A Group-Theoretical Approach to the Non-Relativistic Three-Body Problem

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A group-theoretical method to construct a complete set of angular functions of given angular momentum and definite permutational symmetry is proposed. In order to completely classify those functions, a Hermitian, angular, operator is constructed: its square, W^2, together with $\mathscr{I}_2$ (the 2nd order Casimir operator of R,), U^2 (which takes care of the permutational symmetry) as well as L^2 and L, (the square of the CM angular momentum and its 3rd component), constitute a complete set of commuting Hermitian operators which provide a convenient set of labels. The matrix elements of U and W, in a suitable polynomial basis, are calculated.

Propõe-se, neste trabalho, um método baseado na teoria de grupos para se construir um conjunto completo de funções angulares com dado momento angular e simetria permutacional definida. A fim de classificar completamente tais funções, constrói-se um operador hermitiano angular cujo quadrado, W^2, juntamente com $\mathscr{I}_2$ (o operador de Casimir de 2.ª ordem do grupo R,), U^2 (responsável pela simetria permutacional) assim como L^2 e L, (o quadrado do momento angular no sistema de centro de massa e a sua 3.ª componente), constituem um conjunto completo de operadores hermitianos que comutam entre si e que fornecem um conjunto conveniente de números quânticos. Os elementos de matriz de U e W são calculados em uma base polinomial conveniente.

1. Introduction

One of the basic problems, in the study of the non-relativistic three-body system in Quantum Mechanics, is the construction of the angular part of the eigenfunctions of the interna1 kinetic energy. This problem has been studied by severa1 authors[1,2,3] (see Ref. 3 for other references) using different approaches and different characterizations of the angular functions. The problem becomes rather involved when, in the equal mass case, one looks for functions with definite permutational symmetry[3].

By making use of some standard constructions of the theory of angular momentum and guided by basic group-theoretical considerations, we have constructed angular functions with definite permutational symmetry and given a complete characterization of them. In order to completely characterize those angular functions, one needs five labels which are associated to a complete set of commuting Hermitian operators. In this set we have $\mathscr{I}_2$ (the 2nd order Casimir operator of R,), the square of the operator[3]

U (which will distinguish between different irreducible representations of the permutation group S,) and the operators L^2 and L, (the square of the CM angular momentum and its 3rd component). We have constructed a fifth Hermitian operator, W, which commutes with the above ones and whose square we have used to complete the characterization of the angular functions.

In Sections 2 and 3, the symmetry of the Hamiltonian is discussed and the angular functions are related to the homogeneous and harmonic polynomials of a given degree in six dimensions, following Ref. (1). We construct, in Section 4, a basis of R, in a chain in which the angular momenta with respect to the Jacobi relative coordinates as well as the angular momentum in the CM-frame are diagonal. The calculation of the matrix elements of the R, generators, in the above mentioned basis, is indicated in Section 5. **A** method for the construction of the angular functions with definite permutational symmetry is developped in Section 6 and their complete characterization discussed in some detail. In the last Appendix, the homogeneous and harmonic polynomials of degree up to $\lambda = 6$, with definite permutational symmetry, are exhibited. In the Notes, a few useful notions have been included.

2. The R, Group

In the center of mass frame, the nonrelativistic kinetic energy operator for a system of three particles with equal mass reads

$$H = -\tfrac{1}{2}(\nabla_x^2 + \nabla_y^2), \tag{2-1}$$

$A = m = 1$, where x and y are the Jacobi coordinates of the relative motion,

$$\mathbf{x} = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3), \qquad \mathbf{y} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2), \tag{2-2}$$

$\mathbf{r}_1$, $\mathbf{r}_2$, r, being the particle coordinates in the laboratory frame. In (2-I), ∇_x^2 and ∇_y^2 are the Laplacians with respect to x and y. Note that the coordinates (2-2) are translational invariant, as relative coordinates have to be. The above Hamiltonian is invariant[4] under the rotation group R, acting on the six-dimensional space of the coordinates x and y. The fifteen generators of R, are here realized by[1]:

$$\Lambda_{ij}^{\alpha\beta} = \tfrac{1}{2}\left(x_i^\alpha \frac{\partial}{\partial x_j^\beta} - x_j^\beta \frac{\partial}{\partial x_i^\alpha}\right), \tag{2-3}$$

i, j = 1,2,3 and a, β = 1,2, where $x_i^1 = x_i$ and $x_i^2 = y_i$. Obviously, $\Lambda_{ij}^{\alpha\beta} = -\Lambda_{ji}^{\beta\alpha}$. They satisfy the following commutation relations:

$$[\Lambda_{ij}^{\alpha\beta}, \Lambda_{kl}^{\gamma\rho}] = \tfrac{1}{2}(\Lambda_{il}^{\alpha\rho}\delta^{\beta\gamma}\delta_{jk} - \Lambda_{ik}^{\alpha\gamma}\delta^{\beta\rho}\delta_{jl} + \Lambda_{jk}^{\beta\gamma}\delta^{\alpha\rho}\delta_{il} - \Lambda_{jl}^{\beta\rho}\delta^{\alpha\gamma}\delta_{ik}). \quad (2\text{-}4)$$

Since R, is a rank-three group, its most general irreducible representations (fromnow on, IR = irreducible representation) are characterized by three labels, which one can associate to three functionally independent Casimir invariants. From (2-4), it follows that the operators

$$\mathscr{I}_2 = \Lambda_{ij}^{\alpha\beta}\Lambda_{ji}^{\beta\alpha}, \quad (2\text{-}5)$$
$$\mathscr{I}_3^{(1)} = \Lambda_{ij}^{\alpha\beta}\Lambda_{jk}^{\beta\gamma}\Lambda_{ki}^{\gamma\alpha}, \qquad \mathscr{I}_3^{(2)} = \varepsilon^{(\alpha i)(\beta j)(\gamma k)(\delta l)(\mu m)(\nu n)}\Lambda_{ij}^{\alpha\beta}\Lambda_{kl}^{\gamma\delta}\Lambda_{mn}^{\mu\nu}$$

are Casimir invariants of the group, since they commute with all generators. (In (2-5), ε is the totally antisymmetric tensor of rank six; each pair *(cri)*, (βj), etc., is to be regarded as a single index running from 1 to 6). With the realization (2-3) for the generators, one finds that $\mathscr{I}_3^{(2)} = 0$, while

$$\begin{aligned}\mathscr{I}_2 &= \tfrac{1}{2}[(\mathbf{r}\cdot\nabla)(\mathbf{r}\cdot\nabla + 4) - \mathbf{r}^2\nabla^2],\\ \mathscr{I}_3^{(1)} &= \tfrac{1}{2}[(\mathbf{r}\cdot\nabla)(\mathbf{r}\cdot\nabla + 7) + \mathbf{r}^2\nabla^2],\end{aligned} \quad (2\text{-}6)$$

where $\mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2$ is the square of the six-dimensional distance and $\mathbf{V}^2 = \nabla_x^2 + \nabla_y^2$ the six-dimensional Laplacian.

It is clear from (2-3) that the generators maintain the degree of homogeneous polynomials in the variables x_i, y_i and it follows therefore that the set of homogeneous polynomials of a given degree λ carry a representation of R,. Such a representation is, in general, reducible as we shall see in Section 4. $IR's$ are obtained by requiring that the homogeneous polynomials be harmonic in the six variables, since in this case both Casimir operators $\mathscr{I}_2$ and $\mathscr{I}_3^{(1)}$ have definite values. Moreover, since ($\mathbf{r}\cdot$V) gives the degree λ of the homogeneous polynomials, it is enough to use $\mathscr{I}_2$ to label the $IR's$ we are dealing with. Although **9,** has the eigenvalue $\frac{1}{2}\lambda(\lambda + 4)$, it is simpler to use the integer λ to characterize the $IR's$[5].

3. Schrodinger Equation and Angular Functions

First of all, the treatment which is been developped applies more generally to Hamiltonians with a purely r-dependent potential $V(r)$:

$$H = -\tfrac{1}{2}\nabla^2 + V(r). \quad (3\text{-}1)$$

From *(2-5),* the Hamiltonian *(3-1)* can be rewritten as

$$H = \frac{1}{\mathbf{r}^2}\mathscr{I}_2 - \frac{1}{2\mathbf{r}^2}(\mathbf{r}\cdot\nabla)(\mathbf{r}\cdot\nabla + 4) + V(r). \tag{3-2}$$

Since the generators *(2-3)* are angular operators (i.e., independent of the six-dimensional distance *r),* it follows that the Casimir $\mathscr{I}_2$ is also an angular operator. On the other hand, the operator $(r\cdot\nabla)(r\cdot\nabla + 4) + V(r)$, which is not built up from generators, depends only on ***r***. One can then use the method of separation of variables in the Schrodinger equation $H\psi = E\psi$. Writing[6] $\psi(\mathbf{x}, y) = f(r)F(\chi, \theta_1, \phi_1, \theta_2, \phi_2)$, one gets the equations:

$$\mathscr{I}_2 F(\chi, \theta_1, \phi_1, \theta_2, \phi_2) = \alpha F(\chi, \theta_1, \phi_1, \theta_2, \phi_2) \tag{3-3}$$

and

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(\frac{d}{dr} + 4\right) + (E - V(r)) - \frac{\alpha}{r^2}\right] f(r) = 0. \tag{3-4}$$

From the considerations of Section 2, one can write

$$F(\chi, \theta_1, \phi_1, \theta_2, \phi_2) = \frac{P^{[\lambda]}}{r^\lambda}, \tag{3-5}$$

where $P^{[\lambda]}$ is a homogeneous and harmonic polynomial of degree λ. One gets then the value $\frac{1}{2}\lambda(\lambda + 4)$ for the separation constant α. The radial equation *(3-4)* relates λ to ***E***, this relation depending of course on $V(r)$.

Note that, by *(3-3)* and (3-5), the angular functions for a given energy value E belong to the set of functions carrying an IR of R, (the same holds for the eigenfunctions ψ since ***r*** is an ***R,*** scalar). This is of course a consequence of the invariance of *(3-1)* under R, and is true when the degeneracies of E are "normal", i.e., in the absence of "accidental degeneracies"[7].

One could choose to go on with the method of separation of variables to solve the angular equation *(3-3)*. We shall instead determine the angular function by making use of basic group-theoretical techniques.

