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Quantum Theory of the Magnon-Phonon Interaction in a Time-Dependent Magnetic Field*

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The theory of the interaction between magnons and phonons in a ferromagnetic crystal subject to time-dependent magnetic fields is developed by quantum mechanical methods. The theory has been previously developed only in semiclassical terms and it revealed that under ccrtain wnditions one can convert completely a state of lattice vibration (phonons) into a state of magnetic excitation (magnons). The theory developed here is based on the quantization of the magnetoelastic fields. With the Heisenberg equations of motion for the magnon, phonon and magnetoelastic excitation operators, it is shown that the results of the quantum theory are essentially the same as those previously obtained. In a time-dependent field, a magnetoelastic excitation has an invariant momentum but variable energy. When the field gradient at the crossover region is much smaller than a critical value, an initial elastic excitation can be completely converted into a magnetic excitation, or vice-versa It is shown further that if the system is initially in a coherent state, its coherence properties are maintained regardless of the time-dependence of the field.

Nêste trabalho estudamos quânticamente o problema da interação entre magnons e fonons, em um material ferromagnétiw submetido a um campo magnético externo variável no tempo. Êste problema foi estudado anteriormente no formalismo semi-clássico e seus resultados mostram, entre outros fenômenos, que em certas condições é possível converter completamente um estado de vibrações da rêde (fonons) em vibrações dos spins (magnons) e vice-versa A teoria aqui desenvolvida baseia-se no formalismo de quantização dos campos magnetoelásticos. Através das equações de Heisenberg para os operadores de magnons, de fonons e de excitações magnetoelásticas, mostra-se que a teoria quântica leva a resultados que são essencialmente os mesmos obtidos anteriormente. Em um campo dependente do tempo, uma excitação magnetoelástica propaga-se com momenturn constante e energia variável. Quando o gradiente do campo na região de cruzamento magnon-fonon é muito menor que um valor critico, uma excitação anteriormente elástica é convertida integralmente em excitação magnética e vice-versa Mostra-se também que se o sistema encontra-se inicialmente em um estado coerente, a coerência é mantida durante a variação do campo.

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1. Introduction

The problem of the propagation of waves in a medium whose parameters vary in time in a non harmonic fashion has received little attention in the past. One of the reasons for this fact is, perhaps, that the speed of electromagnetic waves is very large, and this is the type of wave most used for studying propagation phenomena. As a consequence, it is difficult to control the time variation of a material parameter during the short traveling time of a light wavepacket in a sample. With the understanding of the properties of slower types of wave excitations in solids, this problem began to attract some interest. The behavior of an electromagnetic wave in a medium with time-dependent dielectric constant and permeability has been considered[1]. Recently it was investigated[2] the more interesting case of a wave involving excitations of two different natures, namely that of a magnetoelastic wave propagating in a medium subject to a time-dependent magnetizing field. The theoretical analysis of this situation has been carried out semiclassically[2]. Here we investigzte this process quantum mechanically.

Spin waves, whose quanta are called magnons, can be excited in a ferro-, ferri-, or antiferromagnetic material under a static magnetic field, by means of a microwave magnetic field. Due to the magnetostrictive properties of crystals, spin waves are usually coupled to elastic vibrations (phonons), resulting in what are called magnetoeiastic waves[3-6]. These waves can be coherently excited in low-loss materials, with a velocity which is controlled by the magnetizing field, leading to important device applications. The possibility of their technological uses have resulted in the great attention they received recently.

The propagation of magnetoelastic waves in a spatially varying magnetic field was studied theoretically by Schlomann and Joseph[7]. These authors have shown that these waves propagate in the field gradient with constant frequency, constant power flow, but variable momentum and wave number. They also showed that if a spin wavepacket traverses the crossover region (the region where the magnon and phonon wave numbers are comparable) very quickly, wost of the energy stays in the spin wave state. On the other hand, if the field gradient is very small, most energy is converted into the elastic state. They have actually demonstrated that the magnon-phonon conversion efficiency is a continuous function of the ratio between the field gradient and a "critica1 gradient". Experiments confirming the possibility of converting a spin excitation into coherent lattice vibrations with a spatial gradient were realized sometime ago[8], whereas the conversion efficiency as a function of the gradient only recently has been measured[9].

A somewhat analogous situation, namely the propagation of magnetoelastic waves in spatially uniform time-varying magnetic fields, has also been investigated [2,10] both theoretically and experimentally. In this instance it was shown that propagation occurs at constant momentum and wave number, but variable energy and frequency. Magnon-phonon conversion at the crossover is also possible and the wnversion efficiency is a function of the time gradient of the field. The theoretical analysis of this situation was carried out completely with the semiclassical formalism, which is based on the equations of motion for the magnetization[4,11] and the lattice displacement[4]. Obviously some questions are not answered in this treatment, in particular those related to the evolution of the coherence of the quantum state during the magnon-phonon conversion. This paper is devoted to the quantum mechanical analysis of this situation.

In Sec. 2, we review the transformations used to obtain magnon and phonon creation and annihilation operators which diagonalize the Hamiltonians for the magnetic and the elastic systems in the case of a static applied field. The magnetoelastic interaction Hamiltonian is also expressed in terms of these operators.

In Sec. 3, the total magnon-phonon Hamiltonian is diagonalized and the possible states of the system are discussed. In Secs. 4 and 5, we consider the equations of motion and the invariant operators of the system under a time-dependent applied field. Sec. 6 is devoted to the solutions of the Heisenberg equations of motion of the magnon and phonon operators for the case of the time-dependent field, and to the calculation of the momentum conversion efficiencies. Finally, in Sec. 7, we discuss the relations between the results of the quantum and semiclassical treatments of the problem.

2. The Hamiltonian for the Magnon-Phonon System in the Case of a Static Magnetic Field

The analysis presented in this paper applies to a simple Heisenberg ferromagnetic cubic crystal, magnetized to saturation by a uniform magnetic field which is allowed to vary in time. In this section, the field is assumed to be static. The total Hamiltonian of the system can be expressed in terms of the spin operator and the elastic displacement operator at each lattice site. In a first approximation, the Hamiltonian can be written as the sum of three parts, a magnetic component depending only on the spins, a pure elastic one, and a magnetoelastic term depending on both the spin and the elastic displacement.

a) Quantization of the Spin Excitations

The most important contributions of the spin system to the total Hamiltonian arise from the interaction between individual spins with the external field (Zeeman interaction), and the exchange and dipolar interactions between neighboring spins. These components can be written as[6]

$$\mathscr{H}_m = -2\mu \sum_i \mathbf{S}_i \cdot \mathbf{H}_i - \sum_{i \neq j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$$
$$+ \tfrac{1}{2} \sum_{i \neq j} (2\mu)^2 \left[\mathbf{S}_i \cdot \mathbf{S}_j / r_{ij}^3 - 3(\mathbf{r}_{ij} \cdot \mathbf{S}_i)(\mathbf{r}_{ij} \cdot \mathbf{S}_j)/r_{ij}^5\right], \quad (2\text{-}1)$$

where μ is the Bohr magneton, $\mathbf{S}_i$ is the spin at the lattice site i (in units of $\hbar$), assumed to have g-factor of 2, J_{ij} is the exchange constant of spins $\mathbf{S}_i$ and $\mathbf{S}_j$ and $\mathbf{r}_{ij}$ is their relative position vector. $\mathbf{H}_i$ is the applied field at site i, lying in the z-direction of a cartesian coordinate system. The electron spin is taken parallel to its magnetic moment, as assumed in most quantum treatments of spin waves. The Hamiltonian (2-1) can be cast into a diagonal form with a series of canonical transformations performed on the spin operators, which are known as the Holstein-Primakoff transformations [6, 12]. The first transformation is

$$S_i^+ = S_i^x + iS_i^y = (2S)^{1/2}(1 - a_i^\dagger a_i/2S)^{1/2} a_i, \quad (2\text{-}2)$$
$$S_i^- = S_i^x - iS_i^y = (2S)^{1/2} a_i^\dagger (1 - a_i^\dagger a_i/2S)^{1/2}, \quad (2\text{-}3)$$
$$S_i^z = S - a_i^\dagger a_i, \quad (2\text{-}4)$$

