nu^2\omega Z_{7/2}\theta^{7/2} + 0(\theta^{9/2}), \quad (2\text{-}31)$$

where we have used Wallace's notation[2]. Thus, the zeroth-order magnetization is given by

$$\langle S^z\rangle = S - Z_{3/2}\theta^{3/2} - (3/4)\pi\nu Z_{5/2}\theta^{5/2} - \pi^2\nu^2\omega Z_{7/2}\theta^{7/2} + 0(\theta^{9/2}). \quad (2\text{-}32)$$

The magnetization *(2-32)* is equal to that obtained for the independent Bloch spin-waves[2].

Let us now evaluate the perturbation term $\langle \bar{R}_k^\dagger a_k\rangle$. From the first-order basic equation (2-15), we may write:

$$N^{-1}\sum_k \langle a_k^\dagger a_k\rangle = N^{-1}\sum_k \phi_k - (\beta/N)\sum_k (\phi_k - 1)\langle \bar{R}_k^\dagger a_k\rangle. \quad (2\text{-}33)$$

The correlation function $\langle \bar{R}_k^\dagger a_k\rangle$ can be evaluated in zeroth-order by using the basic equation *(2-14):*

$$\langle \bar{R}_k^\dagger a_k\rangle = \phi_k\langle [a_k, \bar{R}_k^\dagger]_-\rangle. \quad (2\text{-}34)$$

But

$$[a_k, \bar{R}_k^\dagger]_- = (2/N)\sum_{k'} (J_{k'} + J_k - J_{k'-k} - J_0)a_{k'}^\dagger a_{k'}. \quad (2\text{-}35)$$

and, therefore, combining Eqs. (2-33), (2-34) and (2-35), we obtain

$$N^{-1}\sum_k \langle a_k^\dagger a_k\rangle = \phi - (3/2S)\pi\nu Z_{3/2}Z_{5/2}\theta^4, \quad (2\text{-}36)$$

where the integrals are evaluated as usual[2,9]. Substituting Eq. *(2-36)* into Eq. (2-29), the iirst-order magnetization is given by

$$\langle S^z\rangle = \phi - Z_{3/2}\theta^{3/2} - (3/4)\pi\nu Z_{5/2}\theta^{5/2} - \pi^2\nu^2\omega Z_{7/2}\theta^{7/2}$$
$$- (3/2S)\pi\nu Z_{3/2}Z_{5/2}\theta^4 + 0(\theta^{9/2}). \quad (2\text{-}37)$$

The magnetization given by *Eq. (2-37)* is equal to that obtained by Tahir-Kheli and ter Haar[4], applying the Green's function method to the Dyson spin-waves. Hence, we may conclude that, for low boson concentrations, the application of the statistical perturbation method to the Dyson spin-waves violates neither local spin kinematics nor the particle-like behavior. We must emphasize that Eq. *(2-37)* shows the equivalence between the statistical perturbation and Green's function methods.

Finally, we must emphasize that Eq. (3-34) of Wallace's paper[2] may be written as follows

$$S^z = S - Z_{3/2}\,\theta^{3/2} - (3/4)Z_{5/2}\,\theta^{5/2} - \pi^2 v^2 \omega Z_{7/2}\,\theta^{7/2} - S^{-1} Z_{3/2}^2\,\theta^3 - 0(\theta^{9/2}). \qquad (2\text{-}38)$$

This means that the perturbation term introduces a spurious T^3 term and cancels the T^4 term in the Bloch spin-wave magnetization. Eq. (2-38) is fundamentally different from the Dyson result[3] and coincides with that obtained by Callen[5] for S = 1/2. Then the local spin kinematics is violated by the careless application of the statistical perturbation method to the Bloch spin-waves.

Dyson[3] and Wortis[1] have shown that, at low temperatures, the kinematical interactions are negligible. This means that the Fourier transform of the S^+ and S^- operators may be considered as boson operators. Consequently, with this ad *hoc* argument we may obtain Eq. (2-37) instead of Eq. (2-38), for the S = 1/2 Bloch 'spin-wave magnetization[2].

3. Symmetric Approximation to the Renormalized Energies

As pointed out by Wallace[2], the relevant equation for the Fourier transform of the spín operator may be rewritten as follows:

$$[\mathrm{H},\ A_k^\dagger]_- = \varepsilon_k A_k^\dagger - P_k, \qquad (3\text{-}1)$$

where

$$\varepsilon_k = \mu h - 2\langle S^z\rangle (J_0 - J_k) \qquad (3\text{-}2)$$

and

$$P_k^\dagger = 2\sum_{k'} (J_{k'-k} - J_k)\, A_{k'}^\dagger (B_{k'-k} - \langle S^z\rangle \delta_{kk'}). \qquad (3\text{-}3)$$

The $A_k^\dagger$, A, and B_k operators are Fourier transforms of the S_j^-, S_j^+ and S^z operators, respectively[2].