4. The Chain R, ⊃ [R, (x) ⊗ R, (y)] ⊃ R, (L)

Before going into the actual determination of the angular functions, one has to choose a classification scheme for the homogeneous and harmonic polynomials of degree λ.

In the set of quantum numbers to be used in the classification of the angular functions, we want of course to have the angular momentum in the *CM*-frame, as well as its z-component M. This is clearly possible since $L =$ $= -i(x \times \nabla_x + y \times \nabla_y)$ can be expressed in terms of the R_6 generators, namely,

$$L_k = -i\,\varepsilon_{kjl}(\Lambda_{jl}^{11} + \Lambda_{jl}^{22}), \tag{4-1}$$

and therefore the R_3 group generated by L, which we denote by $R_3(\mathbf{L})$, is a subgroup of R_6. The angular momentum in the variable x, namely, $\mathbf{L}(\mathbf{x}) = -ix \times \nabla_x$, is just $L_k(\mathbf{x}) = -i\varepsilon_{kjl}\Lambda_{jl}^{11}$ and the subgroup $R_3(\mathbf{x})$ it generates is also a subgroup of R_6. Similarly for $R_3(y)$ which is generated by $L_k(\mathbf{y}) = -i\varepsilon_{kjl}\Lambda_{jl}^{22}$. Moreover, $\mathbf{L}(\mathbf{x})$ and $\mathbf{L}(\mathbf{y})$ commute with each other and we can, therefore, take the following chain of R_6 subgroups:

$$R_6 \supset [R_3(\mathbf{x}) \otimes R_3(\mathbf{y})] \supset R_3(\mathbf{L}) \supset R_2(L_z), \tag{4-2}$$

which will provide us with the quantum numbers λ, l, l', L and M.

The solid harmonics $\mathscr{Y}_m^l(\mathbf{x}) \equiv |\mathbf{x}|^l Y_m^l(\hat{\mathbf{x}})$ are homogeneous and harmonic polynomials of degree l carrying an $IR[l]$ of $R_3(x)$. We couple $\mathscr{Y}_m^l(\mathbf{x})$ and $\mathscr{Y}_{m'}^{l'}(\mathbf{y})$, via Clebsch-Gordan coefficients, to get homogeneous polynomials of degree $(l + l')$, with definite angular momentum L and z-projection $M = m + m'$. Next, we multiply the resulting polynomial by an unknown homogeneous polynomial, of degree $\lambda - l - l' \equiv 2n$, in the variables x^2 and y^2 (which are invariant under $R_3(\mathbf{x}) \otimes R_3(\mathbf{y})$). That is, we write the Ansatz

$$P_{ll'LM}^{[\lambda]} = \mathscr{N}_{ll'}^{\lambda} B_{ll'}^{\lambda}(x^2, y^2) \sum_{m,m'} < lml'm' \,|\, LM > \mathscr{Y}_m^l(\mathbf{x})\mathscr{Y}_{m'}^{l'}(\mathbf{y}), \tag{4-3}$$

and require it to be harmonic in the six variables x, y. This is enough to determine the polynomials B up to a multiplicative constant which we choose in such a way as to make them related to the Jacobi polynomials[5] by

$$B_{ll'}^{\lambda}(\mathbf{x}^2, \mathbf{y}^2) = r^n P_n^{(l+1/2,\, l'+1/2)}(-\cos 2\chi), \tag{4-4}$$

where $n =$ *non negative integer* $= \frac{1}{2}(\lambda - l - l')$ and χ one of the spherical angles (cf. Ref. 3)). The constants $\mathscr{N}_{ll'}^{\lambda}$ are obtained by normalizing the polynomials $P^{[\lambda]}$ in the unit sphere of E_6. One gets

$$\mathscr{N}_{ll'}^{\lambda} = \left[\frac{2^{\lambda+3}(\lambda + 2)n!(n + l + l' + 1)!}{\pi(\lambda + l - l' + 1)!!(\lambda - l + l' + 1)!!}\right]^{1/2}. \tag{4-5}$$

Moreover, the polynomials *(4-3)* are orthogonal to each other:

$$(P^{[\lambda_1]}_{l_1 l'_1 L_1 M_1}, P^{[\lambda_2]}_{l_2 l'_2 L_2 M_2}) \equiv \int d\Omega_6 \left[P^{[\lambda_1]*}_{l_1 l'_1 L_1 M_1} \, P^{[\lambda_2]}_{l_2 l'_2 L_2 M_2}\right]_{r=1}$$
$$= \delta_{\lambda_1 \lambda_2} \, \delta_{l'_1 l'_2} \, \delta_{l_1 l_2} \, \delta_{L_1 L_2} \, \delta_{M_1 M_2}, \qquad (4\text{-}6)$$

where $d\Omega_6$ is the element of solid angle[6] in E, and (*) denotes the operation of complex conjugation. The last four δ' s, in (4-6), arise from the orthonormality of the spherical harmonics, which together with the orthogonality of the Jacobi polynomials[8], gives rise to the first S. On the other hand, the polynornials *(4-3)* exhibit the following property under the exchange $(x \leftrightarrow y)$:

$$(x \leftrightarrow y) P^{[\lambda]}_{ll'LM} = (-)^{n+\lambda-L} \, P^{[\lambda]}_{l'lLM} .$$

From what has bèen said above, it is clear that the polynomials $P^{[\lambda]}_{ll'LM}$ carry the $IR[\lambda]$ of R_6 in the chain *(4-2)*,and one can verify that the labels A, 1, l', L, M are related by the branching laws

$$l + l' = \lambda,\ \lambda - 2, \ldots, \left\{\begin{matrix}0\\1\end{matrix}\right.,$$
$$l + l' \geqslant L \geqslant |l - l'|, \qquad L \geqslant M \geqslant -L. \qquad (4\text{-}8)$$

From the branching laws, one gets:

(i) The wellknown formula for the dimension of the $IR[\lambda]$:

$$\dim [\lambda] = \tfrac{1}{12}(\lambda + 1)(A + 2)^2(\lambda + 3) \qquad (4\text{-}9)$$

(ii) In the *IR [A]*,the values A, A − 1,. . . , 1 are always allowed for L. The value L = 0 occurs only for even A. The multiplicity of a given L value in the $IR[\lambda]$ depends on the relative parity of L and λ, and is given by

$$\mathscr{M}^{\lambda}(L) = \tfrac{1}{8}[2L + 1 + (-)^{\lambda+L}][2(\lambda - L) + 3 + (-)^{\lambda+L}] \qquad (4\text{-}10)$$

In Section *2,* we stated that the representation of *R,* carried by homogeneous polynomials of a given degree λ is, in general, reducible. To prove that, one can start by counting the linearly independent monomials of degree A in six variables, which provide a basis of the vector space of homogeneous polynomials of that degree in six variables. Their number[g] is clearly the number of solutions of the equation $\sum_{i=1}^{6} \alpha_i = \lambda$, with $\alpha_i =$ = *non negative integers,* and is equal to the binomial coefficient $\binom{\lambda+5}{\lambda}$. We next observe that, for any fixed value of A, the polynomials

$$(x^2 + y^2)^{\rho} P^{[\lambda - 2\rho]}_{ll'LM}(x, y), \qquad (4\text{-}11)$$

with $\rho = 0,1,2,\ldots,[\lambda/2]$ (cf. Ref. 10), are homogeneous of degree λ and, for each value of ρ, they provide a basis for the $IR[A - 2\rho]$ of $R,$ because $x^2 + y^2 = r^2$ is an invariant of $R,$. Since, for different values of ρ, they carry different $IR's$ of $R,$, the polynomials (4-11) are linearly independent. Their number is given by

$$\sum_{\rho=0}^{[\lambda/2]} \dim [A - 2\rho] = \binom{\lambda+5}{\lambda}, \tag{4-12}$$

the equality being obtained making use of (4-9). It follows then that the $R,$ basis provided by the homogeneous polynomials of degree A is, in general, reducible into the $IR's [\lambda]$, [A−21,. .., $\{{[0] \atop [1]}$. One can therefore decompose any[11] homogeneous polynomial of degree λ, in six variables, using the bases (4-11):

$$P^\lambda(x, y) = \sum_{\substack{\rho l l' \\ LM}} C^\lambda_{\rho l l' LM} (r^2)^\rho P^{[\lambda - 2\rho]}_{l l' LM}(x, y), \tag{4-13}$$

$\rho = 0,1,2,\ldots[\lambda/2]$, a result which we shall use in Section 6.

If we were not interested in polynomials with definite angular momentum L and projection M, we could take the chain

$$R_6 \supset [R_3(x) \otimes R_3(y)] \supset [R_2(x) \otimes R_2(y)], \tag{4-14}$$

and then instead of (4-3) we would have the polynomials

$$P^{[\lambda]}_{l m l' m'}(x, y) = \mathcal{N}^\lambda_{l l'} B^\lambda_{l l'}(x^2, y^2) \mathcal{Y}^l_m(x) \mathcal{Y}^{l'}_{m'}(y), \tag{4-15}$$

which carry an IR of $R,$ equivalent to the one carried by the functions (4-3), since the two bases are related by a unitary transformation:

$$P^{[\lambda]}_{l l' LM} = \sum_{mm'} \langle l m l' m' | LM \rangle P^{[\lambda]}_{l m l' m'}. \tag{4-16}$$

5. Matriz Elements of the R, Generators

It is simple to classify the generators (2-3) of $R,$ according to their irreducible character with respect to (wrt) the subgroups which are present in the chain (4-2). Indeed, by using the commutation relations (2-4), one can show that the Λ^{11}_{ij} are the components of a vector wrt $R_3(x)$, Λ^{22}_{ij} a vector *wrt* $R_3(y)$, while the generators Λ^{12}_{ij} are the cartesian components of a tensor $T^{[1,1]}$ wrt $R_3(x) \otimes R_3(y)$ which reduces, wrt $R_3(L)$, into a scalar, a vector and a rank-two tensor, according to the formula

$$T_q^k = \sum_{mm'} \langle 1m1m' | kq \rangle \Lambda_{mm'}^{12}, \tag{5-1}$$

where $k = 0,1,2$ and $q = -k, -k+1, \ldots, k-1, k$. In (5-1), the $\Lambda_{mm'}^{12}$ are the spherical components of Λ_{ij}^{12}. The scalar operator T_0^0 plays an important role in connection with the permutational properties of the angular functions (Section 6).