where $a_i^\dagger$ and a_i are creation and annihilation operators that satisfy the usual Bose commutation relations, and are also localized at the lattice site i. The collective excitation Bose operators are introduced by

$$a_i = N^{-1/2} \sum_k \left[u_k e^{i\mathbf{k} \cdot \mathbf{r}_i} c_k - v_k e^{-i\mathbf{k} \cdot \mathbf{r}_i} c_k^\dagger\right], \quad (2\text{-}5)$$

where N is the number of spins and k denotes the wave vector of the excitation. In the indices, the vector sign is kept out for clarity of the notation. The summation in (2-5) extends over the whole Brillouin zone. The parameters u_k and v_k are given by[6]

$$u_k = \cosh \mu_k, \qquad v_k = e^{i2\phi_k} \sinh \mu_k, \quad (2\text{-}6)$$

where

$$\tanh 2\mu_k = |B_k| / A_k \quad (2\text{-}7)$$

and, for a simple cubic lattice, with exchange only between nearest neighbors, in the long wavelength limit, A_k and B_k are[6]

$$A_k = Dk^2 + 2\mu H + \mu 4\pi M \sin^2\theta_k, \qquad (2\text{-}8)$$
$$B_k = \mu 4\pi M \sin^2\theta_k \exp(-i2\phi_k),$$

where $D = 2SJa^2$, a is the lattice parameter, M is the saturation magnetization and θ_k and ϕ_k are the polar and azymuthal angles of the wave vector, shown in Fig. 1. The inverse transformation is

$$c_k = N^{-1/2}\sum_i [u_k e^{-ik\cdot r_i} a_i + v_k e^{ik\cdot r_i} a_i^\dagger], \qquad (2\text{-}9)$$

so that

$$[c_k, c_{k'}] = 0, \quad [c_k, c_{k'}^\dagger] = \delta_{kk'}. \qquad (2\text{-}10)$$

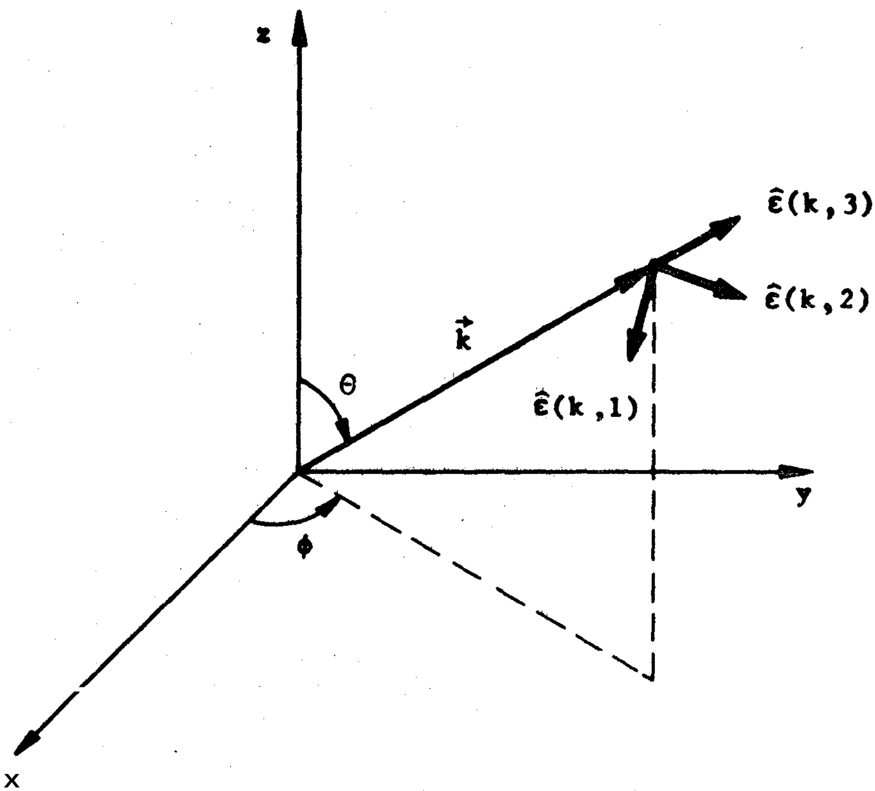

Fig. 1 - Coordinate system and polarization vectors for wavevectors.

Using (2-2)-(2-10)and the closure relation, one can show that the Hamiltonian (2-1) becomes

$$\mathscr{H}_m = \sum_k \hbar\omega_m(\mathbf{k})\,(c_k^\dagger c_k + \tfrac{1}{2}), \qquad (2\text{-}11)$$

where higher order terms are neglected. $\hbar\omega_m(\mathbf{k})$ is the magnon energy given by

$$\begin{aligned} A\omega_k(\mathbf{k}) &= (A_k^2 - |B_k|^2)^{1/2} = \\ &= (2\mu H + Dk^2 + \mu 8\pi M \sin^2\theta_k)^{1/2}(2\mu H + Dk^2)^{1/2}. \end{aligned} \qquad (2\text{-}12)$$

The form of the magnon Hamiltonian (2-11) is the source of a well-known and fruitful analogy between the mode amplitudes of the collective spin excitations and the coordinates of an assembly of one-dimensional harmonic sscillators. The operators $c_k^\dagger$ and c_k are interpreted as creation and annihilation operators of quanta of spin excitation, called magnons. The creation of a magnon corresponds to the flipping of a spin by one unit, and the flipping process propagates through the crystal instead of staying localized. In (2-11), the terms involving three or more magnon operators which were neglected, represent magnon-magnon interactions which are responsible for relaxation mechanisms, saturation effects and other phenomena.

None of the operators presented so far corresponds to an observable variable. In experiments one detects spin excitations by means of the magnetization operator, which is introduced in a continuous description of the crystal through the relation $\mathbf{M}(\mathbf{r}) = 2\mu \sum_i \mathbf{S}_i/\delta V$. Here the summation runs over the sites inside a small volume δV, around the point $\mathbf{r}$, which contains many sites. Using (2-2), (2-3) and (2-5), we can express M in terms of the magnon operators. In the Heisenberg picture, the components transverse to the static field are, to first order in the magnon variables[13] (represented by small letters),

$$\mathbf{m}(\mathbf{r}, t) = \mathbf{m}^{(+)}(\mathbf{r}, t) + \mathbf{m}^{(-)}(\mathbf{r}, t),$$

where

$$\begin{aligned} m_x^{(+)}(\mathbf{r}, t) &= [m_x^{(-)}(\mathbf{r}, t)]^\dagger = (\mu M/V)^{1/2} \sum_k (u_k - v_k^*)\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_m t}\, c_k \\ m_y^{(+)}(\mathbf{r}, t) &= [m_y^{(-)}(\mathbf{r}, t)]^\dagger = i(\mu M/V)^{1/2} \sum_k (u_k + v_k^*)\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_m t}\, c_k, \end{aligned} \qquad (2\text{-}13)$$

where V is the volume of the crystal. The longitudinal component of M is $M_z = M - m_z$, where $m_z \simeq (m_x^2 + m_y^2)/2M$.

