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A New Approach Toward Dual Models*

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An example is given of a partial differential equation for scattering amplitudes leading to solutions with the charactenstics of dual models.

Dá-se um exemplo de equações diferenciais a derivadas parciais para amplitudes de espalhamento cujas soluções têm características de modêlos duais.

1. Introduction

In this paper, we would like to outline a new attempt towards the construction of dual models. The novelty of the approach consists in using partial differential equations to construct amplitudes which i) result from the interpolation of infinitely many resonances, ii) satisfy crossing, iii) are asymptotically Regge behaved in the forward (backward) direction of each relativistic channel and iv) are exponentially decreasing in the non-forward (non-backward) directions in the physical s, t, u regions.

The approach has the inherent limitation represented by the difficulty of dealing with partial differential equations and, furthermore, possesses some of the defects shared by other approaches to dual models like a large arbitrariness in the formulation of the model.

Due to this kind of limitations, we do not attempt here a very general construction of dual models through this new method, but we limit ourselves to giving a specific example of an equation whose solution satisfies the previously stated requirements of duality.

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This kind of approach may turn out to be too complicated to be useful in practice. We believe, however, that it is rather interesting particularly in that it shows the possibility of new ways towards the construction of dual models.

2. Formulation of the Model

The starting point of our approach is a generalization of an idea which was put forward several years ago[1].

In Ref. 1, the attempt was made to write down an equation which should reduce to the Legendre equation (i.e., giving an asymptotic Regge behavior) when one of the three Mandelstam variables is kept fixed (say t) and we take the asymptotic limit $s \to +\infty$ and simultaneously $u \to -\infty$ because of the constraint $s + t + u = M$ (M being the sum of the squared masses of the four particles involved). This limit would correspond to having Regge behavior in the forward direction of the physical s-channel.

Confining for simplicity our analysis to the consideration of an amplitude completely crossing symmetric in the three variables s, t, u, we want to give here an example of an equation whose solution has the following properties: i) Regge-like asymptotic behavior whenever one of the three variables is kept fixed and the other two go to $+\infty$ and $-\infty$ respectively, ii) exponential decrease when all three variables go to $\pm\infty$ (in the physical regions of the s, t, u channels), iii) infinitely many resonances in each of the physical channels.

The wanted equation must, necessarily, be a partial differential equation. Furthermore, if it has to reduce to the Legendre equation in the forward (backward) directions, it must be at least of second order and, for simplicity, we shall consider only linear equations.

The equation we suggest is the following

$$\left\{\frac{(t-c)(u-c)}{f_s}\frac{\partial_s^2}{\partial t^2}+\frac{(s-c)(u-c)}{f_t}\frac{\partial_t^2}{\partial u^2}+\frac{(s-c)(t-c)}{f_u}\frac{\partial_u^2}{\partial s^2}+\right.$$

$$\left.+\frac{u-t}{g_s}\frac{\partial_s}{\partial t}+\frac{s-u}{g_t}\frac{\partial_t}{\partial u}+\frac{t-s}{g_u}\frac{\partial_u}{\partial s}+1\right\}A(s,t,u)=0 \quad (1)$$

where $\frac{\partial_s}{\partial t}\left(\frac{\partial_t}{\partial u}, \frac{\partial_u}{\partial s}\right)$ means derivative with respect to $t(u, s)$ along a line

$s = \text{const}$ ($t = \text{const}$, $u = \text{const}$). In Eq. (1), c is a constant which (for reasons to be discussed later), we choose $\geq M$.

Crossing symmetry in Eq. (1) is guaranteed if we take f, and g_s to be symmetric in the variables u, t and the solution $A(s, t, u)$ will also be crossing symmetric if the boundary conditions are also crossing symmetric.

What we want to show now is that, by appropriately choosing f_s and g_s, the previously stated properties (i, ii, iii) follow for $A(s, t, u)$.