By using the Wigner-Eckart theorem[12] for the R_3 groups of the chain (4-2), we get the following expressions for the matrix elements of the irreducible tensor operators T_q^k in the $P^{[\lambda]}$ basis:

$$(L, \bar{l}\bar{l}'\bar{L}\bar{M} | T_q^k | L, ll'LM) = [(2L+1)(2k+1)(2l+1)(2l'+1)]^{1/2} (-)^{L+\bar{L}} \cdot$$

$$\langle LMkq | \bar{L}\bar{M} \rangle \begin{Bmatrix} 1 & 1 & k \\ l & l' & L \\ \bar{l} & \bar{l}' & \bar{L} \end{Bmatrix} \langle \lambda, \bar{l}\bar{l}' \| T^{[1,1]} \| \lambda, ll' \rangle, \tag{5-2}$$

where the simpler bra-ket notation has been used (note that the scalar product on the *LHS* of (5-2) is understood as in (4-6)). The notation {} stands here for $9-j$ symbols[12] and the last bracket on the *RHS* is the reduced matrix element of $T_{mm'}^{[1,1]} \equiv \Lambda_{mm'}^{12}$ as defined by the Wigner-Eckart theorem applied to both $R_3(\mathbf{x})$ and $R_3(\mathbf{y})$:

$$\langle \lambda, \bar{l}\bar{m}\bar{l}'\bar{m}' | T_{qq'}^{[1,1]} | \lambda, lml'm' \rangle = \langle lm1q | \bar{l}\bar{m} \rangle \langle l'm'1q' | \bar{l}'\bar{m}' \rangle$$

$$\langle \lambda, \bar{l}\bar{l}' \| T^{[1,1]} \| \lambda, ll' \rangle. \tag{5-3}$$

In order to evaluate these reduced matrix elements, it is simpler to use the scalar operator T_0^0, which is given by

$$T_0^0 = -\frac{1}{\sqrt{3}} \sum_m T_{m-m}^{[1,1]} \equiv -\frac{1}{\sqrt{3}} \sum_m \Lambda_{m-m}^{12} = -\frac{1}{\sqrt{3}} \sum_i \Lambda_{ii}^{12} =$$

$$= -\frac{1}{2\sqrt{3}} (x \cdot \nabla_y - y \cdot \nabla_x). \tag{5-4}$$

The matrix elements of T_0^0 are easily calculated using the formulas of Appendix *1.* We get:

$$\langle \lambda, \bar{l}\, \bar{l}' \bar{L} \bar{M} \,|\, T_0^0 \,|\, \lambda,\ ll' LM \rangle = -\frac{1}{2\sqrt{3}} (-)^{l+l'+L} \delta_{L\bar{L}} \delta_{M\bar{M}} \cdot$$

$$\cdot [\![- [(l+1)(l'+1)(\lambda-l-l')(\lambda+l+l'+4)]^{1/2} \begin{Bmatrix} l'+1 & l+1 & L \\ l & l' & 1 \end{Bmatrix} \delta_{\bar{l},l+1} \delta_{\bar{l}',l'+1} +$$

$$+ [(l+1)l'(\lambda-l+l'+1)(\lambda+l-l'+3)]^{1/2} \begin{Bmatrix} l'-1 & l+1 & L \\ l & l' & 1 \end{Bmatrix} \delta_{\bar{l},l+1} \delta_{\bar{l}',l'-1} -$$

$$- [l(l'+1)(\lambda+l-l'+1)(\lambda-l+l'+3)]^{1/2} \begin{Bmatrix} l'+1 & l-1 & L \\ l & l' & 1 \end{Bmatrix} \delta_{\bar{l}\,l-1} \delta_{\bar{l}',l'+1} +$$

$$+ [ll'(\lambda+l+l'+2)(\lambda-l-l'+2)]^{1/2} \begin{Bmatrix} l'-1 & l-1 & L \\ l & l' & 1 \end{Bmatrix} \delta_{\bar{l},l-1} \delta_{\bar{l}',l'-1}]\!], \qquad (5\text{-}5)$$

where the notation $\{\}$ denotes here $6-j$ symbols[12].

We now compare (5-5) and *(5-2)* and use again the Wigner-Eckart theorem in the LHS of (5-5) to obtain the four reduced matrix elements of $T^{[1,1]}$:

$$\langle \lambda, l+1\, l'+1 \,\|\, T^{[1,1]} \,\|\, \lambda,\ ll' \rangle = \tfrac{1}{2} \left[\frac{(l+1)(l'+1)(\lambda-l-l')(\lambda+l+l'+4)}{(2l+3)(2l'+3)} \right]^{1/2},$$

$$\langle \lambda,\ l+1\, l'-1 \,\|\, T^{[1,1]} \,\|\, \lambda,\ ll' \rangle = -\tfrac{1}{2} \left[\frac{(l+1)l'(\lambda-l+l'+1)(\lambda+l-l'+3)}{(2l+3)(2l'-1)} \right]$$

$$\langle \lambda,\ l-1\, l'+1 \,\|\, T^{[1,1]} \,\|\, \lambda,\ ll' \rangle = \tfrac{1}{2} \left[\frac{l(l'+1)(\lambda+l-l'+1)(\lambda-l+l'+3)}{(2l-1)(2l'+3)} \right]^{1/2},$$

$$\langle \lambda,\ l-1\, l'+1 \,\|\, T^{[1,1]} \,\|\, \lambda,\ ll' \rangle = \tfrac{1}{2} \left[\frac{ll'(\lambda+l+l'+2)(\lambda-l-l'+2)}{(2l-1)(2l'-1)} \right]^{1/2}. \qquad (5\text{-}6)$$

6. Angular Functions with Permutational Symmetry

The next step is to construct, from the R_6 basis $P^{[\lambda]}_{ll'LM}$, functions with definite permutational symmetry. That this is actually possible comes from the fact that the above polynomial basis also carries an IR of O_6, the orthogonal group in six dimensions, the symmetry group of the general Hamiltonian we are dealing with, namely, *(3-1);* cf. Ref. *4.* Indeed, one cannot consider S_3 (the permutation group of operators acting on particle indices 1,2,3) as a subgroup of R_6 since the matrices corresponding to transpositions have determinant equal to (-1). It is also clear that one can take S_3 as a subgroup of O,. Since O, can be written as R, $\oplus \alpha R_6$, where α denotes the transformation $(\mathbf{x} \rightarrow \mathbf{x},\ \mathbf{y} \rightarrow -\mathrm{y})$, which is diagonal in the basis $P^{[\lambda]}_{ll'LM}$, we see that this basis also carries an IR of O,.

The mixed representation of S_3 is chosen as[13]

$$(1,2)\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}, \quad (1,3)\begin{pmatrix} F \\ G \end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}, \tag{6-1}$$

where **(i, j)** denotes the transposition $(\mathbf{r}_i \leftrightarrow \mathbf{r}_j)$. From **(4-3),** one can show that

$$(1,2)\, P^{[\lambda]}_{ll'LM} = (-)^{l'} P^{[\lambda]}_{ll'LM}. \tag{6-2}$$

The effect of the generator (1,3) on these polynomials is, however, very complicated. The usual procedure to bypass this difficulty is to introduce the more convenient (complex) variables

$$\xi = \frac{1}{\sqrt{2}}(\mathrm{x} - i\mathrm{y}), \qquad \eta = \frac{1}{\sqrt{2}}(\mathrm{x} + i\mathrm{y}). \tag{6-3}$$

In these new variables, we have

$$(1,2)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad (1,3)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{6-4}$$

$\phi = \frac{2\pi}{3}$, and the **CM** angular momentum L reads

$$\mathbf{L} = -i(\xi \times \nabla_{\xi} + \eta \times \nabla_{\eta}) = \mathbf{L}(\xi) + \mathbf{L}(\eta). \tag{6-5}$$

It follows then that the polynomials

$$Q^{\lambda}_{\mu\nu jj'LM}(\xi, \eta) = (\xi^2)^{\mu}(\eta^2)^{\nu} \sum_{mm'} \langle jmj'm' \,|\, LM \rangle\, \mathscr{Y}^{j}_{m}(\xi)\, \mathscr{Y}^{j'}_{m'}(\eta), \tag{6-6}$$

where μ, ν, **j**, j' are non-negative integers satisfying the condition $2(\mu + \nu) + \mathbf{j} + \mathbf{j}' = \lambda$, are homogeneous polynomials of degree λ with total angular momentum L and projection M. These polynomials form a basis for the vector space of hornogeneous polynomials of degree λ in six variables. They carry a most degenerate IR of U_6, the unitary group in six dimensions[9]. Since the Q's are homogeneous separately in ξ and η, they exhibit very simple permutational properties:

$$(1,2)\, Q^{\lambda}_{\mu\nu jj'LM} = (-)^{j+j'-L} Q^{\lambda}_{\nu\mu j'jLM},$$

$$(1,3)\, Q^{\lambda}_{\mu\nu jj'LM} = (-)^{j+j'-L} e^{i(2\mu-2\nu+j-j')\phi} Q^{\lambda}_{\nu\mu j'jLM} \tag{6-7}$$

These relations show that for $\mu \neq$ v or j $\neq j'$, the pair

$$Q^\lambda_{\mu\nu jj'LM}, \quad Q^\lambda_{\nu\mu j'jLM} \tag{6-8}$$

carry a two-dimensional representation of S_3, which however may be reducible. Taking two linear combinations of the pair (6-8), with arbitrary coefficients, we can by using **(6-7)** find the conditions for which such a two dimensional representation is exactly the mixed representation (6-1) or else reduces into symmetric and antisymmetric representations. The outcome is that the linear combinations

$$Q^\lambda_{\sigma\mu\nu jj'LM} \equiv Q^\lambda_{\mu\nu jj'LM} + \sigma\,(-)^{j+j'-L}\,Q^\lambda_{\nu\mu j'jLM}\,, \tag{6-9}$$

with o $= \pm 1$, have definite permutational symmetry depending on the number

$$u = 2(\mu - \nu) + j - j'. \tag{6-10}$$

For u E **0** *(mod* 3), $Q^\lambda_{+1\mu\nu jj'LM}$ is symmetric and $Q^\lambda_{-1\mu\nu jj'LM}$ antisymmetric, while for $u \equiv 0$ (*mod* 3) we have (cf. Eq. **6-1)**

$$\alpha Q^\lambda_{+1\mu\nu jj'LM} = F, \quad \beta Q^\lambda_{-1\mu\nu jj'LM} = G, \tag{6-11}$$

with $(\beta/\alpha) = (e^{i\lambda\phi}/\sqrt{3})\,[e^{i(j'-\nu)\phi} - e^{i(j-\mu)\phi}]$, α being an arbitrary constant. That is: for $u \not\equiv 0$ *(mod* 3), Q^λ_{+1} and Q^λ_{-1} are proportional to the up and down components of the mixed representation *(6-1)*, respectively. Note that o $= \pm 1$ are the eigenvalues of the operator (1,2) of S,.