b) Quantization of the Lattice Vibrations

Let us consider that the ferromagnetic crystal is a continuous solid, elastically isotropic, with average mass density *p*. We also assume that it is a cubic crystal so that, within the linear approximation, the relation between the stress tensor and the strain tensor involves only two different elastic constants[14], c_{12} and c„ . The elastic deformations of the solid are expressed in terms of the vector displacement R = r' – r, where r is the initial position of an atom or of a volume element, and *r'* is the position after deformation. The contributions of the elastic system to the Hamiltonian anse from the kinetic and potential energies. In the linear approximation, the elastic Hamiltonian can be written as[15]

$$\mathcal{H}_e = \int d^3 r \left[\frac{\rho}{2} \frac{\partial R_i}{\partial t} \frac{\partial R_i}{\partial t} + \frac{\alpha}{2} \frac{\partial R_i}{\partial x_i} \frac{\partial R_j}{\partial x_j} + \frac{\beta}{2} \frac{\partial R_i}{\partial x_j} \frac{\partial R_i}{\partial x_j} \right], \quad (2\text{-}14)$$

where the repeated indices indicate summation, and $\alpha =$ c,, + c„ and $\beta =$ *c,,*. The cartesian coordinate system has its axes lying along the [100] crystallographic directions. It is useful to introduce the canonical momentum density through the relation $\Pi_i = \mathcal{L}/\dot{R}_i = \rho \dot{R}_i$, where $\mathcal{L}$ is the Lagrangian density. In order to obtain the collective excitation operators for the elastic system, we make the canonical transformation

$$R_i(\mathbf{r}, t) = \sum_{k,\mu} \varepsilon_{i\mu}(\mathbf{k}) \left(\frac{\hbar}{V}\right)^{1/2} Q_k^\mu(t)\, e^{ik\cdot r},$$
$$\Pi_i(\mathbf{r}, t) = \sum_{k,\mu} \varepsilon_{i\mu}(\mathbf{k}) \left(\frac{\hbar}{V}\right)^{1/2} P_k^\mu(t)\, e^{-i\mathbf{k}\cdot \mathbf{r}}, \quad (2\text{-}15)$$

where $\varepsilon_{i\mu}(\mathbf{k}) = \hat{\mathbf{x}}_i \cdot \hat{\varepsilon}(\mathbf{k}, \mu)$, and the $\hat{\varepsilon}(\mathbf{k}, \mu)$ are unitary polarization vectors defined for the wave vector k, illustrated in Fig. 1. This new basis is introduced because in an elastically isotropic crystal the eigensolutions of (2-14) may be rigorously classified as longitudinal or transverse. We choose $\hat{\varepsilon}(\mathbf{k}, 3)$ as the longitudinal polarization vector. Notice that from hermiticity it follows that $Q_k^i = Q_{-k}^{i\dagger}$ and $P_k^i = P_{-k}^{i\dagger}$.

The quantization of the elastic vibrations is made through the commutation relations involving $R^i(r)$ and $\Pi^i(\mathbf{r})$. The only noncommuting pair is such that

$$[R^i(\mathrm{r}), \Pi^j(r')] = i\hbar\delta_{ij}\,\delta(\mathbf{r} - \mathrm{r}'), \quad (2\text{-}16)$$

which leads to

$$[Q_k^\mu, P_{k'}^\nu] = i\hbar\, \delta_{kk'}\, \delta_{\mu\nu}. \quad (2\text{-}17)$$

As in the case of the spin excitations, one can find another canonical transformation which diagonalizes the elastic Hamiltonian. The transformation is

$$Q_k^\mu = \left[\frac{\hbar}{2\rho\omega_{p\mu}(k)}\right]^{1/2} (a_{\mu-k}^\dagger + a_{\mu k}) \tag{2-18}$$

$$P_k^\mu = i\left[\frac{\rho\hbar\omega_{p\mu}(k)}{2}\right]^{1/2} (a_{\mu k}^\dagger - a_{\mu-k})$$

where

$$\omega_{p\mu}(k) = k\left[\frac{\beta + \alpha\delta_{\mu 3}}{\rho}\right]^{1/2}, \tag{2-19}$$

is the phonon frequency. With this transformation, the Hamiltonian (2-14) becomes

$$\mathscr{H}_e = \sum_{k,\mu} \hbar\omega_{p\mu}(k)\,(a_{\mu k}^\dagger a_{\mu k} + \tfrac{1}{2}). \tag{2-20}$$

The new operators satisfy the commutation relations

$$[a_{\mu k},\, a_{\nu k'}] = 0, \quad [a_{\mu k},\, a_{\nu k'}^\dagger] = \delta_{\mu\nu}\,\delta_{kk'} \tag{2-21}$$

and are interpreted as creation and annihilation operators of lattice vibrations, whose quanta are called phonons. In terms of these operators, the displacement and the momentum density operators are

$$R_i = \sum_{k,\mu} \varepsilon_{i\mu}(\mathbf{k})\left[\frac{\hbar}{2\rho V\omega_{p\mu}}\right]^{1/2} (a_{\mu k}^\dagger e^{-ik\cdot r} + a_{\mu k} e^{ik\cdot r})$$

$$\Pi_i = \sum_{k,\mu} i\varepsilon_{i\mu}(\mathbf{k})\left[\frac{\rho\hbar\omega_{p\mu}}{2V}\right]^{1/2} (a_{\mu k}^\dagger e^{ik\cdot r} - a_{\mu k} e^{-ik\cdot r}) \tag{2.22}$$

c) The Magnetoelastic Interaction

In Fig. 2, we show the dispersion curves for magnons and phonons. The curve for a magnon with a given direction of propagation θ lies between the two parâbolic curves shown, and it is important to notice that the frequency depends on the intensity of the applied static field. For small values of the wave number, the frequency is of the order of $\omega_s \sim 2\mu H/\hbar$,

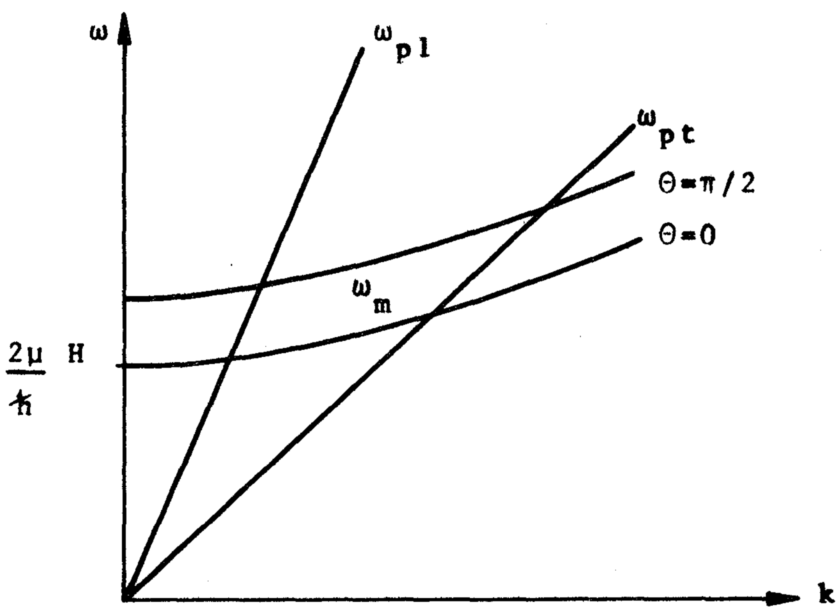

Fig. 2 - Magnon and phonon dispersion curves.

and, for values of H $\sim 1kOe$, ω_m lies in the microwave range ($10^9 - 10^{10}$ Hz). Because the velocity of elastic waves in a crystal is typically in the range $10^5 - 10^6$ cm/sec, their wave numbers at microwave frequencies are $k \sim 10^4 - 10^5\ \mathrm{cm}^{-1}$. This implies that both transverse and longitudinal phonon curves intersect the magnon curves at low values of **k** (center of the Brillouin zone). Due to the magnetostrictive properties of a crystal, the elastic displacement is coupled to the spin. As a result, a spin wave with frequency close to the intersection region in Fig. 2 is strongly coupled to an elastic wave. This magnon-phonon interaction can be expressed by a phenomenological Hamiltonian, which is a function of M and R. For a cubic crystal, with the static field applied along one of the [100] directions, the lowest order term of the interaction Hamiltonian is given by[5,6]

$$\mathscr{H}_{me} = \int d^3 r \, \frac{b_2}{2M^2} M_i M_j \left(\frac{\partial R_i}{\partial x_j} + \frac{\partial R_j}{\partial x_i} \right), \tag{2-23}$$

where the repeated indices indicate summation with $i \neq j$, and b_2 is one of the magnetoelastic constants. Using the expansions (2-13) and (2-22), this Hamiltonian can be written in terms of the boson operators. We will assume that the wave vectors of interest lie on the xz plane of the cartesian system, and that $\hat{\varepsilon}(\mathbf{k}, 1) \mathbf{r} \hat{\mathbf{y}}$. The component of *(2-23)* quadratic in the boson operators is given by

$$\mathscr{H}_{me} = i\left[\frac{b_2^2\,\mu\hbar}{2\rho M}\right]^{1/2} \sum_k \big[k\omega_{pt}^{-1/2}\,(u_k - v_k)\cos 2\theta\,(c_k + c_{-k}^\dagger)(a_{2k}^\dagger + a_{2-k})$$

$$- k\omega_{pl}^{-1/2}\,(u_k - v_k)\sin 26\,(c_k + c_{-k}^\dagger)(a_{3k}^\dagger + a_{3-k})$$

$$- ik\omega_{pt}^{-1/2}\,(u_k + v_k)\cos\theta\,(c, - c_{-k}^\dagger)(a_{1k}^\dagger + a,_{-k})\big], \qquad (2\text{-}24)$$

where ω_{pt} and ω_{pl} are the shear and longitudinal phonon frequencies. We shall now confine our attention only to waves propagating along the magnetic field *(8 = O).* The main reason for this assumption is that in this case the equations for the field variables are simple to solve. Besides, this is the most important situation in experiments because, due to focusing effects, z-directed magnetoelastic waves are easier to excite and control. One of the simplifications in this case results from the absence of the dipolar interaction in the spin system and, as a consequence, the ground state does not depend on the applied field[16]. Finally, it is important to note that the physical aspects of the general case are essentially the same inferred in this particular situation.