A) Regge-Like Asymptotic Behavior

In order to recover asymptotic Regge behavior when one of the variables is kept fixed and the other two go to $+\infty$ and $-\infty$ respectively, we consider the case

$$t = \text{fixed and negative}, \quad s \to +\infty, \quad u \to -\infty,$$

which corresponds to the asymptotic behavior in the forward direction for the s-channel. We impose that

$$\begin{aligned} \lim_{s,t,u} (g_s, t_s) &= f_0(s), \\ \lim_{s,t,u} (g_t, f_t) &= f_0(t), \\ \lim_{s,t,u} (g_u, f_u) &= f_0(u) \end{aligned} \tag{2}$$

(where $\lim\limits_{s,t,u}$ means the limit for one or several variables going to infinity), and that

$$f_0(s) = \alpha(s)[\alpha(s) + 1], \tag{3}$$

where $\alpha(s)$ is the Regge trajectory about which we assume the usual analyticity condition

$$\alpha(s) = a_0 + a_1 s + \frac{s}{\pi} \int_{s_0}^{\infty} \frac{Im\, \alpha(s')}{s'(s'-s)} ds' \tag{4}$$

(with $a_1 > 0$ and $0 < a_0 \leq 1$). Since, asymptotically,

$$\alpha(s) \xrightarrow[s \to \infty]{} a_1 s, \tag{5}$$

Eq. (3) implies

$$\lim_{s \to \infty} f_0(s) = a_1^2 s^2 . \tag{6}$$

Because of Eq. (2), the same limit obtains for $f_s(g_s)$.

Under the above conditions, Eq. (1) becomes as $s \to +\infty$, at fixed t,

$$\left[-\frac{s^2}{f_0(t)}\frac{\partial^2}{\partial s^2}-\frac{2s(t-c)}{f_0(u)}\frac{\partial^2}{\partial s\,\partial t}+s(t-c)\left(\frac{1}{f_0(u)}-\frac{1}{f_0(s)}\right)\frac{\partial^2}{\partial t^2}-\right.$$

$$\left.-\frac{2s}{f_0(t)}\frac{\partial}{\partial s}+s\left(\frac{1}{f_0(u)}-\frac{1}{f_0(s)}\right)\frac{\partial}{\partial t}+1\right]A(s,t)=0. \qquad (7)$$

From **Eq.** (7) (and keeping Eqs. (2, 3) in mind), one can see that the following asymptotic solution obtains:

$$A_1(s,t) \underset{\substack{s\to\infty \\ t\ fixed}}{\sim} g(t)\, s^{\alpha(t)}\,. \qquad (8)$$

This is the same solution admitted asymptotically by the Legendre equation, i.e., by

$$\left[s^2\frac{\partial^2}{\partial s^2}+2s\frac{\partial}{\partial s}-\alpha(t)\,(\alpha(t)+1)\right]A(s,t)=0. \qquad (9)$$

Thus, as $s \to \infty$ at fixed t, there exists a Regge-behaved solution of our equation.

Because of the crossing properties previously mentioned, analogous behaviors obtain whenever other forward and backward limits are taken; for instance as $s \to +\infty$ at fixed u,.

$$A(s,u) \underset{\substack{s\to+\infty \\ u\ fixed}}{\rightarrow} g(u)\, s^{\alpha(u)}\,. \qquad (10)$$

B) Non-Forward Behavior

In order to investigate the asymptotic behavior when all the three Mandelstam variables go to infinity (which we, loosely speaking, refer to as "non forward" behavior) it is convenient to use polar coordinates. Namely, we set

$$s=\alpha_1\rho+\frac{M}{3},\qquad t=\alpha_2\rho+\frac{M}{3},\qquad u=\alpha_3\rho+\frac{M}{3}, \qquad (11)$$

where

$$\alpha_1=\cos\varphi,\ \alpha_2=\cos\left(\varphi-\frac{2\pi}{3}\right),\ \alpha_3=\cos\left(\varphi+\frac{2\pi}{3}\right). \qquad (12)$$