Finally, for $\mu =$ v and $j = j'$ we see at once that, for even L, Q^λ_{+1} is symmetric and $Q^\lambda_{-1} \equiv$ **0,** while, for odd L, $Q^\lambda_{+1} \equiv$ **0** and Q^λ_{-}**,** is antisymmetric.

It was rather simple to obtain homogeneous polynomials, with deiinite permutational symmetries, in terms of the $Q^{\lambda'}s$ (6-6). The polynomials, however, are not in general harmonic and therefore the IR of U_6 they carry is reducible with respect to R,, according to (4-13). In order to get harmonic polynomials out of the $Q^{\lambda'}_\sigma s$, we use the projection operator

$$\mathscr{P}^{[\lambda]} = \frac{\left[\frac{\lambda+3}{2}\right]!}{2^{[\lambda/2]}(\lambda+1)!\,[\lambda/2]!}\prod_{\omega=1}^{[\lambda/2]}\left[\mathscr{I}_2 - \tfrac{1}{2}(\lambda-2\omega)(\lambda-2\omega+4)\right] \tag{6-12}$$

where $\mathscr{I}_2$ is the Casimir given in **(2-6)** and the notation [ ] in the RHS is defined in Ref. **10.** Of course, if a harmonic component is present in a given Q^λ_σ, it will be a linear combination of the polynomials $P^{[\lambda]}_{ll'LM}$, (4-3), with the same L and M, since $[\mathscr{I}_2, \mathrm{L}] = 0$. On the other hand, since $\mathscr{I}_2$

commutes with the elements of S_3, the above projector preserves the permutational symmetry. The number of harmonic polynomials one gets by such a procedure is however greater than *dim* [A], what indicates that they are not linearly independent.

If we take the set (with v = 0)

$$\mathscr{P}^{[\lambda]} Q^{\lambda}_{\sigma\mu o j j' LM} \tag{6-13a}$$

and restrict the L values to

$$\mathbf{L} = \lambda - 2\mu - 4k \quad \text{and} \quad \mathbf{L} = \lambda - 2\mu - 4k - 1, \qquad k = 0,1,2,\dots, \tag{6-13b}$$

we get dim $[\lambda]$ polynomials[14]. In what follows we shall designate the polynomials (6-13a), restricted by conditions (6-llb), simply by (6-13).

Since the numbers μ, v, j, j' are not preserved by the projector $\mathscr{P}^{[\lambda]}$, it is necessary to look for a new set of labels. To assure that polynomials with different sets of labels are orthogonal, we have to take labels associated to eigenvalues of Hermitian operators.

First of all, the number u, (6-10), is just the difference of the degrees of homogeneity in ξ and η for the Q^{λ} polynomials. This number is the eigenvalue of the Hermitian, scalar[15], operator

$$U = \xi \cdot \nabla_{\xi} - \eta \cdot \nabla_{\eta} = i(\mathbf{x} \cdot \nabla_y - \mathbf{y} \cdot \nabla_x) = -2i\sqrt{3}\, T^0_0, \tag{6-14}$$

where T^0_0 was given in (5-4) and its matrix elements in (5-5). It is easy to verify that the polynomials Q^{λ}_{σ}, defined by (6-9), are eigenfunctions of the operator U^2 with eigenvalues u^2 and of course the same holds for the harmonic set (6-13) since $[U, \mathscr{I}_2] = 0$. We note however that the operator U^2 does not distinguish between up and down components of a mixed representation of S_3: this is done by the transposition (1,2). It is then clear that, instead of U^2, we have to use the product $(1,2)U^2$.

For $\lambda > 5$, degeneracies occur and one needs an extra operator to classify completely the basic harmonic polynomials. To that purpose, we constructed the Hermitian, scalar, operator[16]

$$\begin{aligned} W = {} & 4[\xi^2(\eta \cdot \nabla_{\xi}) - \eta^2(\xi \cdot \nabla_{\eta})](\nabla_{\xi} \cdot \nabla_{\eta}) - 2(\xi^2 \nabla^2_{\xi} - \eta^2 \nabla^2_{\eta})(\mathbf{r} \cdot \nabla + 1) + \\ & + 4(\xi \cdot \eta)[(\xi \cdot \nabla_{\eta})\nabla^2_{\xi} - (\eta \cdot \nabla_{\xi})\nabla^2_{\eta}] - 2[2(\mathbf{r} \cdot \nabla) + 3]\, U, \end{aligned} \tag{6-15}$$

or, in the $\mathbf{x}$, $\mathbf{y}$ variables:

$$\begin{aligned}\mathbf{W} = i\{[(\mathbf{x}^2-\mathbf{y}^2)(\mathbf{x}\cdot\nabla_y+\mathbf{y}\cdot\nabla_x)-2(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}\cdot\nabla_x-\mathbf{y}\cdot\nabla_y)]\nabla^2 + \\ + 2[(\mathbf{x}\cdot\mathbf{y})(\nabla_x^2-\nabla_y^2)-(\mathbf{x}^2-\mathbf{y}^2)(\nabla_x\cdot\nabla_y)](\mathbf{r}\cdot\nabla+1)- \\ -\mathbf{r}^2[(\mathbf{x}\cdot\nabla_y+\mathbf{y}\cdot\nabla_x)(\nabla_x^2-\nabla_y^2)-2(\mathbf{x}\cdot\nabla_x-\mathbf{y}\cdot\nabla_y)(\nabla_x\cdot\nabla_y)]\}- \\ -2[2(\mathbf{r}\cdot\nabla)+3]U.\end{aligned}$$

This operator was obtained by requiring its commutation with $\mathscr{I}_2$, U and $\mathbf{L}$. In order to commute with $\mathbf{L}$, it has to be built up from scalars with respect to $R_3(\mathbf{L})$; to commute with U, we take it homogeneous separately in ξ and η and with the same degree of homogeneity in these variables. To preserve the degree of homogeneity λ of the polynomials, the operator has to have the same degree in the coordinates and derivatives. For an operator Ansatz of second degree in the coordinates and derivatives, the three conditions together with its commutation with $\mathscr{I}_2$, give an operator which is a function of $\mathscr{I}_2$, U, $\mathbf{L}^2$, $(\mathbf{r}\cdot\nabla)$, and therefore will not provide us with a new label. If instead we start with an operator Ansatz built up from terms of first, second and third degrees in the coordinates and the derivatives, we find that there exist solutions that are not functions of $\mathscr{I}_2$, U, L^2 and $(\mathbf{r}\cdot\nabla)$, and any of them can therefore be used to give us the extra label we need. These different solutions, however, differ only by a function of those four operators. We chose our solution by requiring that the operator be angular (i.e., it has to commute with r) and to have factorized matrix elements in the basis $P^{[\lambda]}$ (cf. Appendix 2): this is the origin of the terms $-2[2(\mathbf{r}\cdot\nabla)+3]U$ in **(6-15)**. Note also that since $\mathbf{W}$ will operate in a space of harmonic polynomials, the terms which factorize the six-dimensional Laplacian ∇^2 (which equals $2\nabla_\xi\cdot\nabla_\eta$) on the right, can be omitted.

Since W, like U, is antisymmetric under S_3, we take $\mathbf{W}^2$ to label the harmonic polynomials **(6-13)**. These polynomials are already eigenfunctions of $\mathbf{W}^2$ if no degeneracies with respect to the remaining labels occur and therefore the operator $\mathbf{W}^2$ is necessary only for $\lambda > 5$, since up to $\lambda = 4$ degeneracies are not present. When a degeneracy occurs, we have of course to take linear combinations of the polynomials involved to diagonalize $\mathbf{W}^2$. Fortunately, these degeneracies are rather rare and only for large λ are greater than two.

We have shown that the polynomials (6-13) are *dim* $[\lambda]$ in number and have also exhibited a set of operators to label these polynomials. To show that they provide a basis for an **IR** of O_6, it is enough to prove that when we apply the projector $\mathscr{P}^{[\lambda]}$ to the set $Q^{\lambda}_{\sigma\mu o j j' LM}$, with the restriction (6-13b), we get no zeros and, moreover, that the resulting harmonic polynomials are linearly independent. The proof that, in the set (6-13), there are no elements identically zero is given in Appendix 2. We have not however been able to give a complete proof of the linear independence for any value of A Indeed, we have shown that we get the same multiplicities for the L values as given by (4-10), namely, the same as in the IR[A] of O, carried by the polynomials $P^{[\lambda]}$. On the other hand, it can also be shown that the set (6-13), for fixed values of A and L, split into multiplets of S_3 with the right multiplicities[17]. We also verified that, for λ up to seven, the polynomials (6-11) are linearly independent.

In order to exhibit the chain of O_6 subgroups which provide the set of labels which characterize the polynomials with permutational symmetry, (6-13), we recall that the operator $(1,2)\,U^2$ labels representations of S_3. Therefore, the set of commuting operators $(1,2)\,U^2$, L^2 and L, label $IR's$ of $S_3 \otimes R_3(\mathrm{L})$ and $S_3 \otimes R_2(L_z)$. We are then in the chain

$$O_6 \supset [S_3 \otimes R_3(L)] \supset [S_3 \otimes R_2(L_z)]. \qquad (6\text{-}16)$$

Since we need four labels[18] besides λ, we see'that the above chain is not complete: the extra label is then provided by W^2.

The matrix elements of U in the $P^{[\lambda]}$ basis are, up to a constant, given by (5-5), since U and T_0^0 are proportional to each other (cf. 6-14). The matrix elements of W in the same basis are explicitly exhibited in Appendix 2.