Taking $6 = 0$ in (2-24), we obtain

$$\mathscr{H}_{me} = i\left[\frac{b_2^2\,\mu\hbar}{2\rho M}\right]^{1/2} \sum_k \big[k\omega_{pt}^{-1/2}\,c_k\,(a_{2k}^\dagger - ia_{1k}^\dagger + a_{2-k} - i\,a_{1-k}) - c.c.\big]. \qquad (2\text{-}25)$$

Note that longitudinal phonons do not couple with magnons propagating along the magnetic field. In order to simplify *(2-25)* further, we introduce creation and annihilation operators of transverse circularly polarized phonons

$$a_{k(+)}^\dagger = 2^{-1/2}\,(a_{xk}^\dagger + ia_{yk}^\dagger) = 2^{-1/2}\,(-a_{2k}^\dagger + ia_{1k}^\dagger)$$

$$a_{k(-)}^\dagger = 2^{-1/2}\,(a_{xk}^\dagger - ia_{yk}^\dagger) = 2^{-1/2}\,(-a_{2k}^\dagger - ia_{1k}^\dagger)\,. \qquad (2\text{-}26)$$

Using the polarization index μ as (+) or (–), it is easy to show that the elastic Hamiltonian *(2-20)* and the commutation relations *(2-21)* have the same form for the circular polarization operators (2-26). In terms of the new operators, the total Hamiltonian for the magnon-phonon system becomes

$$\begin{aligned}\mathscr{H} &= \mathscr{H}_m + \mathscr{H}_e + \mathscr{H}_{me} = \\ &= \sum_k \hbar\omega_m(k) c_k^\dagger c_k + \sum_{k,\mu} \hbar\omega_{pt}(k) a_{k\mu}^\dagger a_{k\mu} + \\ &+ \sum_k ikL_k \left[c_k^\dagger (a_{k(+)} + a_{-k(-)}^\dagger) - c_k (a_{k(+)}^\dagger + a_{-k(-)})\right], \end{aligned} \tag{2-27}$$

where

$$L_k = \left[\frac{b_2^2 \mu\hbar}{\rho\omega_{pt} M}\right]^{1/2}$$

3. Eigenstates of the Magnon-Phonon System

In this section we study some properties of the normal mode collective excitations of a magnetoelastic crystal under a static magnetic field. The basic assumption of the last section is maintained, so that only z-directed spin waves coupled to shear elastic waves are considered. In order to simplify a little further the total Hamiltonian of the system, let us consider the equations of motion of the magnon and phonon operators in the Heisenberg representation. Using

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{1}{i\hbar}[A, \mathscr{H}], \tag{3-1}$$

we obtain

$$\dot{c}_k^\dagger = i\omega_m c_k^\dagger + kL_k \hbar^{-1} a_{k(+)}^\dagger + kL_k \hbar^{-1} a_{-k(-)}, \tag{3-2}$$

$$\dot{a}_{k(+)}^\dagger = i\omega_p a_{k(+)}^\dagger - kL_k \hbar^{-1} c_k^\dagger, \tag{3-3}$$

$$\dot{a}_{k(-)}^\dagger = i\omega_p a_{k(-)}^\dagger - kL_k \hbar^{-1} c_{-k}. \tag{3-4}$$

In the stationary state all operators have a $\exp(i\omega t)$ variation, and the magnetoelastic dispersion relation resulting from *(3-2)-(3-4)* is

$$(\omega^2 - \omega_p^2)(\omega - \omega_k) - \tfrac{1}{2}\omega_p \sigma_k^2 = 0, \tag{3-5}$$

where

$$\sigma_k = 2kL_k \hbar^{-1}, \tag{3-6}$$

which is a well-known result[4]. The dispersion curve is shown in Fig. 3, with the frequency splitting at the crossover region greatly exaggerated, because in usual situations this splitting is of the order of 10^{-2} compared

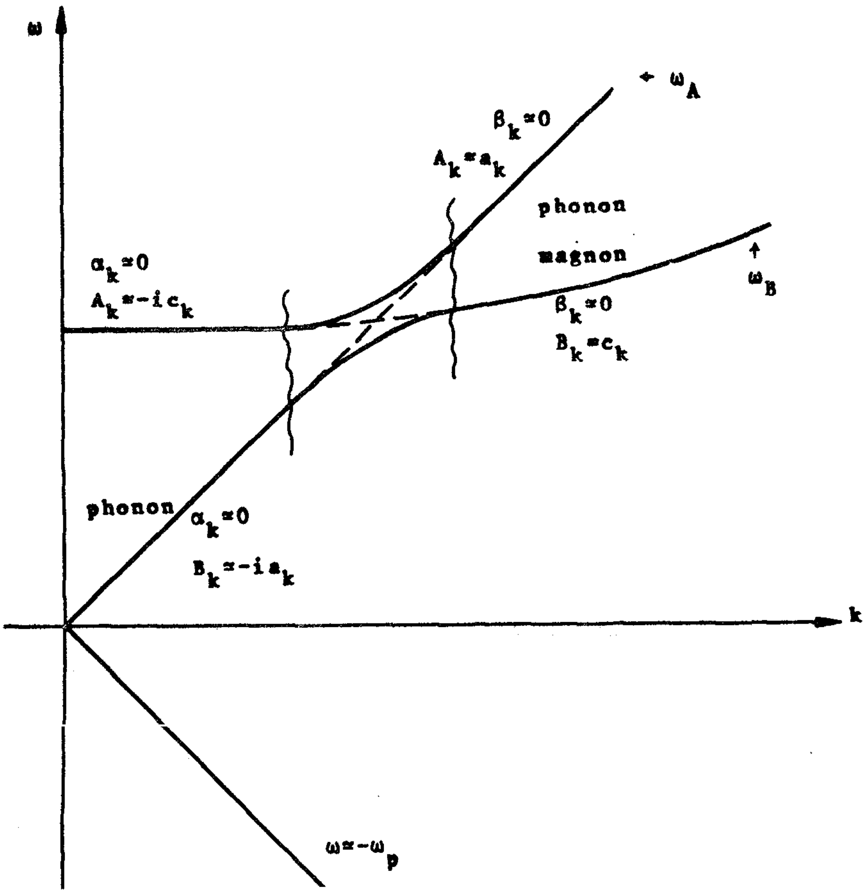

Fig. 3 - Magnetoelastic dispersion curves for z-directed waves.

to the magnon frequency. Note that the two positive frequency branches in Fig. 3 correspond to positive circularly polarized modes, propagating in the z-direction, whereas the negative branch corresponds to a negative circularly polarized wave in the opposite direction. The analysis of the equations of motion (3-2)-(3-4) shows that an excitation with frequency and wave number far from the crossover can have an almost pure magnon or phonon character. However, in the crossover region the normal modes are mixtures of magnetic and elastic excitations. The interesting phenomenon that we investigate in this paper is the possible change of character of an excitation, from magnetic to elastic or vice-versa, caused by the time variation of the applied field.