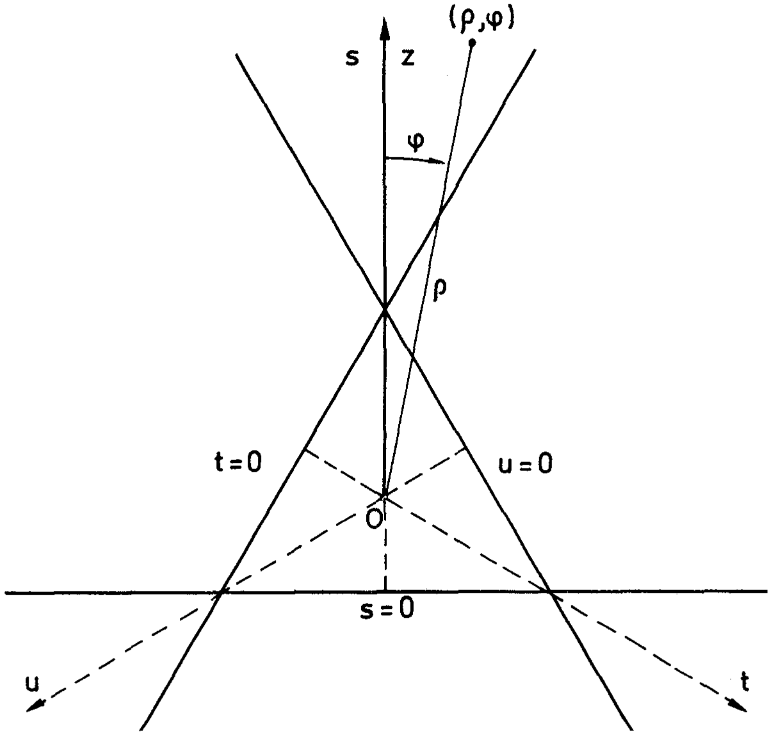

Fig. 1 - The polar coordinate system in the (s, t, u) plane, as defined by Eqs. (11) and (12).

The choice of coordinates (11, 12) has the only advantage of maintaining the crossing properties in the asymptotic behavior $\rho \to \infty$. So, for instance, as $\rho \to \infty$, the physical s-channel domain corresponds to $-\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{6}$, the physical u-channel to $-\frac{5\pi}{6} \leq \varphi \leq -\frac{\pi}{2}$ and the physical t-channel to $\frac{\pi}{2} \leq \varphi \leq \frac{5\pi}{6}$ (see Fig. 1). From Eqs. (11, 12) we have

$$\rho = \tfrac{2}{3}(s^2 + t^2 + u^2 - st - su - tu)^{1/2},$$

$$\varphi = \mathrm{arctg}\,\frac{\sqrt{3}\,(t-u)}{3s-M} \qquad (13)$$

Notice that a straight-line passing through the origin of the polar system chosen, corresponds asymptotically to a line $\theta = \mathrm{const}$ where θ is the c.m. scattering angle.

Using Eqs. *(11, 12, 13)*, in the limit $p \to \infty$, Eq. *(1)* becomes

$$\left[\frac{80(\alpha_1\alpha_2\alpha_3)^2-9}{16(\alpha_1\alpha_2\alpha_3)^2 a_1^2}\frac{\partial^2}{\partial\rho^2}-\frac{6\beta_1\beta_2\beta_3}{\alpha_1\alpha_2\alpha_3 a_1^2\rho}\frac{\partial^2}{\partial\rho\,\partial\varphi}-\frac{1}{a_1^2\rho}\frac{\partial^2}{\partial\varphi^2}-\right.$$
$$\left.-\frac{4}{a_1^2\rho}\frac{\partial}{\partial\rho}+1\right]A(\rho,\varphi)\underset{\rho\to\infty}{=}0, \qquad (14)$$

where a, was defined in Eqs. *(4,* 5) and

$$\beta_i = -\sin\varphi_i, \qquad (15)$$

if $\alpha_i = \cos\varphi_i$ (see Eq. *(12))*.