As a final comment, we should add that the projection technique presented above is convenient, in practice, when we are interested in getting only a few polynomials of a given $IR[\lambda]$ of O_6 with permutational symmetry. When, however, we want to calculate a large number of polynomials we found it more convenient to use another method. We, first, construct the $P^{[\rho]}$ bases (4-3) for $\rho = A, A-2, \ldots, \{{}^0_1$ and then express the $Q_\sigma^{[\lambda]'}s$ in terms of the $P^{[\lambda]'}s$. The effect of the projection operator $\mathscr{P}^{[\lambda]}$ is just to omit, in such an expansion, the $P^{[\rho]'}s$ with $\rho < A$, the result being a linear combination of $P^{[\lambda]'}s$. Since the $P^{[\lambda]'}s$ are orthonormal, the final expansion will tell us directly the norm of the harmonic polynomials (6-13). Use of the expressions for the matrix elements of W (Appendix 2) gives us its matrix elements in the basis (6-13). In this way we have calculated the normalized harmonic polynomials with permutational symmetry, (6-13), up to $\lambda = 6$. They are exhibited in Appendix 4.

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Appendix 1. Basic Formulas for the Calculation of Matrix Elements in the $P^{[\lambda]}$ Basis

$$x_q P^{[\lambda]}_{lml'm'} =$$

$$\tfrac{1}{2}\left[\frac{l+1}{2l+3}\right]^{1/2} \langle lm\, 1q \,|\, l+1\, m+q\rangle \cdot$$

$$\cdot\left\{\left[\frac{(\lambda+l+l'+4)(\lambda+l-l'+3)}{(\lambda+2)(\lambda+3)}\right]^{1/2} P^{[\lambda+1]}_{l+1\,m+q\,l'm'} - \left[\frac{(\lambda-l+l'+1)(\lambda-l-l')}{(\lambda+1)(\lambda+2)}\right]^{1/2} (\mathbf{x}^2+\mathbf{y}^2) P^{[\lambda-1]}_{l+1\,m+q\,l'm'}\right\}$$

$$+\tfrac{1}{2}\left[\frac{l}{2l-1}\right]^{1/2} \langle lm\, 1q \,|\, l-1\, m+q\rangle \cdot$$

$$\cdot\left\{\left[\frac{(\lambda-l-l'+2)(\lambda-l+l'+3)}{(\lambda+2)(\lambda+3)}\right]^{1/2} P^{[\lambda+1]}_{l-1\,m+q\,l'm'} - \left[\frac{(\lambda+l-l'+1)(\lambda+l+l'+2)}{(\lambda+1)(\lambda+2)}\right]^{1/2} (\mathbf{x}^2+\mathbf{y}^2) P^{[\lambda-1]}_{lml'+1\,m'+q}\right\},$$

$$(\nabla_x)_q P^{[\lambda]}_{lml'm'} =$$

$$-\left[\frac{(\lambda+2)(l+1)(\lambda-l-l')(\lambda-l+l'+1)}{(\lambda+1)(2l+3)}\right]^{1/2} \langle lm\, 1q \,|\, l+1\, m+q\rangle P^{[\lambda-1]}_{l+1\,m+q\,l'm'}$$

$$-\left[\frac{(\lambda+2)l(\lambda+l+l'+2)(\lambda+l-l'+1)}{(\lambda+1)(2l-1)}\right]^{1/2} \langle lm\, 1q \,|\, l-1\, m+q\rangle P^{[\lambda-1]}_{l-1\,m+q\,l'm'},$$

where $q = \pm 1, 0$ and $x_{\pm} = \mp \frac{1}{\&}(x_1 \pm i x_2)$, $x_0 = x_3$ and similarly for the spherical components of the gradient. To get the corresponding expressions in the other variable, one can use a relation which gives the result of exchanging **x** and y, l and l', m and m', in the $P^{[\lambda]}$ polynomials, namely,

$$\begin{bmatrix} \mathbf{x} \leftrightarrow \mathbf{y} \\ l \leftrightarrow l' \\ m \leftrightarrow m' \end{bmatrix} P^{[\lambda]}_{lml'm'} = (-)^n P^{[\lambda]}_{lml'm'},$$

where $n = \frac{1}{2}(\lambda - l - l')$; note that, by (4-8), $2n$ is an even number. The basic formulas above are obtained by using the corresponding formulas for solid harmonics as well as recursion relations for Jacobi polynomials[S].

Appendix 2. Matrix Elements of W in the $P^{[\lambda]}$ Basis

$$(I,,\ \bar{l}\bar{l}' LM \,|\, W \,|\, A,\ ll' LM\rangle = 2i(-)^{l+l'+L} \cdot$$

$$\cdot\Big[(\lambda+l-l'+2)(\lambda-l+l'+2)[(l+1)(l'+1)(\lambda-l-l')(\lambda+l+l'+4)]^{1/2} \begin{Bmatrix} l+1 & 1 & l \\ l' & L & l'+1 \end{Bmatrix} \delta_{\bar{l},l+1}\delta_{\bar{l}',l'+1}$$

$$-(\lambda+l+l'+3)(\lambda-l-l'+1)[(l+1)l'(\lambda-l+l'+1)(\lambda+l-l'+3)]^{1/2} \begin{Bmatrix} l+1 & 1 & l \\ l' & L & l'-1 \end{Bmatrix} \delta_{\bar{l},l+1}\delta_{\bar{l}',l'-1}$$

$$+ (\lambda+l+l'+3)(\lambda-l-l'+1)[l(l'+1)(\lambda+l-l'+1)(\lambda-l+l'+3)]^{1/2}\begin{Bmatrix} l-1 & 1 & l \\ l' & L & l'+1 \end{Bmatrix}\delta_{\bar{l},l-1}\,\delta_{\bar{l}',l'+1}$$

$$-(\lambda+l-l'+2)(\lambda-l+l'+2)[ll'(\lambda+l+l'+2)(\lambda-l-l'+2)]^{1/2}\begin{Bmatrix} l-1 & 1 & l \\ l' & L & l'-1 \end{Bmatrix}\delta_{\bar{l},l-1}\,\delta_{\bar{l}',l'-1}\Bigg].$$

In the above expressions, { } stands for $6-j$ symbols. To obtain those matrix elements, we have made use of the formulas of Appendix-1. Note that W is diagonal in L and **M** since it is a scalar wrt R, (L).

Appendix 3.

To prove that the polynomials (6-13) have a harmonic part, we first of all observe that the polynomials (cf. 6-6)

$$Q^{\lambda}{}_{oojj'LL} = \sum_{mm'} < jmj'm' \mid LL > \mathscr{Y}^{j}_{m}(\xi)\mathscr{Y}^{j'}_{m'}(\eta)$$

can be rewritten by making use of the following Ansatz:

$$Q^{\lambda}_{oojj'LL} = \sum_{qq'} f^{\lambda L\, jj'}_{qq'}(\xi^2, \eta^2, \xi\cdot\eta) K^{L}_{qq'LL}\,, \tag{A-1}$$

where the $K^{L}_{qq'LL}$ constitute a basis for polynomials of angular momentum Lin six variables:

$$K^{L}_{qq'LL} = \begin{cases} Q^{L}_{ooqq'LL} = (\xi_+)^q(\eta_+)^{q'} \text{ for } (-)^{\lambda+L} = 1, \\ Q^{L+1}_{ooqq'LL} \sim (\xi_+)^{q-1}(\eta_+)^{q'-1}(\xi_+\eta_0 - \xi_0\eta_+), \text{ for } (-)^{\lambda+L} = -1 \text{ and } q, q' \geq 1, \end{cases} \tag{A-2}$$

where $\xi_{\pm} = \mp\frac{1}{\sqrt{2}}(\xi_1 \pm i\,\xi_2)$ and $\xi_0 = \xi_3$ are the speherical components of ξ (and similarly for η). Since $Q^{\lambda}_{oojj'LL}$ is harmonic and homogeneous separately in ξ and η (with degrees of homogeneity j and j', respectively), we see that the functions f in (A-1) can be written as

$$f^{\lambda Ljj'}_{qq'} = \sum_{\alpha\beta\gamma} A^{\lambda Ljj'qq'}_{\alpha\beta\gamma}(\xi^2)^{\alpha}(\eta^2)^{\beta}(\xi\cdot\eta)^{\gamma} \tag{A-3}$$

where α, β, γ and the angular momenta q, q' are the non-negative integral solutions of

$$2\alpha + \gamma + q = j, \quad 2\beta + \gamma + q' = j',$$

with

$$\begin{aligned} &q + q' = L \quad \text{for} \quad \lambda + \mathrm{L} = \text{even}, \\ &q + q' = \mathrm{L} + 1 \quad \text{for} \quad \lambda + \mathrm{L} = \text{odd}, \end{aligned} \tag{A-4}$$

and, of course, we have the relation $j + j' = \lambda$. In (A-3), the A coefficients are determined imposing that the polynomials $Q^{\lambda}_{oojj'LL}$ be harmonic. We shall deal with conditions (A-4) in three steps.

(i) For L = O, λ is even and $j = j'$ and one has (since $q = q' = 0$)

$$Q^{\lambda}_{oojjoo} = \sum_{\alpha\beta\gamma} A_{\alpha\beta\gamma}(\xi^2)^{\alpha}(\eta^2)^{\beta}(\xi\cdot\eta)^{\gamma},$$

with $2\alpha + \gamma = 2\beta + \gamma = j$. This implies $\alpha = \beta$ and $\gamma = j, j-2, \ldots, \{^0_1$. For odd j, therefore, $\gamma \neq 0$ which means that Q^{λ}_{oojjoo} factors $(\xi\cdot\eta)$ out. Since $(\xi\cdot\eta) = (1/2)(x^2 + y^2)$, we conclude by recalling Eq. (4-13) in the text that Q^{λ}_{oojjoo} does not have a harmonic part. Such cases are, however, excluded by restriction (6-13b), since odd values of j entail $\lambda = 2j = 4k + 2$ ($k = 0, 1, 2, \ldots$) and therefore $\lambda - 2\mu - L = 4k + 2$, which contradicts (6-13b). For even j, however, γ can be zero and this means that Q^{λ}_{oojjoo} does not factor $(\xi \cdot \eta)$ out, i.e., that polynomial has always a harmonic part.