From (3-2)-(3-4) one can find that, in the stationary state, the expectation values of the positive and negative circularly polarized phonon operators are related by

$$\langle a^{\dagger}_{k(-)} \rangle = \left[\frac{\omega - \omega_p}{\omega + \omega_p} \right] \langle a^{\dagger}_{k(+)} \rangle \,, \tag{3-7}$$

which shows that, in a large portion of the two upper branches of the dispersion diagram, the influence of the negative circularly polarized phonons is small. Therefore, we can neglect the negative phonon operators in *(2-27).* Dropping the (+) index in the phonon operators left, we can write the Hamiltonian as

$$\mathscr{H} = \sum_k [\text{Ao,}\ c^{\dagger}_k c_k + \hbar\omega_p a^{\dagger}_k a_k + i\tfrac{1}{2} \hbar\sigma_k (c^{\dagger}_k a_k - a^{\dagger}_k c_k)]. \tag{3-8}$$

This Hamiltonian can be diagonalized by new operators obtained from linear combinations of the magnon and the phonon operators,

$$A_k = \alpha_k a_k - i\beta_k c_k \,, \ B_k = \alpha_k c_k - i\beta_k a_k \,, \tag{3-9}$$

where

$$\alpha_k = \left[\frac{\omega_b + \omega_d}{2\omega_b} \right]^{1/2}, \quad \beta_k = \left[\frac{\omega_b - \omega_d}{2\omega_b} \right]^{1/2},$$

and

$$\omega_d = \tfrac{1}{2}(\omega_p - \omega_m), \quad \omega_b = \left[\omega_d^2 + \frac{\sigma_k^2}{4} \right]^{1/2}. \tag{3-10}$$

The transformation *(3-9)* is such that the new operators satisfy the boson commutation relations

$$\begin{aligned} [A_k, A^{\dagger}_{k'}] &= [B_k, B^{\dagger}_{k'}] = \delta_{kk'} \,, \\ [A_k, B_{k'}] &= [A_k, B^{\dagger}_{k'}] = 0, \\ [A_k, A_{k'}] &= [B_k, B_{k'}] = 0, \end{aligned} \tag{3-11}$$

and the Hamiltonian becomes

$$\mathscr{H} = \sum_k [\hbar\omega_A(k) A^{\dagger}_k A_k + \hbar\omega_B(k) B^{\dagger}_k B_k] \,, \tag{3-12}$$

where

$$\begin{aligned} \omega_A(k) &= \tfrac{1}{2}(\omega_p + \omega_m) + \omega_b \,, \\ \omega_B(k) &= \tfrac{1}{2}(\omega_p + \omega_m) - \omega_b \,, \end{aligned} \tag{3-13}$$

which are the normal mode frequencies corresponding to the two upper branches of Fig. 3. These frequencies can also be obtained from (3-5) by elimination of the negative root. Eqs. (3-9)-(3-12) lead to the interpretation of $A_k^\dagger$ and $B_k^\dagger$ being the creation operators of quanta of collective magnetoelastic excitations, with energy $\hbar\omega_A$ and $\hbar\omega_B$, A_k and B_k are the annihilation operators. Note that far from the crossover region, i.e., when the difference between the magnon and the phonon frequencies is much larger than the splitting of the two branches ($|\omega_p - \omega_m| \gg a_k$), we have the following limits:

$$\begin{array}{llcll} \omega_p > \omega_m & \omega_A \to \omega_p & \text{and} & \omega_B \to \omega_m & \\ (\beta_k \to 0) & A_k \to a_k & & B_k \to c_k & (3\text{-}14) \\ \omega_m > \omega_p & \omega_A \to \omega_m & \text{and} & \omega_B \to \omega_p & \\ (\alpha_k \to 0) & A_k \to -ic_k & & B_k \to -ia_k. & (3\text{-}15) \end{array}$$

The stationary states of the Hamiltonian (3-12) may be obtained by applying integral powers of the creation operators to the vacuum state. The single mode states can be written in normalized form as

$$\begin{aligned} |n_{Ak}\rangle &= \frac{(A_k^\dagger)^{n_k}}{(n_k!)^{1/2}}|0\rangle, \\ |n_{Bk}\rangle &= \frac{(B_k^\dagger)^{n_k}}{(n_k!)^{1/2}}|0\rangle. \end{aligned} \tag{3-16}$$

It is not difficult to show that the mean occupation numbers of magnons and phonons in these states are given by

$$\begin{aligned} \langle n_{Ak}|c_k^\dagger c_k|n_{Ak}\rangle &= \langle n_{Bk}|a_k^\dagger a_k|n_{Bk}\rangle = \beta_k^2 n_k, \\ \langle n_{Ak}|a_k^\dagger a_k|n_{Ak}\rangle &= \langle n_{Bk}|c_k^\dagger c_k|n_{Bk}\rangle = \alpha_k^2 n_k, \end{aligned} \tag{3-17}$$

which are in agreement with the limits (3-14) and (3-15). Note also that, as $\alpha_k^2 + \beta_k^2 = 1$, the mean number of magnons plus the mean number of phonons in any state is the total number of the magnetoelastic quanta in that state.

The stationary states (3-16) can also be expanded in terms of the pure magnon and the pure phonon eigenstates. The magnetic eigenstates describe systems with well defined number of magnons and uncertain phase. They have been used in nearly all quantum treatments of thermodynamic properties, relaxation mechanisms and magnon interaction processes in ferromagnets. On the other hand, they do not correspond to the macroscopic spin waves used in the semiclassical treatments. This is clear from the fact that the first order components of the transverse magnetization (2-13) have zero expectation values in the stationary states. In addition, a system that behave nearly classically should involve a large and uncertain number of magnons, with well defined phases. It has been indicated[17,18] that in order to establish a correspondence between classical and quantum spin waves one should use the concept of coherent magnon states, defined by analogy to the photon coherent states[19]. In the same way we introduce the magnetoelastic coherent states. The single mode coherent states are defined as the eigenstates of the annihilation operators

$$A_k |u_A , k\rangle = u_A |u_A , k\rangle , \quad B_k |u_B , k\rangle = u_B | u_B , k\rangle . \tag{3-18}$$

They can be expanded in terms of the eigenstates of the Hamiltonian

$$|u, k\rangle = e^{-1/2|u|^2} \sum_{n_k} \frac{(u)^{n_k}}{(n_k !)^{1/2}} |n_k\rangle , \tag{3-19}$$

where u stands for u_A or u_B

4. The Magnon-Phonon Interaction in a Time-Dependent Magnetic Field

In this section, we assume that the ferromagnetic lattice is subject to an uniform magnetic field which varies in time. In order to understand one of the important aspects of this situation, let us assume that prior to an instant of time t_1 the field is constant, between t_1 and t_2 it increases monotonicaily in time, and after t_2 it remains constant, at a larger value than before. We assume also that, prior to t_1, a magnetoelastic wave with essentially pure phonon character was propagating in the crystal, with the frequency and wave number illustrated in Fig. 4. We now ask what happens to this phonon excitation as the magnetic field increases (resulting in a motion of the magnon curve), and what the final state is after t_2. As will be shown later, the wave number of the excitation remains constant during the process, because the field is spatially uniform. As a result, as the field changes the frequency of the excitation changes and goes through the crossover region, Therefore, the final state will be a superposition of states in the same branch as the initial and final states in the other branch, i.e., a superposition of magnon and phonon states. The conversion from a pure phonon excitation produced in Yttrium Iron Garnet by a piezoelectric iransducer into a magnetic excitation, detected by the current induced on a nearby fine wire, has been observed[2]. The magnon-phonon conver-