Eq. (14) admits the asymptotic solution

$$A_2(\rho,\varphi)\underset{\rho\to\infty}{\simeq}0(e^{-p(\varphi)\rho}), \qquad (16)$$

where $p(\varphi)$ satisfies the equation

$$\left(\frac{dp(\varphi)}{d\varphi}+\frac{3\beta_1\beta_2\beta_3}{\alpha_1\alpha_2\alpha_3}p(\varphi)\right)^2+4p^2(\varphi)-a_1^2=0. \qquad (17)$$

Setting

$$p(\varphi)=\alpha_1\alpha_2\alpha_3\,q(\varphi), \qquad (18)$$

Eq. (17) becomes

$$q'^2+4q^2-\left(\frac{a_1}{\alpha_1\alpha_2\alpha_3}\right)^2=0. \qquad (19)$$

Analysing the solution of Eq. *(19)* in the physical region of the s-channel $\left(|\varphi| \leq \frac{\pi}{6}\right)$, one can see that there exists a solution which is even and positive for all values of $|\varphi| \leq \frac{\pi}{6}$ and such that

$$q(0)=\frac{a_1}{2\alpha_1\alpha_2\alpha_3}\bigg|_{\varphi=0}=2a_1>0,$$
$$q(\varphi)\xrightarrow[\varphi\to\pm\frac{\pi}{6}]{}\frac{2a_1}{3}\ln\left(k\frac{1+4\beta_1\beta_2\beta_3}{1-4\beta_1\beta_2\beta_3}\right), \qquad (20)$$

where $k > 0$ is an integration constant.

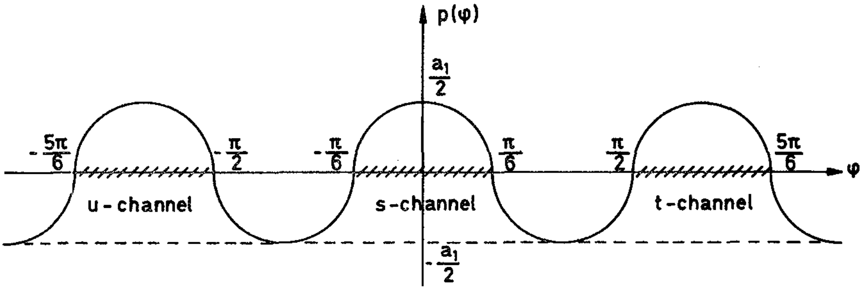

Fig. 2 - The behavior of $p(\varphi)$, the parameter of the exponential in Eq. (16), as φ varies.

Since $\lim_{\varphi \to \pm\frac{\pi}{6}} \beta_1 \beta_2 \beta_3 = \frac{1}{4}$, $q(\varphi)$ diverges logarithmically to $+\infty$ as φ tends to the boundary of the physical s-channel (forward and backward directions). Because of Eq. (18), this implies that $p(\varphi)$ is always positive in the physical s-channel with a maximum for $\varphi = 0$ and is monotonically decreaising as we go from $\varphi = 0$ to $\varphi \to \pm \frac{\pi}{6}$.

The same solution continues outside the physical s-channel to the unphysical regions and to the t-and u-channels according to the scheme of Fig. 2. This shows that if the boundary conditions are assumed such that the solution of Eq. (1) outside the forward (backward) direction is decreasing as $\rho \to \infty$ inside one of the physical regions, then i) the rate of decrease is exponential and ii) in the non-physical regions the same solution diverges exponentially.

C) According to the previous results, we can attempt to impose boundary conditions of the form

$$A(s,t) \underset{s\to\infty}{\longrightarrow} C_1\, A_1\,(s,t) + C_2\, A_2\,(s,t). \tag{21}$$

We then have:

i) $\varphi = \pm \frac{\pi}{6}$ ($t =$ const or $u =$ const),

$$A_1 \underset{s\to\infty}{\sim} \begin{cases} 0(\rho^{\alpha(t)}), \text{ if } \varphi = \frac{\pi}{6}, \\ 0(\rho^{\alpha(u)}), \text{ if } \varphi = -\frac{\pi}{6}, \end{cases} \tag{22}$$

$$A, \sim \text{const.}$$

Since for $\varphi = \frac{\pi}{6}$ we can always choose the polar system so that t be very close to zero where $\alpha(t)$ is positive, along the forward direction the solution (21) has Regge behavior. The same conclusions hold in the backward direction of the s-channel and, by crossing, in the forward and backward directions of the t-and u-channels as well.

ii) $\varphi \neq \pm \frac{\pi}{6}$, then in the physical region

$$\begin{aligned} A_1 &\sim 0(e^{a_1 \alpha_2 \rho \ln(\alpha_1 \rho)}), \\ A_2 &\sim 0(e^{-p(\varphi)\rho}), \end{aligned} \tag{23}$$

and A_2 becomes negligible as $\rho \to \infty$ since $a_1 > 0$ and $a_2 < 0$.

