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Analyticity and Dual Absorptive Models*

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Realizations of the dual absorptive model of Haran by the sum of a Regge recurrence in the direct channel are discussed and in particular a construction proposed by Ingraham is criticized on the light of some general properties of Regge trajectories that follow from analyticity.

Propriedades de trajetórias de Regge que são consequências de sua analiticidade são usadas para analisar construções de amplitudes duais com absorção (propostas por Harari) através da soma de ressonâncias localizadas sôbre uma trajetória de Regge no canal direto. Em particular, é feita uma crítica à construcão de Ingraham.

1. Introduction

A decade of analysis of 4-particle processes, by Regge methods of increasing degree of sophistication[1] and models based on diffraction[2] and on internal symmetries[3], suggested to Harari[4] a synthetic model for 4-particle-non-diffractive scattering which gives the t-structure of the imaginary part of the amplitude at high energies in terms of some more-or-less known functions.

The most natural way to build up such amplitudes would be to sum up resonances located on Regge trajectories in the s-channel. This has been explicitly performed by Ingraham[5] who was able to show that the assumption of a single trajectory required its asymptotic behavior at high energies to depend linearly on the mass of the resonance, to be compareci with the linear dependence on s required by Veneziano type models. In this way, the correct t-distribution obtains and the resonances are not stable.

Problems arise, however, when these features are confronted with recent results coming from analyticity and unitarity[6] which enforce a close rela-

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tion between the trajectory and width functions. It is shown that Ingraham's choice of $\alpha(s)$ (trajectory) and $\Gamma(s)$ (width) is incompatible with analyticity.

In Sec. 2, the highlights of Harari's work[4] are presented and Ingraham's model[5] is discussed. Sec. 3 reviews ihe results[6] on resonance widths and applies them to the case of Ref. 5.

2. Dual Absorptive Models

Harari[4] proposed a "qualitative theory" of 2-body scattering at high-energies which describes remarkably well the t-structure of the scattering amplitude by writing its imaginary part, in the center of mass system, as

$$Im\, f = Im\, f_I + Im\, f_{II}, \tag{2-1}$$

where f_{II} is the diffractive part, structureless in t, and $Im\, f_I$ is the sum of resonances (or the exchange of Regge poles) in the direct channel, assumed to depend on t through Bessel-like functions:

$$Im\, f_I(s,t) = \beta(s)\ \text{``}J_{|\Delta\lambda|}(R\sqrt{-t})\text{''}, \tag{2-2}$$

where "$J_{|\Delta\lambda|}(z)$" is a function which has the same zeroes, maxima and minima as $J_{|\Delta\lambda|}(z)$. The process considered is $a + b \rightarrow c + d$, and $\Delta\lambda = \lambda c - \lambda d - \lambda a + \lambda b$; R is a "radius of interaction" of about 1 fermi and $\beta(s)$ is asymptotically constant. We restrict our attention to processes which involve the exchange of something different from the vacuum, called non-diffractive, so that we will be concerned with f_I. Suffice it to say that the diffractive part is observed to be $0(\sqrt{s})$. The essential point in the discussion to follow is that (2-2) gives a t-structure independent of s, a fact well verified experimentally.

To explicitly construct an amplitude of Harari's type, Ingraham assumed the resonances in the s-channel to lie on a Regge trajectory, and then studied the conditions under which the amplitude had the aspect of (2-2). We describe briefly his procedure, restricting the calculations to equal-mass spinless particles ($|\Delta\lambda| = 0$). The generalization, on the light of recent results on fermion trajectories[7], brings nothing new.

Start with the partial-wave expansion of f_I:

$$f_I(s,t) = \frac{1}{2k}\sum_J (2J+1)\, f_J(s)\, P_J(\cos\theta), \tag{2-3}$$

$s \equiv 4(k^2 + M^2)$, $t \equiv -2k^2(1-\cos\theta)$, M being the mass of the external particles, and now assume each f_J to be dominated by a resonance of a recurrence family of signature r:

$$f_J(s) = \theta_J(s) \frac{\Gamma_J}{\mu_J - i\frac{\Gamma_J}{2} - \sqrt{s}}, \tag{2-4}$$

where μ_J and Γ_J are mass and width of the resonance of spin J. In (2-3), the exchange potentials require the replacement $P_J(\cos\theta) \rightarrow \frac{1}{2}[P_J(\cos\theta) + \tau P_J(-\cos\theta)]$. $\theta_J(s)$ is a factor that allows for deviations from the Breit-Wigner form which could come in the high energy region, where many-particle resonating states play a role. In any case, $\theta_J(s)$ should approach 1 in the low energy region, where a Breit-Wigner form can be derived[8]. As the resonances lie on a trajectory, there is a relation between mass and spin. This is taken as

$$\mu_J^2 = 4\left(J + \frac{1}{2}\right)^2 / R^2 + 4M^2, \tag{2-5}$$

which corresponds to

$$J + \tfrac{1}{2} = k_J R \tag{2-6}$$

for some "radius of interaction" R. k_J is the CM momentum such that $s_J = 4(k_J^2 + M^2) = \mu_J^2$.