(ii) For $L = 1$, the K basis is

$$K^1_{1011} = \xi_+, \qquad K^1_{0111} = \eta_+ \quad \textit{for odd } \lambda$$

and

$$K^2_{1111} \sim (\xi_+ \eta_0 - \xi_0 \eta_+) \quad \textit{for even } \lambda.$$

Then, for odd i, we have according to (A-2):

$$Q^{\lambda(odd)}_{oojj'11} = f^{\lambda 1 jj'}_{10}\, \xi_+ + f^{\lambda 1 jj'}_{01}\, \eta_+$$

Since, in this case, j and j' must have opposite parities, from conditions (A-4) we see that, in f_{10}, $y = 0$ implies odd j and even j', while in f_{01}, $y = 0$ implies even j and odd j'. So, in any case, one of the f'_s does not factor $(\xi \cdot q)$ out: $Q^{\lambda}_{oojj'11}$, odd L has always a harmonic part. On the other hand, for even λ, $j = j'$ and we have

$$Q^{\lambda(even)}_{oojj11} \sim f^{\lambda 1 jj}_{11} (\xi_+ \eta_0 - \xi_0 \eta_+).$$

In f_{11}, $y = 0$ implies odd j. Therefore, for even j, the polynomial factors $(\xi \cdot \eta)$ out and so does not contain a harmonic part. Those cases are excluded, since for even j we shall have $\lambda - 2\mu - L = 4k - 1$, which violates the restriction (6-13b). It is clear that, for odd j, the corresponding polynomial has a harmonic part.

(iii) For any $L \geq 2$, the K basis has elements with even as well as odd values of q. Conditions (A-4) imply then that, regardless of the parities of j and j', in some $f's$ of the expansion **(A-i)** there will be terms with $y = 0$, which therefore will not factor $(\xi \cdot \eta)$ out. We then conclude that for $L \geq 2$ any polynomial $Q^{\lambda}_{oojj'LL}$ has a harmonic part.

Note now that clearly (cf. 6-6)

$$Q^{\lambda}_{\mu\nu jj'LL} = (\xi^2)^{\mu} (\eta^2)^{\nu}\, Q^{\lambda - 2\mu - 2\nu}_{oojj'LL}$$

Assuming that the Q polynomial on the RHS of the above relation satisfies the restriction (6-13b), in which case the discussion above has shown that it contains a harmonic part, it is clear that $Q^{\lambda}_{\mu\nu jj'LL}$ also contains a harmonic part since the RHS factors do not involve $(\xi \cdot \eta)$.

Finally, as $Q^{\lambda}_{\mu\nu jj'LM} = (L_-)^{L-M}\, Q^{\lambda}_{\mu\nu jj'LL}$, where L_- is the (-1) spherical component of L, all the conclusions above hold for any value of the z-component M since L_- commutes with $(\xi \cdot \eta)$.

Appendix 4. Harmonic Polynomials with Permutational Symmetry for λ up to 6.

The polynomials with permutational symmetry are here designated by S (symmetric), A (antisymmetric), F (up component of a mixed representation), or G (the down component). Along with their explicit expressions in terms of the Jacobi coordinates, we have given their expressions in terms of the $P^{[\lambda]}_{ll'LM}$ polynomials, in which the upper index λ has been ommited and, to make the reading easier, we used parentheses to separate the 1, i' labels from the others. When, for given values of A and L, a certain S_3 representation appears more than once, we have used dashes to distinguish them. Their linear independence follows from the fact that they have different values for the pair (u^Z, w^2). For $A = 5$ and 6, pairs of polynomials corresponding to the mixed representation, with the same A, u^2 and L, occur: they are not eigenfunctions of W^Z and their w^Z eigenvalues have to be obtained by diagonalization. We listed only polynomiais with M = L since those with M < L can be obtained by successive applications of the ladder operator L.. We finally recall that

$$x_{\pm} = \mp \frac{1}{\sqrt{2}}(x_1 \pm ix_2), \qquad x_0 = x_3$$

$\lambda = 0$

$$S_{00} = P_{(00)00} = (1/\pi^3)^{1/2}\,;\ u^2 = w^2 = 0$$

$\lambda = 1$

$$F_{11} = P_{(10)11} = (6/\pi^3)^{1/2} x_+\,;\ u^2 = 1,\ w^2 = (10)^2$$

$$G_{11} = P_{(01)11} = (6/\pi^3)^{1/2} y_+\,;\ u_2 = 1,\ w^2 = (10)^2$$

$\lambda = 2$

$$F_{22} = \frac{1}{\sqrt{2}}(P_{(20)22} - P_{(02)22}) = 2(3/\pi^3)^{1/2}(x_+^2 - y_+^2)\,;\ u^2 = 4,\ w^2 = (28)^2$$

$$G_{22} = -P_{(11)22} = -4(3/\pi^3)^{1/2} x_+ y_+\,;\ u^2 = 4,\ w^2 = (28)^2$$

$$S_{22} = \frac{1}{\sqrt{2}}(P_{(20)22} + P_{(02)22}) = 2(3/\pi^3)^{1/2}(x_+^2 + y_+^2)\,;\ u^2 = w^2 = 0$$

$$A_{11} = P_{(11)11} = 2(6/\pi^3)^{1/2}(x_+ y_0 - x_0 y_+)\,;\ u^2 = w^2 = 0$$

$$F_{00} = P_{(00)00} = 2(1/\pi^3)^{1/2}(\mathbf{x}^2 - \mathbf{y}^2)\,;\ u^2 = 4,\ w^2 = (64)^2$$

$$G_{00} = P_{(11)00} = -4(1/\pi^3)^{1/2}(\mathbf{x}\cdot \mathbf{y})\,;\ u^2 = 4,\ w^2 = (64)'$$

$\lambda = 3$

$$S_{33} = \tfrac{1}{2}(P_{(30)33} - \sqrt{3}\,P_{(12)33}) = 2(5/\pi^3)^{1/2} x_+(x_+^2 - 3y_+^2)\,;\ u^2 = 9,\ w^2 = (54)^2$$

$$A_{33} = \tfrac{i}{2}(P_{(03)33} - \sqrt{3}\,P_{(21)33}) = 2i(5/\pi^3)^{1/2} y_+(y_+^2 - 3x_+^2)\,;\ u^2 = 9,\ w^2 = (54)^2$$

$$F_{33} = \tfrac{1}{2}(\sqrt{3}\,P_{(30)33} + P_{(12)33}) = 2(15/\pi^3)^{1/2} x_+(x_+^2 + y_+^2)\,;\ u^2 = 1,\ w^2 = (18)^2$$

$$G_{33} = \tfrac{1}{2}(\sqrt{3}\,P_{(03)33} + P_{(21)33}) = 2(15/\pi^3)^{1/2} y_+(x_+^2 + v:)\,;\ u^2 = 1,\ w^2 = (18)^2$$

$$F_{22} = P_{(12)22} = 4(10/\pi^3)^{1/2} y_+(x_+ y_0 - x_0 y_+)\,;\ u^2 = 1,\ w^2 = (18)^2$$

$$G_{22} = -P_{(21)22} = -4(10/\pi^3)^{1/2} x_+(x_+ y_0 - x_0 y_+)\,;\ u^2 = 1,\ w^2 = (18)^2$$

$$S_{11} = \frac{1}{\sqrt{3}}(P_{(12)11} - \sqrt{2}\,P_{(10)11}) = 2(3/\pi^3)^{1/2}\left[(\mathbf{x}^2 - \mathbf{y}^2)x_+ - 2(\mathbf{x}\cdot\mathbf{y})y_+\right]\,;\ u^2 = 1,\ w^2 = (134)^2$$

$$A_{11} = \frac{1}{\sqrt{3}}(P_{(21)11} + \sqrt{2}\,P_{(01)11}) = -2i(3/\pi^3)^{1/2}[(\mathbf{x}^2-\mathbf{y}^2)y_+ + 2(\mathbf{x}\cdot\mathbf{y})x_+]\,;\ u^2 = 1,\ w^2 = (134)^2$$

$$F_{11} = -\frac{1}{\sqrt{3}}(\sqrt{2}\,P_{(12)11} + P_{(10)11}) = -(6/\pi^3)^{1/2}[(3\mathbf{y}^2-\mathbf{x}^2)x_+ - 4(\mathbf{x}\cdot\mathbf{y})y_+]\,;\ u^2 = 1,\ w^2 = (58)^2$$

$$G_{11} = \frac{1}{\sqrt{3}}(-\sqrt{2}\,P_{(21)11} + P_{(01)11}) = -(6/\pi^3)^{1/2}[(3\mathbf{x}^2-\mathbf{y}^2)y_+ - 4(\mathbf{x}\cdot\mathbf{y})x_+]\,;\ u^2 = 1,\ w^2 = (58)^2$$

$\lambda = 4$

$$F_{44} = \tfrac{1}{4}(\sqrt{2}\,P_{(40)44} + \sqrt{2}\,P_{(04)44} - 2\sqrt{3}\,P_{(22)44}) = (30/\pi^3)^{1/2}(x_+^4 + y_+^4 - 6x_+^2 y_+^2)\,;\ u^2 = 16,\ w^2 = (88)^2$$

$$G_{44} = \frac{1}{\sqrt{2}}(P_{(13)44} - P_{(31)44}) = 4(30/\pi^3)^{1/2} x_+ y_+ (x_+^2 - y_+^2)\,;\ u^2 = 16,\ w^2 = (88)^2$$

$$F'_{44} = \frac{1}{\sqrt{2}}(P_{(40)44} - P_{(04)44}) = 2(30/\pi^3)^{1/2}(x_+^4 - y_+^4)\,;\ u^2 = 4,\ w^2 = (44)^2$$

$$G'_{44} = \frac{-1}{\sqrt{2}}(P_{(31)44} + P_{(13)44}) = -4(30/\pi^3)^{1/2} x_+ y_+ (x_+^2 + y_+^2)\,;\ u^2 = 4,\ w^2 = (44)^2$$

$$S_{44} = \tfrac{1}{4}(\sqrt{6}\,P_{(40)44} + \sqrt{6}\,P_{(04)44} + 2P_{(22)44}) = 3(10/\pi^3)^{1/2}(x_+^2 + y_+^2)^2\,;\ u^2 = w^2 = 0$$

$$F_{33} = P_{(22)33} = 12(10/\pi^3)^{1/2} x_+ y_+ (x_+ y_0 - x_0 y_+)\,;\ u^2 = 4,\ w^2 = (44)^2$$

$$G_{33} = \frac{1}{\sqrt{2}}(P_{(31)33} - P_{(13)33}) = 6(10/\pi^3)^{1/2}(x_+^2 - y_+^2)(x_+ y_0 - x_0 y_+)\,;\ u^2 = 4,\ w^2 = (44)^2$$