In conclusion,

$$\begin{aligned} A(s,t) &\xrightarrow[\rho \to \infty]{} A_1 , \\ A(s,t) &\xrightarrow[\substack{\rho \to \infty \\ |\varphi| < \frac{\pi}{6}}]{} A_2 . \end{aligned} \tag{24}$$

D) Poles of the Amplitude

The previous results depend only on the form of Eq. (1) and on the asymptotic behavior of f_s, g_s, etc. We can now proceed to specify a form for f_s and g_s which, while preserving the asymptotic limit (2) in every direction of the physical regions of the s, t, u channels, be also such that the amplitude possesses an infinity of resonances in each channel. Whereas there must exist a large number (presumably infinite) of forms for f_s, g_s that satisfy these requirements, we will show that there exists at least one. We choose

$$f_s = f_0(s) - (t-c)(u-c)\left[\alpha'(t)\,\beta\left(1-\frac{\alpha(t)}{2}\right) + \alpha'(u)\,\beta\left(1-\frac{\alpha(u)}{2}\right)\right]^2 + \\ + \beta^2\left(1-\frac{\alpha(s)}{2}\right) e^{-(t^2+u^2)} \tag{25}$$

$$g_s = f_0(s) - \left[\alpha'(t)\beta\left(1-\frac{\alpha(t)}{2}\right) + \alpha'(u)\beta\left(1-\frac{\alpha(u)}{2}\right)\right]^2 + \beta^2\left(1-\frac{\alpha(s)}{2}\right) e^{-(t^2+u^2)}, \tag{26}$$

where

$$\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z+n} = \psi(z) - \psi\left(\frac{z}{2}\right) - \ln 2 \tag{27}$$

and $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$:

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z). \tag{28}$$

First we notice that, as required, f_s and g_s are symmetric by interchange of u and t.

Secondly, from Eq. (27), when $z \to -n$,

$$\lim_{z \to -n} (z + n)^2 \beta^2 (z) = 1, \tag{29}$$

thus f_s and g_s have double poles whenever one of the Regge trajectories ($\alpha(s)$, $\alpha(t)$ or $\alpha(u)$) is equal to $2(n+1)$ with $n = 0, 1, 2, \dots$. For instance,

$$\lim_{\alpha(t) \to 2(n+1)} (-\alpha(t) + 2u)^2 f_s = -4d^2(t)(t-c)(u-c). \tag{30}$$

More exactly,

$$\begin{aligned}
&\lim_{\alpha(t) \to 2(n+1)} (t - t_n)^2 f_s = -4(t-c)(u-c), \quad (\alpha(t_n) = 2n+2),\\
&\lim_{\alpha(u) \to 2(n+1)} (u - u_n)^2 f_s = -4(t-c)(u-c), \quad (\alpha(u_n) = 2n+2),\\
&\lim_{\alpha(s) \to 2(n+1)} (s - s_n)^2 f_s = \frac{4}{\alpha'^2(s_n)} e^{-(t^2+u^2)}, \quad (\alpha(s_n) = 2n+2),
\end{aligned} \tag{31}$$

and analogously for f_t and f_u (the similar relations for g_s, g_t, g_u follow at once from Eqs. (25, 26, 31)).