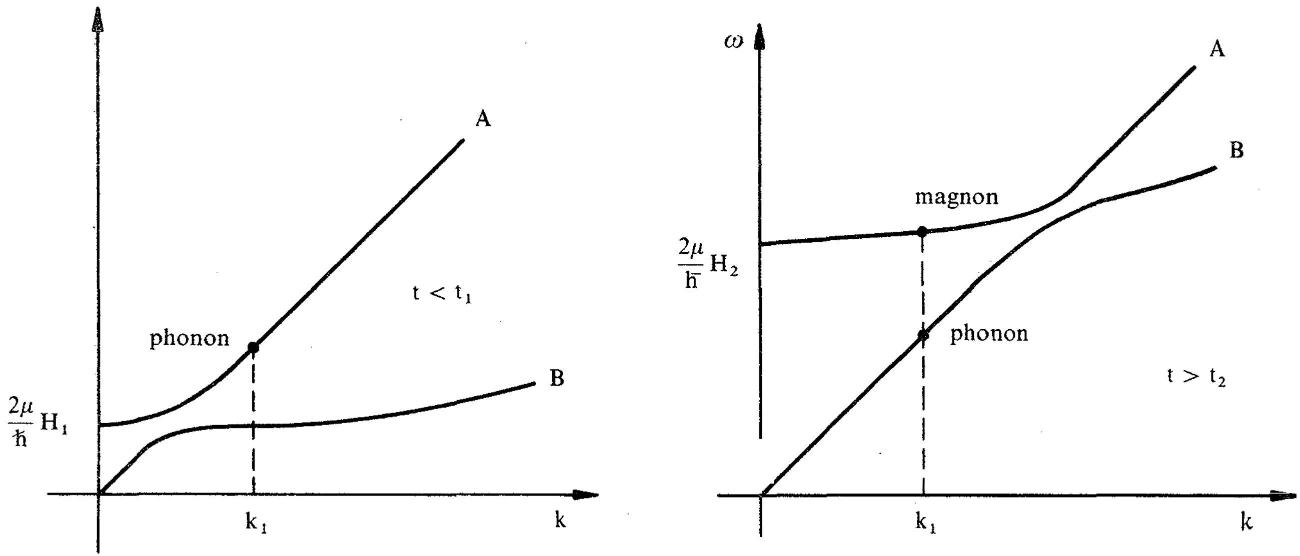

Fig. 4 - Behavior of an **initially** phonon excitation in a time varying magnetic field.

sion efficiency, which will be defined later, has also been measured as a function of the time rate of change of the field[9]. In the following sections, we study the evolution of the quantum system in this process.

The Hamiltonian of the system with $H(t)$ as an explicit function of time, can be obtained directly from (2-27) by letting o, be a function of time. Note that in the case of z-directed waves, o, $= (2\mu H + Dk^2)/\hbar$, so that o, is proportional to $H(t)$. Therefore, with the assumptions made in Sec. 3, we can write

$$\mathscr{H}(t) = \sum_k \left[\hbar\omega_m(t) c_k^\dagger c_k + \hbar\omega_p a_k^\dagger a_k + \frac{1}{2}\hbar\sigma_k (c_k^\dagger a_k - a_k^\dagger c_k) \right]$$

$$= \sum_k \left[\hbar\omega_A(t) A_k^\dagger A_k + \hbar\omega_B(t) B_k^\dagger B_k \right]. \tag{4-1}$$

The equations of motion for the magnon and the phonon operators can be obtained from (3-1) and (4-1). As none of the transformations used to define c, and a, involve time-dependent quantities, these operators are not explicit functions of time. Therefore we have

$$\begin{aligned} \dot{c}_k^\dagger &= i\omega_m(t)\, c_k^\dagger + \tfrac{1}{2}\sigma_k a_k^\dagger, \\ \dot{a}_k^\dagger &= i\omega_p a_k^\dagger - \tfrac{1}{2}\sigma_k c_k^\dagger. \end{aligned} \tag{4-2}$$

The equations of motion for the normal-mode magnetoelastic operators can be obtained from (3-9) and (4-2), or directly from the diagonal operator (4-1). In this case, one has to note that the partial derivatives of the operators with respect to time are not zero. We have

$$\dot{A}_k^\dagger = i\omega_A(t) A_k^\dagger + i\frac{\dot{\theta}}{2} B_k^\dagger, \quad \dot{B}_k^\dagger = i\omega_B(t) B_k^\dagger + i\frac{\dot{\theta}}{2} A_k^\dagger \tag{4-3}$$

where $9 = 2 \arccos \left[\dfrac{\omega_b + \omega_d}{2\omega_b} \right]^{1/2}$ and

$$\dot{\theta} = \frac{\mu\sigma_k}{2\hbar\omega_b^2} \dot{H}(t). \tag{4-4}$$

Equations (4-2) have the same form as the semiclassical equations[2] for the transformed magnetization and elastic displacement variables, whereas (4-3) are the same as for the normal-mode magnetoelastic variables. Notice that if $\partial H/\partial t = 0, 9$ is zero and the equations for A, and B_k are not coupled to each other. In this case the states corresponding to the two branches of the dispersion diagram of Fig. 3 are orthogonal to each other at all instants of time. However, if $\theta \neq 0$, one can couple the excitations of the two branches and the situation illustrated in Fig. 4 is plausible.

The foregoing equations have been formulated in the Heisenberg picture, which is characterized by the time-dependent operators, 'and by a time independent state vector. Therefore, if the system is initially in a state for which the expectation values of the magnon and the phonon operators are not zero, the time evolution of the expectation values are governed by Eqs. (4-2) and (4-3). As a consequence, a quantum mechanical analysis of this process in terms only of the expectation values of the operators will give the same results as the previous classical treatment[2]. Our aim, however, is to obtain also some information about the system with respect to the time evolution of its possible states of excitation.

5. Explicit Time-Dependent Invariants

The invariance properties of a system play a large role in Quantum as well as in Classical Mechanics. In the problem we are considering, an invariant with respect to time is expected to play two important roles. First, for the situation illustrated in Fig. 4, we have to define a magnon-phonon conversion efficiency in terms of a quantity which is conserved in the process. As the system is not conservative, the efficiency cannot be defined as the ratio between the energies of the two states. Second, it is possible to study the evolution of the state of a system with a time-dependent Hamiltonian, by means of a simple theory[20] based on the expansion of the state in terms of the eigenstates of invariant operators.

In the semiclassical theory of the magnetoelastic crystal in a time varying magnetic field, it was found that, due to the spatially uniformity of the field, a quantity identified as the quasi-momentum density of the system was conserved[2]. The momentum density which was found was the sum of the spin-wave and the elastic-wave momenta. We construct the quantum momentum density operators for magnons and phonons by replacing, in the classical expressions, the magnetization and the elastic displacement by the corresponding operators.

$$g_m^i = \frac{\hbar}{4\mu M}\left(\mathbf{m} \times \frac{\partial \mathbf{m}}{\partial x_i}\right) \cdot \hat{\mathbf{z}}, \tag{5-1}$$

$$g_p^i = \frac{\rho}{2}\left(\frac{\partial^2 \mathbf{R}}{\partial x_i\, \partial t} \cdot \mathbf{R} - \frac{\partial \mathbf{R}}{\partial t} \cdot \frac{\partial \mathbf{R}}{\partial x_i}\right) \tag{5-2}$$

The total momentum of the system is

$$\mathbf{P} = \int d^3 r\, \mathbf{g}(\mathbf{r}) = \int d^3 r\, [\mathbf{g}_m(\mathbf{r}) + \mathbf{g}_p(\mathbf{r})], \tag{5-3}$$

and, with the aid of the commutation relations for the magnon and phonon operators, it can be shown that

$$\mathbf{P} = \sum_k \hbar \mathbf{k}\, (c_k^\dagger c_k + a_k^\dagger a_k), \tag{5-4}$$

provided that terms with three or more operators are discarded. Eq. *(5-4)* has the expected form for the momentum of an elementary excitation in solids. In addition, the total momentum of magnons and phonons is equal to the total momentum of the quanta of the magnetoelastic system:

$$\mathbf{P} = \sum_k \hbar \mathbf{k}\, (A_k^\dagger A_k + B_k^\dagger B_k). \tag{5-5}$$

We can see also that the momentum for each **k**-mode is proportional to the occupafion number of quanta of the mode. The equation of motion for $\mathbf{P}(t)$ is

$$\frac{d\mathbf{P}}{dt} = \frac{\partial \mathbf{P}}{\partial t} + \frac{1}{i\hbar}\, [\mathbf{P}, \mathscr{H}]. \tag{5-6}$$

The commutator which appears in *(5-6)* is zero, a conclusion easily drawn from the expressions of $\mathscr{H}$ and **P** in terms of the normal-mode magnetoelastic operators, (3-12) and (5-5). The partial derivative of P with respect to time is also zero, which can be proved with the aid of *(4-3)*, or directly from (5-4), because a_k and c_k are not explicit functions of time. Therefore, $d\mathbf{P}/dt = 0$ and P is an explicit time-dependent invariant. Notice further that it is a Hermitian operator. Under the assumptions we are considering, the total number of quanta is also conserved. This latter property is true only because the "reflected particles" represented by the operator a_{-k} in *(3-2)-(34)* were neplected. However, the conclusion for **P** holds true in general. Finally, one can see that as the modes with different **k** are not coupled by the time variation of the inagnetic field, the wave vector of an excitation is also conserved.