$$A_{33} = \frac{i}{\sqrt{2}}(P_{(31)33} + P_{(13)33}) = 6i(10/\pi^3)^{1/2}(x_+^2 + y_+^2)(x_+ y_0 - x_0 y_+)\,;\ u^2 = w^2 = 0$$

$$F_{22} = \frac{1}{2\sqrt{3}}(\sqrt{2}\,P_{(22)22} - \sqrt{5}\,P_{(20)22} + \sqrt{5}\,P_{(02)22}) = 6(5/7\pi^3)^{1/2}[(\mathbf{x}^2-\mathbf{y}^2)(x_+^2 - y_+^2) - 4(\mathbf{x}\cdot\mathbf{y})x_+ y_+]\,;\ u^2 = 16,\ w^2 = (228)^2$$

$$G_{22} = \frac{1}{2\sqrt{5}}(-\sqrt{3}\,P_{(31)22} + \sqrt{3}\,P_{(13)22} - \sqrt{14}\,P_{(11)22}) = 12(5/7\pi^3)^{1/2}[(\mathbf{x}^2-\mathbf{y}^2)x_+ y_+ + (\mathbf{x}\cdot\mathbf{y})(x_+^2 - y_+^2)]\,;\ u^2 = 16,\ w^2 = (228)^2$$

$$F'_{22} = -\frac{1}{\sqrt{2}}(P_{(20)22} + P_{(02)22}) = 2(6/7\pi^3)^{1/2}[(3\mathbf{x}^2 - 7\mathbf{y}^2)x_+^2 + (7\mathbf{x}^2 - 3\mathbf{y}^2)y_+^2]\,;\ u^2 = 4,\ w^2 = (128)^2$$

$$G'_{22} = \frac{1}{\sqrt{2}}(P_{(31)22} + P_{(13)22}) = 4(6/7\pi^3)^{1/2}[2(\mathbf{x}^2 + \mathbf{y}^2)x_+ y_+ - 5(\mathbf{x}\cdot\mathbf{y})(x_+^2 + y_+^2)]\,;\ u^2 = 4,\ w^2 = (128)^2$$

$$S_{22} = -\frac{1}{2\sqrt{3}}(P_{(20)22} + \sqrt{10}\,P_{(22)22} - P_{(02)22}) = (6/7\pi^3)^{1/2}[20(\mathbf{x}\cdot\mathbf{y})\,x_+ y_+ + (\mathbf{x}^2 - 9\mathbf{y}^2)x_+^2 + (\mathbf{y}^2 - 9\mathbf{x}^2)y_+^2]\,;\ u^2 = 0,\ w^2 = 1680$$

$$A_{22} = \frac{i}{2\sqrt{5}}(\sqrt{7}\,P_{(31)22} - \sqrt{7}\,P_{(13)22} - \sqrt{6}\,P_{(11)22}) = 4i(15/\pi^3)^{1/2}[(\mathbf{x}^2 - \mathbf{y}^2)x_+ y_+ - (\mathbf{x}\cdot\mathbf{y})(x_+^2 - y_+^2)]\,;\ u^2 = 0,\ w^2 = 1680$$

$$F_{11} = -P_{(22)11} = 24(1/\pi^3)^{1/2}(\mathbf{x}\cdot\mathbf{y})(x_+ y_0 - x_0 y_+)\,;\ u^2 = 4,\ w^2 = (144)^2$$

$$G_{11} = -P_{(11)11} = 12(1/\pi^3)^{1/2}(\mathbf{x}^2 - \mathbf{y}^2)(x_+ y_0 - x_0 y_+)\,;\ u^2 = 4,\ w^2 = (144)^2$$

$$S_{00} = -\frac{1}{\sqrt{3}}(\sqrt{2}\,P_{(22)00} + P_{(00)00}) = -(3/\pi^3)^{1/2}[8(\mathbf{x}\cdot\mathbf{y})^2 + \mathbf{x}^4 + \mathbf{y}^4 - 6\mathbf{x}^2\mathbf{y}^2]\,;\ u^2 = w^2 = 0$$

$$F_{00} = \frac{1}{\sqrt{3}}(-P_{(22)00} + \sqrt{2}\,P_{(00)00}) = 6(1/\pi^3)^{1/2}[(\mathbf{x}^2 - \mathbf{y}^2)^2 - 4(\mathbf{x}\cdot\mathbf{y})^2];\ u^2 = 16,\ w^2 = (288)^2$$

$$G_{00} = P_{(11)00} = 4(6/\pi^3)^{1/2}(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2 - \mathbf{y}^2)\,;\ u^2 = 16,\ w^2 = (288)^2$$

$\lambda = 5$

$$F_{55} = \tfrac{1}{4}(P_{(50)55} - \sqrt{10}\,P_{(32)55} + \sqrt{5}\,P_{(14)55}) = (42/\pi^3)^{1/2}(x_+^5 - 10x_+^3 y_+^2 + 5x_+ y_+^4)\,;\ u^2 = 25,\ w^2 = (130)^2$$

$$G_{55} = -\tfrac{1}{4}(P_{(05)55} - \sqrt{10}\,P_{(23)55} + \sqrt{5}\,P_{(41)55}) = -(42/\pi^3)^{1/2}(5x_+^4 y_+ - 10x_+^2 y_+^3 + y_+^5)\,;\ u^2 = 25,\ w^2 = (130)^2$$

$$S_{55} = \tfrac{1}{4}(\sqrt{5}\,P_{(50)55} - \sqrt{2}\,P_{(32)55} - 3P_{(14)55}) = (210/\pi^3)^{1/2}(x_+^5 - 2x_+^3 y_+^2 - 3x_+ y_+^4)\,;\ u^2 = 9,\ w^2 = (78)^2$$

$$A_{55} = \frac{i}{4}(\sqrt{5}\,P_{(05)55} - \sqrt{2}\,P_{(23)55} - 3\,P_{(41)55}) = i(210/\pi^3)^{1/2}(-3x_+^4 y_+ - 2x_+^2 y_+^3 + y_+^5)\,;\ u^2 = 9,\ w^2 = (78)^2$$

$$F'_{55} = \frac{1}{2\sqrt{5}}(\sqrt{5}\,P_{(50)55} + P_{(14)55} + \sqrt{2}\,P_{(32)55}) = 2(105/\pi^3)^{1/2}(x_+^5 + 2x_+^3 y_+^2 + x_+ y_+^4)\,;\ u^2 = 1,\ w^2 = (26)^2$$

$$G'_{55} = \frac{1}{2\sqrt{2}}(\sqrt{5}\,P_{(05)55} + P_{(41)55} + \sqrt{2}\,P_{(23)55}) = 2(105/\pi^3)^{1/2}(x_+^4 y_+ + 2x_+^2 y_+^3 + y_+^5)\,;\ u^2 = 1,\ w^2 = (26)^2$$

$$S_{44} = \tfrac{1}{2}(\sqrt{3}\,P_{(32)44} - P_{(14)44}) = 4(42/\pi^3)^{1/2}\,y_+(3x_+^2 - y_+^2)(x_+ y_0 - x_0 y_+)\,;\ u^2 = 9,\ w^2 = (78)^2$$

$$A_{44} = \tfrac{i}{2}(-\sqrt{3}\,P_{(23)44} + P_{(41)44}) = 4i(42/\pi^3)^{1/2}\,x_+(x_+^2 - 3y_+^2)(x_+ y_0 - x_0 y_+)\,;\ u^2 = 9,\ w^2 = (78)^2$$

$$F_{44} = \tfrac{1}{2}(P_{(32)44} + \sqrt{3}\,P_{(14)44}) = 12(14/\pi^3)^{1/2}\,y_+(x_+^2 + y_+^2)(x_+ y_0 - x_0 y_+)\,;\ u^2 = 1,\ w^2 = (26)^2$$

$$G_{44} = -\tfrac{1}{2}(P_{(23)44} + \sqrt{3}\,P_{(41)44}) = -12(14/\pi^3)^{1/2}\,x_+(x_+^2 + y_+^2)(x_+ y_0 - x_0 y_+)\,;\ u^2 = 1,\ w^2 = (26)^2$$

$$F_{33} = \frac{1}{\sqrt{140}}(-\sqrt{35}\,P_{(30)33} + 9P_{(12)33} + \sqrt{14}\,P_{(32)33} - \sqrt{10}\,P_{(14)33}) = 2(35/3\pi^3)^{1/2}[(\mathbf{x}^2-\mathbf{y}^2)x_+(x_+^2-3y_+^2) + 2(\mathbf{x}\cdot\mathbf{y})y_+(y_+^2-3x_+^2)]\,;\ u^2 = 25,\ w^2 = (346)^2$$

$$G_{33} = -\frac{1}{\sqrt{140}}(\sqrt{35}\,P_{(03)33} - 9P_{(21)33} + \sqrt{14}\,P_{(23)33} - \sqrt{10}\,P_{(41)33}) = -2(35/3\pi^3)^{1/2}[(\mathbf{x}^2-\mathbf{y}^2)y_+(3x_+^2-y_+^2) + 2(\mathbf{x}\cdot\mathbf{y})x_+(x_+^2-3y_+^2)]\,;\ u^2 = 25,\ w^2 = (346)^2$$

$$S_{33} = \frac{1}{\sqrt{140}}(-\sqrt{70}\,P_{(30)33} - 3\sqrt{2}\,P_{(12)33} + \sqrt{7}\,P_{(32)33} + 3\sqrt{5}\,P_{(14)33}) = 2(70/3\pi^3)^{1/2}[(\mathbf{x}^2-2\mathbf{y}^2)x_+^3 - 3(\mathbf{x}\cdot\mathbf{y})y_+(x_+^2+y_+^2) + 3\mathbf{x}^2 x_+ y_+^2]\,;\ u^2 = 9,\ w^2 = (222)^2$$

$$A_{33} = \frac{i}{\sqrt{140}}(\sqrt{70}\,P_{(03)33} + 3\sqrt{2}\,P_{(21)33} + \sqrt{7}\,P_{(23)33} + 3\sqrt{5}\,P_{(41)33}) = 2i(70/3\pi^3)^{1/2}[-3(\mathbf{x}\cdot\mathbf{y})x_+(x_+^2+y_+^2) + 3\mathbf{y}^2 x_+^2 y_+ - (2\mathbf{x}^2-\mathbf{y}^2)y_+^3]\,;\ u^2 = 9,\ w^2 = (222)^2$$