This choice is favored by the semi-classical interpretation. In fact, at high energies, small impact parameters correspond mostly to production processes[9], the 2-body channels being obtained by peripheral collisions of large, roughly constant, impact parameters. In this case, the angular momentum is given by (2-6), R being the range of the forces, the maximum impact parameter allowed. It has been argued before[10] that this does not imply that Regge trajectories should behave asymptotically as $\sqrt{s}$. In fact, the leading trajectory could have a quite different asymptotic limit (for instance, linear), the interplay of a large number of nondominant trajectories giving rise to an "effective" angular momentum growing like $\sqrt{s}$. However, the cleanest situation would certainly be that of a "parabolic" Regge trajectory such as (2-6).

Performing now a Sommerfeld-Watson transformation on (2-3), we get

$$f = \frac{i}{8k} \int_C dJ(2J + 1)\,\theta(J, s) \frac{\Gamma}{D(J)} \left[\frac{P_J(-\cos\theta) + \tau P_J(\cos\theta)}{\sin \pi J}\right] \tag{2-7}$$

$D(J) = \mu(J) - i\Gamma/2 - \sqrt{s}$, where C encloses the points $J = 0, 1, 2, \ldots$. Distorting C to the contour $ReJ = -1/2$ plus an infinite semicircle to the right, picking up the pole at $J = a$, the zero of $D(J)$, one has

$$f = f_{pole} + f_{background}$$

with

$$f_{pole} = -\frac{\pi}{4k}(2\alpha + 1) R(\alpha)\, \theta(\alpha, s)\left[\frac{P_\alpha(-\cos\theta) + \tau P_\alpha(\cos\theta)}{\sin\pi\alpha}\right] \tag{2-8}$$

The position of the pole and the residue $R(\alpha)$ are

$$\alpha \simeq kR - \frac{1}{2} + i\frac{\Gamma R\omega}{4k} \equiv \alpha_R + i\alpha_I, \quad \omega = \sqrt{k^2 + M^2}, \tag{2-9}$$

$$(2\alpha + 1) R(\alpha) \simeq \frac{R^2\Gamma}{2}\left(\sqrt{s} + i\frac{\Gamma}{2}\right), \tag{2-10}$$

where $\left(\frac{\Gamma\omega}{2k^2}\right)^2$ was neglected. In the Regge limit, $\theta \simeq \sqrt{-t}/k \to 0$, the approximation $P_\alpha(\cos\theta) \simeq J_0(a + \frac{1}{2})\theta$ can be used. With the hypothesis that $(a, \sqrt{-t}/k)^2$ and $\alpha_I/(\alpha_R + 1)$ are very small, the amplitude is finally written as

$$Im\, f \simeq -\frac{\pi}{4k}(2\alpha + 1) R(\alpha)\, \theta(\alpha, s)\, Im\, S\,(a).$$

$$J_0\left[\left(\alpha_R + \frac{1}{2}\right)\frac{\sqrt{-t}}{k}\left(1 - \frac{\tau\alpha_I}{\sinh\pi\alpha_I}\cdot\frac{\sin\pi\alpha_R}{\alpha_R + \frac{1}{2}}\right)\right] \tag{2-11}$$

where $Im\, S_\tau(a) = \dfrac{\tau\cos\pi\alpha_R + \cosh\pi\alpha_I}{\sinh^2\pi\alpha_I + \sin^2\pi\alpha_R}(\tau\sin\pi a, -i\sinh\pi a_I)$.

Using now (2-9), it is seen that the dominant term is precisely of the form $J_0(R\sqrt{-t})$, giving a t-strncture which is s-independent. This is Ingraham's argument to choose "parabolic trajectories". It is important to observe that we are in the domain of high energies, so that all the assumptions made are asymptotic.

There is now the need of determining the behavior of $\Gamma(s)$. Being constrained by (2-11), it must be $O(k^\varepsilon)$, $0 < \varepsilon < 1$. Now, as we will see below, there is a connection, coming from analyticity, between the asymptotic behavior of the trajectory and that of the width. Ingraham chose the width to be

$$\Gamma_J = 2\gamma\, k_J, \tag{2-12}$$

γ constant, which is consistent with the approximation (2-11) and also with the results from analyticity. However, parabolic trajectories are queer objects, as it will become clear soon.