6. Solutions of the Heisenberg Equations: the Magnon-Phonon Conversion Efficiency

This section is devoted to the presentation of the solutions of the Heisenberg equations of motion introduced in Sec. *4.* As mentioned previously, Eqs. (4-2) and *(4-3)* have the same form as the corresponding semiclassical equations, and therefore we can apply here the solutions already known.

Although Eqs. *(4-2)* and *(4-3)* are operator equations, their linear character means that they can be solved in terms of c-numbers linear equations. The solutions to the coupled equations *(4-2)* may be written in the form

$$\begin{aligned} c_k^{\dagger}(t) &= q(t)\, c_k^{\dagger}(t_0) + p(t)\, a_k^{\dagger}(t_0), \\ a_k^{\dagger}(t) &= s(t)\, c_k^{\dagger}(t_0) + r(t)\, a_k^{\dagger}(t_0), \end{aligned} \qquad (6\text{-}1)$$

where the momentum invariance implies that

$$|q|^2 + |s|^2 = 1, \quad |p|^2 + |r|^2 = 1, \quad qp^* + sr^* = 0 \qquad (6\text{-}2)$$

and the initial conditions are

$$q(t_0) = r(t_0) = 1, \quad p(t_0) = s(t_0) = 0. \qquad (6\text{-}3)$$

In the c-number functions introduced we have omitted the index k to simplify the notation. From (4-2) and (6-1), we obtain for two of the functions:

$$\begin{aligned} \dot{q}(t) &= i\omega_m(t)\, q(t) + \tfrac{1}{2}\sigma_k\, s(t), \\ \dot{s}(t) &= i\omega_p\, s(t) - \tfrac{1}{2}\sigma_k\, q(t). \end{aligned} \qquad (6\text{-}4)$$

Similarly, for Eq. *(4-3)* we have:

$$\begin{aligned} A_k^{\dagger}(t) &= x(t)\, A_k^{\dagger}(t_0) + w(t)\, B_k^{\dagger}(t_0), \\ B_k^{\dagger}(t) &= y(t)\, A_k^{\dagger}(t_0) + z(t)\, B_k^{\dagger}(t_0) \end{aligned} \qquad (6\text{-}5)$$

where

$$|x|^2 + |y|^2 = 1, \quad |w|^2 + |z|^2 = 1, \quad xw^* + yz^* = 0, \qquad (6\text{-}6)$$

and

$$x(t_0) = z(t_0) = 1, \quad w(t_0) = y(t_0) = 0. \qquad (6\text{-}7)$$

Analogously we obtain:

$$\begin{aligned} \dot{x}(t) &= \mathbf{io}\,(t)x(t) + i\frac{e}{2}\, y(t), \\ \dot{y}(t) &= i\omega_B(t)y(t) + i\frac{a}{2}\, x(t). \end{aligned} \qquad (6\text{-}8)$$

Some of the functions introduced in *(6-1)* and (6-5) have special significance. To see this let us assume, for example, that at the instant t_0 we have in the system a pure magnon excitation characterized by a state $|\psi_0\rangle$, with mean momentum $\bar{\mathbf{g}}_m = \langle\psi_0|\hbar\mathbf{k}\, c_k^\dagger c_k|\psi_0\rangle = \hbar\mathbf{k}\bar{n}$. Obviously, this is only an approximation because it is not possible to have a magnon excitation without some phonon admixture. However, if $\mathbf{k}$ is very far from the crossover region, this approximation may be very good. If after t_0 the applied field varies in time, there will be some momentum transfer to phonon excitations, as revealed by the second of Eqs. (6-1). As the sum of the magnon and the phonon mean momenta is conserved, it is convenient to define a conversion efiiciency from the magnon to the phonon state as the ratio between the mean momenta in the two states. Therefore, using (5-4), (6-1) and the fact that $\langle\psi_0|a_k|\psi_0\rangle = \langle\psi_0|a_k^\dagger a_k|\psi_0\rangle = 0$, we find

$$\eta_{mp}(t) = \bar{g}_p(t)/\bar{g}_m(t_0) = |s(t)|^2. \tag{6-9}$$

Notice that this is valid for any magnon state $|\psi_0\rangle$. Analogously, we see that if the system is initially in a phonon state, the phonon-magnon conversion efiiciency is given by $|p|^2$. In the same way, we can define a momentum conversion factor for the two magnetoelastic normal-mode excitations, which represents the transfer of momentum between the two branches of Fig. 3. It can be shown that

$$\eta_{AB}(t) = \bar{g}_B(t)/\bar{g}_A(t_0) = |y(t)|^2, \tag{6-10}$$

$$\eta_{BA}(t) = \bar{g}_A(t)/\bar{g}_B(t_0) = |w(t)|^2, \tag{6-11}$$

which are valid for conditions analogous to those used to derive (6-9), i.e., to obtain (6-10) we assume that the system is initially in a pure *A* state, and for (6-11) it is initially in a pure B state.

The systems of linear equations (6-4) and (6-8) cannot be solved for a general time dependence of the applied field. However, it is possible to find their solution for particular cases of interest. In the case of a slowly varying field (the slowness condition will be specified later), i.e., in an adiabatic approximation, it is convenient to work with Eqs. (4-3) and (6-8), because in this case the coupling between the *A* and *B* modes is small. Consider, for instance, that in this approximation we have the situation depicted in Fig. 4. The system is initially in a phonon state in branch *A* and the field increases so that the frequency goes through the crossover region. The phonon-magnon conversion efficiency is therefore given by $1 - |y(t)|^2$. The solution of (6-8), in the limit where $2\mu\dot{H}/\hbar \ll \sigma_k^2$, is identical to the solution of the semiclassical equations[2]. In this case, one finds

$$\eta_{pm} = 1 - |y(\infty)|^2 \simeq 1 - \frac{\pi^2}{9}\exp(-\dot{H}_{crit}/\dot{H}_{cross}), \tag{6-12}$$

where H_{cross} is the field gradient at the instant the frequency goes through the crossover point, and

$$\dot{H}_{crit} = \frac{\pi\hbar\sigma_k^2}{4\mu} \qquad (6\text{-}13)$$

is a critical field gradient evalyated ai the wave number of the excitation.

Another situation of interest is that of the sudden change of the field, characterized by the condition $2\mu\dot{H}/\hbar \gg \sigma_k^2$. In this case the coupling between modes *A* and B is strong, so that their concept as quasi-normal modes loses its meaning. In this case, however, the coupling between the magnon and phonon operators is small, and Eqs. (4-2) and *(6-4)* can be solved approximately. Again, considering the situation of Fig. 4, we see that the phonon-magnon conversion efficiency is given by $|p(t)|^2$, where $p(t)$ is a solution of the equations for *p* and r which are identical to *(6-4).* The analogy with the semiclassical case gives immediately[2]

$$\eta_{pm} = |p(\infty)|^2 \simeq \dot{H}_{crit}/\dot{H}_{cross}\,. \qquad (6\text{-}14)$$

Finally, we note that Doane[20] has integrated semiclassical equations identical to *(6-4)* exactly, under the assumption that the magnetic field has a time variation $H(t) = $ H, $+\ \delta H \tanh(\alpha t/2)$. This is a rounded step-