For the renormalized energies, the basic equation (2-6) reads

$$\langle A_k^\dagger A_k\rangle = \psi_{1k}\langle [A_k,\ A_k^\dagger]_-\rangle, \qquad (3\text{-}4)$$

where

$$\psi_{1k} = (\exp \beta(\varepsilon_k - \varepsilon_{1k}) - 1)^{-1} \qquad (3\text{-}5)$$

and

$$\varepsilon_{1k} = \langle P_k^\dagger A_k\rangle / \langle A_k^\dagger A_k\rangle. \qquad (3\text{-}6)$$

Then the $S = 1/2$ magnetization is the selfconsistent solution of the following equation:

$$\langle S^z \rangle (1 - \psi_1/S) = S, \tag{3-7}$$

where

$$\psi_1 = N^{-1} \sum_k \psi_{1k}. \tag{3-8}$$

Let us determine the renormalized energies in the symmetric approximation. Callen[5] has proposed the following S^z representation:

$$S_j^z = \langle S^z \rangle + (1/2)(1-\alpha) S_j^+ S_j^- - (1/2)(1+\alpha) S_j^- S_j^+, \tag{3-9}$$

where $a = \langle S^z \rangle / S$ for $S = 1/2$. The Fourier transform of Eq. *(3-9)* gives

$$\begin{aligned} B_k = \langle S^z \rangle \delta_{k0} &+ (1/2N)(1-\alpha) \sum_{k'} A_{k'} A_{k'-k}^\dagger \\ &- (1/2N)(1+\alpha) \sum_{k'} A_{k'-k}^\dagger A_{k'}. \end{aligned} \tag{3-10}$$

Substituting Eq. (3-10) into *(3-3)*, the $P_k^\dagger$ operator is now given by

$$\begin{aligned} P_k^\dagger = N^{-1} &\sum_{k'k''} (J_{k'-k} - J_{k'}) \cdot \\ &\cdot \{(1-\alpha) A_{k'}^\dagger A_{k''} A_{k''-k'+k}^\dagger - (1+\alpha) A_{k'}^\dagger A_{k''-k'+k}^\dagger A_{k''}\}. \end{aligned} \tag{3-11}$$

Using *Eq.* (3-11), we may evaluate the numerator of Eq. *(3-6.)*.The result is

$$\begin{aligned} \langle P_k^\dagger A_k \rangle = N^{-1} &\sum_{k'k''} (J_{k'-k} - J_{k'}) \cdot \\ &\cdot \{(1-\alpha) \langle A_{k'}^\dagger A_{k''} A_{k''-k'+k}^\dagger A_k \rangle - (1+\alpha) \langle A_{k'}^\dagger A_{k''-k'+k}^\dagger A_{k''} A_k \rangle\}. \end{aligned} \tag{3-12}$$

For the representative correlation function of the right hand side of Eq. (3-12), let us use the following approximation:

$$\begin{aligned} \langle A_{k'}^\dagger A_{k''} A_{k''-k'+k}^\dagger A_k \rangle \simeq &\langle A_{k'}^\dagger A_{k''} \rangle \langle A_{k''-k'+k}^\dagger A_k \rangle + \\ &+ \langle A_{k'}^\dagger A_k \rangle \langle A_{k''} A_{k''-k'+k}^\dagger \rangle. \end{aligned} \tag{3-13}$$

The approximation *(3-13)* corresponds to the Callen symmetric decoupling[5], which is easily verified using the $A_k^\dagger$ and A_k operator definitions. With this approximation, the correlation function $\langle P_k^\dagger A_k \rangle$ reads

$$\begin{aligned} \langle P_k^\dagger A_k \rangle = &-(2\alpha/N) \sum_{k'k''} (J_{k'-k} - J_{k'}) \langle A_{k'}^\dagger A_{k''} \rangle \langle A_{k''-k'+k}^\dagger A_k \rangle - \\ &- N^{-1} \sum_{k'k''} (J_{k'-k} - J_{k'}) \{(1-\alpha) \langle A_{k''} A_{k''-k'+k}^\dagger \rangle - (1+\alpha) \langle A_{k''-k'+k}^\dagger A_{k''} \rangle\}. \end{aligned} \tag{3-14}$$

The second term of the right side of Eq. (3-14) is zero, which is verified by taking the average of Eq. (3-9). In this case, using Eq. (3-4) we obtain

$$\langle P_k^\dagger A_k\rangle = -\{2\langle S^z\rangle^2/NS^2\}\langle A_k^\dagger A_k\rangle \sum_{k'} (J_{k'-k} - J_{k'})\psi_{1k'}. \qquad (3\text{-}15)$$

Combining Eqs. (3-15) and (3-6), we obtain

$$\varepsilon_{1k} = \{2\langle S^z\rangle^2/NS^2\} \sum_{k'} (J_{k'} - J_{k'-k})\psi_{1k'}. \qquad (3\text{-}16)$$

Finally, the renormalized energy in the symmetric approximation reads

$$\begin{aligned}\varepsilon_k + \varepsilon_{1k} = \mu h &+ 2\langle S^z\rangle(J_0 - J_k)\\ &+ \{2\langle S^z\rangle^2/NS^2\} \sum_{k'} (J_{k'} - J_{k'-k})\psi_{1k'},\end{aligned} \qquad (3\text{-}17)$$

which is exactly the Callen result[5]. Eq. (3-17) is a generalization of the Wallace result[2].

Eqs. (3-17), (3-8) and (3-7) can also be obtained by the Green's function method for S = 1/2 (Ref. 5). This means that the low temperature magnetization has the spurious T^3 term and does not have the T^4 Dyson term[5]. A critica1 discussion of this point is given in Ref. 7. Once again, the equivalente between the statistical perturbation and Green's function methods is shown.

4. Conclusion

In conclusion, we may say that one must be careful in calculating the thermodynamics of the Bloch spin-waves by the statistical perturbation method. Unless ad *hoc* arguments are introduced (e.g., kinematical interactions can be neglected), spurious terms are present in the low temperature magnetization, although the application of this method to the Dyson spin-waves violates neither the local spin kinematics nor the particle-like behavior.

Finally, we may say that the $S = 1/2$ Callen renormalized energy can be obtained by the statistical perturbation method. The results of this paper show the equivalence between the statistical perturbation and Green's function methods.

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