With the choice (2-12) and with

$$\theta_J(s) = \frac{2M(\sqrt{s})^{1+\varepsilon}}{\mu_J^{2+\varepsilon}} \qquad (\varepsilon > 0) \tag{2-13}$$

which is strongly suggested by the asymptotic behavior of $\beta(s)$ in (2-2) and by the convergence conditions of (2-3), the final expression for the leading term in the energy is given by

$$Im\, f \simeq \frac{\pi}{2} R^2 \gamma M J_0 (R\sqrt{-t}). \tag{2-14}$$

A comment is due here on the method. Resonances were introduced as Breit-Wigner partial-wave amplitudes with a width Γ which is a function of s. By looking at (2-8), we recognize in the denominator the term sin na which, as a goes near a physical value of the angular momentum, gives rise to a Breit-Wigner denominator with $\Gamma = \frac{a_l}{\sqrt{s}\alpha'_R}$, where the prime denotes differentiation with respect to s. This width function must, of course, be consistent with the Γ introduced in the partial wave amplitude. We show here that this is in fact the case, provided there is a relation between the mass of the resonance and its spin (in other words, that the resonances lie on a trajectory). The denominator of (2-4) vanishes when $\mu(J) = \sqrt{s} + i\Gamma_J/2$. Call μ^{-1} the inverse function of $\mu(J)$. We have

$$J = \mu^{-1}(\sqrt{s} + i\Gamma_J/2).$$

Assuming $\Gamma_J/2$ to be much smaller than $\sqrt{s}$, a Taylor expansion gives

$$J \simeq \mu^{-1}(\sqrt{s}) + i\frac{\Gamma_J}{2}\frac{d}{d\sqrt{s}}\mu^{-1}(\sqrt{s}). \tag{2-15}$$

It follows that

$$\frac{\Gamma_J}{2} = \frac{Im\, J}{\frac{d}{d\sqrt{s}} Re J} = \frac{Im\, J}{2\sqrt{s}\frac{d}{ds} Re J}, \tag{2-16}$$

as it should.

3. Resonance widths

A connection between the asymptotic behaviors of Regge trajectories and resonance widths follows from the use of the analyticity properties of the trajectories which are assumed (mesonic trajectories) to be real analytic in the s-plane cut along the real axis from the physical threshold to infinity. Though fermionic trajectories have a different domain of analyticity, the results we will obtain are true also for them provided they are exchange degenerate[7], what seems to be the case for all known trajectories. We review here briefly the results for the reader's convenience, referring, for more details, to Refs. 6 and 7. We assume that the trajectory does not grow as fast as an exponential along any direction of the upper halfplane and that the function $\alpha(s)/(-s)^{\varepsilon}$ has constant limits as $s \rightarrow \pm\infty$ along the real axis (E is a real positive number). The Phragmèn-Lindelöff's theorem then asserts that the limits must be equal. So, if

$$\lim_{s\rightarrow -\text{ao}} \frac{\alpha(s)}{(-s)^{\varepsilon}} -- A, \tag{3-1}$$

with A real, because of the analyticity, we must have

$$\lim_{s\rightarrow +\infty} \alpha(s) = -Ae^{-i\pi\varepsilon} s^{\varepsilon}. \tag{3-2}$$

Consequently, for large s,

$$\alpha_R(\mathrm{s}) = -A\cos(\pi\varepsilon)\, s^{\varepsilon}, \tag{3-3}$$

$$\mathrm{a,}\ (\mathrm{s}) = A\sin(\pi\varepsilon)\, s^{\varepsilon}, \tag{3-4}$$

and

$$\Gamma(s) = \frac{\alpha_I(s)}{\sqrt{s}\,\alpha'_R(s)} = -\frac{\tan(\pi\varepsilon)}{\varepsilon}\sqrt{s}, \tag{3-5}$$

so that the width function is asymptotically linear in the mass of the interpolated resonances. This is, in fact, the behavior assumed by Ingi-aham, so that everything seems consistent. However, let us examine in detail the case E = 1/2. Eq. (3-3) then tells us that either a, (s) $\sim$ 0 or else that it is dominated by a power of s lower than 1/2. Both cases are to be discarded, for then (2-11) will not give an s-independent t-structure, as required by experiment.

As a possible way out let us look for a trajectory that, behaving asymptotically in a different way as a whole, still has a $0(\sqrt{s})$ real part above threshold. A little thought shows that there is, indeed, such a trajectory. Consider a trajectory which behaves, for large negative s, as

$$-A\sqrt{(-s)}\ln(-s).$$

For large positive s we will then have

$$\alpha(s) = iA\sqrt{s}\,(\ln s - i\pi),$$

$$\alpha_R(s) = \pi A\sqrt{s}\,, \tag{3-6}$$

$$\alpha_I(s) = A\sqrt{s}\,\ln s. \tag{3-7}$$

It follows that

$$\Gamma(s) = \frac{2}{\pi}\sqrt{s}\,\ln s. \tag{3-8}$$

This is sufficient to show that the approximations necessary to get (2-11), viz., $\left(\frac{\Gamma\omega}{2k^2}\right)^2$, $(\alpha_I\sqrt{-t})^2/k^2$ and $\alpha_I/(\alpha_R + 1)$ very small, are not true at high energies, so that (2-11) does not follow.

4. Conclusion

We conclude that parabolic trajectories do not provide, contrary to a common belief, a mechanism to cancel out the s-dependence of the t-structure of 2-body non-diffractive high-energy scattering amplitudes.

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