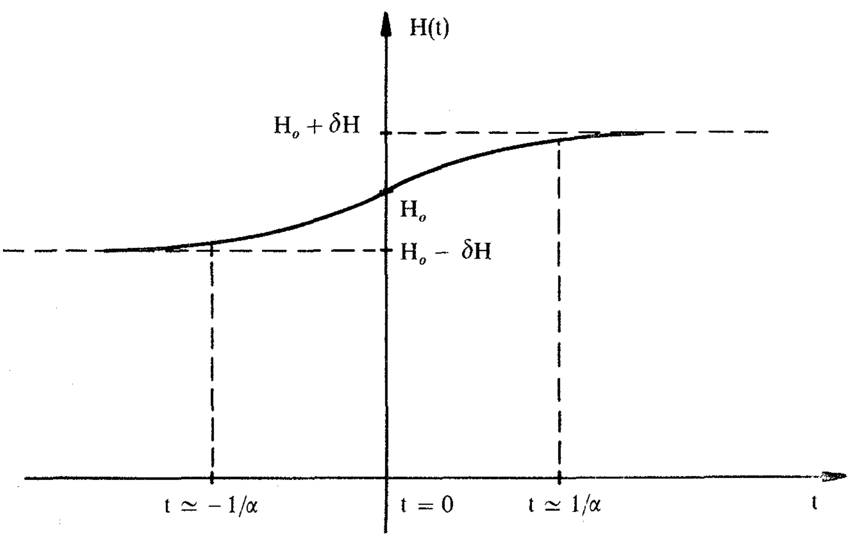

Fig. 5 - Variation of the field with a *tanh*$(\alpha t/2)$ time dependénce.

like variation, just of the type necessary to produce the conversion illustrated in Fig. 4. In Fig. 5 we show a plot of the time variation assumed. Doane[21] showed that Eqs. (6-4) can be transformed into a hypergeometric equation exactly soluble in this case. For the same situation previously considered, it can be shown that the phonon-magnon conversion efficiency, defined as the ratio between the magnon momentum at $t \to \infty$ and the phonon momentum at $t \to -\infty$, is given exactly by

$$\eta_{pm} = 1 - \exp(-\dot{H}_{crit}/\dot{H}_{cross}). \qquad (6\text{-}15)$$

Fig. 6 shows plots of the phonon-magnon conversion efficiencies expressed by Eqs. (6-12), (6-14) and (6-15), valid respectively for $\dot{H} \ll \dot{H}_{crit}$, $\dot{H} \gg \dot{H}_{crit}$ and $H \sim \tanh(\alpha t/2)$. The condition $\dot{H} \ll \dot{H}_{crit}$ is called the strong magnon-

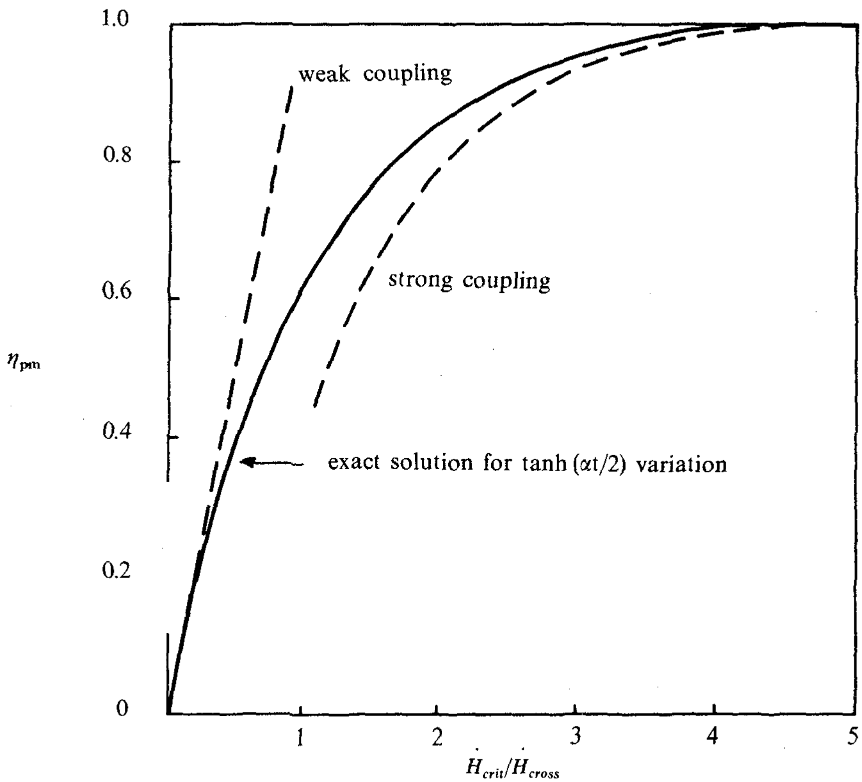

Fig. 6 - Phonon-magnon conversion efficiency as a function of the field gradient at the crossover.

phonon coupling situation, because in this case there is little transfer of momentum between the two magnetoelastic normal modes, and therefore as the frequency passes the crossover there is a large conversion from an elastic excitation into a magnetic excitation. The opposite situation, where $H \gg H_{crit}$, represents a weak magnon-phonon coupling, because in this case there is a large transfer of momentum between the two normal modes.

7. Conclusion: Comparison between Quantum and Semiclassical Theories

In the previous sections we have analyzed the behavior of a ferromagnetic crystal in a time-dependent magnetic field, with respect to the interaction between its phonon and magnon excitations. In particular we have been interested in the process whereby an initially phonon excitation is partially converted into a magnon excitation by means of a proper field variation. As noted in Sec. 6, the Heisenberg equations of motion for the creation and annihilation operators are identical to the classical equations for the variables derived from the magnetization and the elastic displacement vectors. As a result, the calculation of the expectation values of the operators of interest, including the phonon-magnon conversion efficiency, by means of quantum mechanical methods, leads to the same solutions obtained by semiclassical treatments. In this respect, the only difference between the present treatment and the semiclassical one is that here the excitations are quantized. However, one advantage of the present analysis is that the various transformations made to obtain the boson operators form a natural approach in quantum theory, whereas analogous transformations made in the classical treatment bear no physical meaning, and were performed with the sole purpose of simplifying the equations.

There are some aspects of the problem, however, which cannot be explained in semiclassical terms. One of them is the amplification of the zero point (vacuum) oscillations, which is a source of quantum noise. This effect did not arise in our study because in the magnetic Hamiltonian we neglected the dipolar interaction, which is responsible for the coupling between the longitudinal field variation and the magnetic vacuum[16]. Another one is that of the precision of the measurement of the expectation value of an observable. To investigate this point, let us consider the situation where the system is initially in a "pure" phonon state $|\psi_0\rangle$. Due to a time variation of the applied field, the system may end up in a "pure" magnon state. As shown in Sec. 6, the conversion efficiency is given by $|p|^2$, and this result is valid for any initial state which has a nonzero mean momentum. It can be, for instance, a stationary state of the Hamiltonian, or a coherent state. The stationary states have zero expectation values for the magnetization and the elastic displacement, and therefore there is no good reason for studying the precision of their measurements. The coherent states, on the other hand, are the states most closely related to the classical properties of a system. Furthermore, it has been shown[17,18] previously that in experiments, a macroscopic excitation of the system results in the creation of coherent states. Hence, let us consider that the system is initially in a coherent phonon state, which is the type expected to be generated by a piezoelectric transducer in typical ultrasonics experiments. In other words,

$$|\psi_0\rangle = |u, k\rangle, \qquad \text{where} \qquad a_k|u, k\rangle = u|u, k\rangle.$$

Before the field starts changing in time, the variance of the operator $(a_k^\dagger + a_k)$, which is related to the displacement operator, is given by

$$\Delta^2 (a_k^\dagger + a_k) = \langle\psi_0|(a_k^\dagger + a_k)^2|\psi_0\rangle - \langle\psi_0|a_k^\dagger + a_k|\psi_0\rangle^2 = 1. \quad (7\text{-}1)$$

From this result, it is possible to show that the product of the variances of $(a_k^\dagger + a_k)$ and of its canonical conjugate is given by $h^2/4$, which is the minimum value allowed by the uncertainty principle. This is a well known property of coherent states[19]. An indication of the time evolution of the coherence of the system, and consequently of the precision of measurements, is given by the time dependence of the variances of observables. Using the Heisenberg equations of motion (6-1) and the commuting properties of the magnon and phonon operators, we find that for an initial coherent phonon state we have

$$\begin{aligned} \Delta^2 [a_k^\dagger (t) + a_k (t)] &= 1, \\ \Delta^2 [c_k^\dagger (t) + c_k (t)] &= 1. \end{aligned} \quad (7\text{-}2)$$

Therefore, a system described by a Hamiltonian of the type (4-1), which is initially in a coherent state, maintains its coherence properties regardless of the time dependence of the field. This result emphasizes the conclusion that, under the assumptions made in this paper, nothing essentially new arises from the quantization of the fields in a magnetoelastic system under a time dependent magnetic field.

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