Thirdly, remembering that

$$\psi(z) \xrightarrow[|z| \to \infty]{} \ln z - \frac{1}{2z} - 0\left(\frac{1}{z^2}\right), \quad (z \neq \text{real negative}), \tag{32}$$

and using Eq. (27) we have

$$\beta(z) \xrightarrow[|z| \to \infty]{} \frac{1}{2z} + 0\left(\frac{1}{z^2}\right), \quad (z \neq \textit{real negative}). \tag{33}$$

Thus, with the exclusion of $\alpha(s)$, $\alpha(t)$ or $\alpha(u)$ real and positive, we see that the lirnits (2) hold. Since, because of the analyticity conditions (4) imposed on the Regge trajectory, in the Mandelstarn plane, none of the a's can be real and positive, the limits (2) hold uniformly in the entire s, t, u plane and all the previous results remain valid.

Considering now Eq. (1) in the neighborhood of a pole $s = s_n$, we have (if t and u are not equal to c):

$$\left\{\frac{(t-c)(u-c)\,\alpha'^2(s)(s-s_n)^2}{4e^{-(t^2+u^2)}}\frac{\partial^2}{\partial v^2}-\frac{(s-s_n)^2}{4}\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial v}\right)^2-\frac{(s-s_n)^2}{4}\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial v}\right)^2+\right.$$

$$+\frac{(u-t)\,\alpha'^2(s)(s-s_n)^2}{4e^{-(t^2+u^2)}}\frac{\partial}{\partial v}+\frac{(s-u)(s-s_n)^2}{4}\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial v}\right)-$$

$$\left.-\frac{(t-s)(s-s_n)^2}{4}\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial v}\right)+1\right\}A(s,v)=0, \qquad (34)$$

where $v = t - u$. Thus, near a pole, $A(s, v)$ must be either of the form

$$A \sim \frac{h_n(s,v)}{s-s_n}, \qquad (35)$$

where $h_n(s,v)$ is regular for $s = s_n$ or of the form

$$A \sim \tilde{h}_n(s,v)(s-s_n)^2. \qquad (36)$$

It is easy to convince oneself that if either $t = c$ or $u = c$, there are no poles. The points $s = s_n$ and $t = c$ (or $u = c$) lie, however, outside the physical regions.

Solutions of the form (35), i.e., amplitudes with an infinity of resonances in each of the physical regions thus may exist. Inserting Eq. (35) into Eq. (1), we have for the residue

$$\frac{\partial h_n}{\partial s}+\frac{C(s_n,t)+C(s_n,u)}{2}h_n-\frac{3s-M}{4}h_n=0, \qquad (37)$$

where

$$C(s_n,u)=\frac{\alpha''(s_n)}{\alpha'(s_n)}+(-1)^n\left[\alpha'(s_n)\,\beta_n\left(1-\frac{\alpha(s_n)}{2}\right)+\alpha'(u)\,\beta\left(1-\frac{\alpha(u)}{2}\right)\right], \qquad (38)$$

$$\beta_n(z)=\beta(z)-\frac{(-1)^n}{z+n},$$

with the solution

$$h_n(s,v)=k_n(v)\exp\left[\frac{3s^2}{8}-\left(\frac{M}{4}+\frac{C(s_n,t)+C(s_n,u)}{2}\right)s\right], \qquad (39)$$

where $k_n(v)$ is a function of v only.

From Γ,' (39) we see that i) if the boundary conditions are assumed that $k_n(\nu)$ is a real positive function, then the residues of all the infinitely many resonances (poles in the second sheet) are positive and ii) ancestors appear. This last point follows from the observation that the t, u dependence of the exponential in Eq. (39) cannot be killed by the ν dependence of $k_n(\nu)$.

E) Other Singularities

The investigation of the singularites other than poles of the solution of our eqiiation is extremely difficult due to its nature. Quite generally, however, we can expect branch points whenever the Regge trajectories have branch point singularities (starting at $s = M$ or $t = M$ or $u = M$). Due to the exponentially increasing behavior of the solution in the unphysical regions, customary analyticity properties (like Mandelstam representation) are ruled out.

3. Conclusions

In this paper we have shown an example of partial differential equation by which one can meet the requirements of a dual model. The main drawback of the approach consists in the extreme complexity of a problem in which one has to deal with this kind of equations.

Like in other approaches, one of the nice features of this attempt to construct dual models from differential equations is the flexibility provided by this method of which the one illustrated here is only an example.

Reference

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