$$F_{33} = -\frac{1}{\sqrt{1540}}(\sqrt{315}\,P_{(30)33} + 17P_{(12)33} + 3\sqrt{14}\,P_{(32)33} + 9\sqrt{10}\,P_{(14)33}) = 2(105/11\pi^3)^{1/2}[(\mathbf{x}^2-5\mathbf{y}^2)x_+ + 6(\mathbf{x}\cdot\mathbf{y})y_+](x_+^2+y_+^2)\,;\ u^2 = 1$$

$$G_{33} = \frac{1}{\sqrt{1540}}(\sqrt{315}\,P_{(03)33} + 17P_{(21)33} - 3\sqrt{14}\,P_{(23)33} - 9\sqrt{10}\,P_{(14)33}) = 2(105/11\pi^3)^{1/2}[6(\mathbf{x}\cdot\mathbf{y})x_+ - (5\mathbf{x}^2-\mathbf{y}^2)y_+](x_+^2+y_+^2)\,;\ u^2 = 1$$

$$F'_{33} = \frac{1}{\sqrt{140}}(4\sqrt{2}\,P_{(12)33} - 3\sqrt{7}\,P_{(32)33} + 3\sqrt{5}\,P_{(14)33}) = 2(210/\pi^3)^{1/2}[(\mathbf{x}\cdot\mathbf{y})(3x_+^2-y_+^2)y_+ - (2\mathbf{x}^2-\mathbf{y}^2)x_+y_+^2 - \mathbf{y}^2 x_+^3]\,;\ u^2 = 1$$

$$G'_{33} = \frac{1}{\sqrt{140}}(-4\sqrt{2}\,P_{(21)33} - 3\sqrt{7}\,P_{(23)33} + 3\sqrt{5}\,P_{(41)33}) = 2(210/\pi^3)^{1/2}[-(\mathbf{x}\cdot\mathbf{y})(x_+^2-3y_+^2)x_+ + (\mathbf{x}^2-2\mathbf{y}^2)x_+^2y_+ - \mathbf{x}^2 y_+^3]\,;\ u^2 = 1$$

$$S_{22} = -\frac{1}{\sqrt{30}}(2\sqrt{3}\,P_{(32)22} + 3\sqrt{2}\,P_{(12)22}) = 4(30/\pi^3)^{1/2}[(\mathbf{x}^2-\mathbf{y}^2)y_+ + 2(\mathbf{x}\cdot\mathbf{y})x_+](x_+y_0 - x_0y_+)\,;\ u^2 = 9,\ w^2 = (246)^2$$

$$A_{22} = \frac{i}{\sqrt{30}}(2\sqrt{3}\,P_{(23)22} - 3\sqrt{2}\,P_{(21)22}) = 4(30/\pi^3)^{1/2}[(\mathbf{x}^2-\mathbf{y}^2)x_+ - 2(\mathbf{x}\cdot\mathbf{y})y_+](x_+y_0 - x_0y_+)\,;\ u^2 = 9,\ w^2 = (246)^2$$

$$F_{22} = \frac{1}{\sqrt{5}}(-\sqrt{3}\,P_{(32)22} + \sqrt{2}\,P_{(12)22}) = 8(5/\pi^3)^{1/2}[3(\mathbf{x}\cdot\mathbf{y})x_+ - (2\mathbf{x}^2-\mathbf{y}^2)y_+](x_+y_0 - x_0y_+)\,;\ u^2 = 1,\ w^2 = (138)^2$$

$$G_{22} = \frac{1}{\sqrt{5}}(\sqrt{3}\,P_{(23)22} + \sqrt{2}\,P_{(21)22}) = 8(5/\pi^3)^{1/2}[(\mathbf{x}^2-2\mathbf{y}^2)x_+ + 3(\mathbf{x}\cdot\mathbf{y})y_+](x_+y_0 - x_0y_+)\,;\ u^2 = 1,\ w^2 = (138)^2$$

$$F_{11} = \frac{1}{\sqrt{15}}(3P_{(32)11} + P_{(12)11} + \sqrt{5}\,P_{(10)11}) = 2(3/\pi^3)^{1/2}\{[-\mathbf{x}^4 - 3\mathbf{y}^4 + 8\mathbf{x}^2\mathbf{y}^2 - 12(\mathbf{x}\cdot\mathbf{y})^2]x_+ + 2(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2+\mathbf{y}^2)y_+\}\,;\ u^2 = 1,\ w^2 = (82)^2$$

$$G_{11} = \frac{-1}{\sqrt{15}}(3P_{(23)11} - P_{(21)11} + \sqrt{5}\,P_{(01)11}) = 2(3/\pi^3)^{1/2}\{[-3\mathbf{x}^4 - \mathbf{y}^4 + 8\mathbf{x}^2\mathbf{y}^2 - 12(\mathbf{x}\cdot\mathbf{y})^2]\,y_+ + 2(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2 + \mathbf{y}^2)x_+\};\ u^2 = 1,\ w^2 = (82)^2$$

$$F'_{11} = \frac{1}{\sqrt{10}}(-P_{(32)11} - 2P_{(12)11} + \sqrt{5}\,P_{(10)11}) = 3(2/\pi^3)^{1/2}\{[(\mathbf{x}^2 - \mathbf{y}^2)^2 - 4(\mathbf{x}\cdot\mathbf{y})^2]\,x_+ - 4(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2 - \mathbf{y}^2)y_+\};\ u^2 = 25,\ w^2 = (466)^2$$

$$G'_{11} = \frac{1}{\sqrt{10}}(P_{(23)11} - 2P_{(21)11} - \sqrt{5}\,P_{(01)11}) = 3(2/\pi^3)^{1/2}\{[-(\mathbf{x}^2 - \mathbf{y}^2)^2 + 4(\mathbf{x}\cdot\mathbf{y})^2]\,y_+ - 4(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2 - \mathbf{y}^2)x_+\};\ u^2 = 25,\ w^2 = (466)^2$$

$$S_{11} = \frac{1}{\sqrt{30}}(-3P_{(32)11} + 4P_{(12)11} + \sqrt{5}\,P_{(10)11}) = (6/\pi^3)^{1/2}\{[\mathbf{x}^4 + 5\mathbf{y}^4 - 6\mathbf{x}^2\mathbf{y}^2 - 12(\mathbf{x}\cdot\mathbf{y})^2]\,x_+ + 8(\mathbf{x}\cdot\mathbf{y})(2\mathbf{x}^2 - \mathbf{y}^2)y_+\};\ u^2 = 9,\ w^2 = (302)^2$$

$$A_{11} = \frac{i}{\sqrt{30}}(-3P_{(23)11} - 4P_{(21)11} + \sqrt{5}\,P_{(01)11}) = i(6/\pi^3)^{1/2}\{-8(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2 - 2\mathbf{y}^2)x_+ + [5\mathbf{x}^4 + \mathbf{y}^4 - 6\mathbf{x}^2\mathbf{y}^2 - 12(\mathbf{x}\cdot\mathbf{y})^2]\};\ u^2 = 9,\ w^2 = (302)^2$$

$\lambda = 6$

$$S_{66} = \frac{1}{4\sqrt{2}}(P_{(60)66} - P_{(06)66} + \sqrt{15}\,P_{(24)66} - \sqrt{15}\,P_{(42)66}) = 2(14/\pi^3)^{1/2}(x_+^6 - 15x_+^4\,y_+^2 + 15x_+^2\,y_+^4 - y_+^6);\ u^2 = 36,\ w^2 = (180)^2$$

$$A_{66} = \frac{i}{4}(\sqrt{10}\,P_{(33)66} - \sqrt{3}\,P_{(51)66} - \sqrt{3}\,P_{(15)66}) = 4i(14/\pi^3)^{1/2}(-3x_+^4 + 10x_+^2\,y_+^2 - 3y_+^4)x_+\,y_+;\ u^2 = 36,\ w^2 = (180)^2$$

$$F_{66} = \frac{1}{4}(\sqrt{3}\,P_{(60)66} - \sqrt{5}\,P_{(42)66} - \sqrt{5}\,P_{(24)66} + \sqrt{3}\,P_{(06)66}) = 4(21/\pi^3)^{1/2}(x_+^4 - 6x_+^2\,y_+^2 + y_+^4)(x_+^2 + y_+^2);\ u^2 = 16,\ w^2 = (120)^2$$

$$G_{66} = \frac{1}{\sqrt{2}}(P_{(51)66} - P_{(15)66}) = 16(21/\pi^3)^{1/2}(x_+^4 - y_+^4)x_+\,y_+;\ u^2 = 16,\ w^2 = (120)^2$$

$$F'_{66} = \frac{1}{4\sqrt{2}}(\sqrt{15}\,P_{(60)66} - \sqrt{15}\,P_{(06)66} + P_{(42)66} - P_{(24)66}) = 2(210/\pi^3)^{1/2}(x_+^4 - y_+^4)(x_+^2 - y_+^2);\ u^2 = 4,\ w^2 = (60)^2$$

$$G'_{66} = -\frac{1}{4}(\sqrt{5}\,P_{(51)66} + \sqrt{5}\,P_{(15)66} + \sqrt{6}\,P_{(33)66}) = -4(210/\pi^3)^{1/2}(x_+^2 + y_+^2)^2\,x_+\,y_+;\ u^2 = 4,\ w^2 = (60)^2$$

$$S_{66} = \frac{1}{4}(\sqrt{5}\,P_{(60)66} + \sqrt{5}\,P_{(06)66} + \sqrt{3}\,P_{(42)66} + \sqrt{3}\,P_{(24)66}) = 4(35/\pi^3)^{1/2}(x_+^2 + y_+^2)^3;\ u^2 = w^2 = 0$$

$$F_{55} = \frac{1}{\sqrt{2}}(P_{(42)55} - P_{(24)55}) = 16(70/\pi^3)^{1/2}(x_+^2 - y_+^2)\,x_+\,y_+\,(x_+\,y_0 - x_0\,y_+);\ u^2 = 16,\ w^2 = (120)^2$$

$$G_{55} = -\frac{1}{2\sqrt{2}}(P_{(51)55} + P_{(15)55} - \sqrt{6}\,P_{(33)55}) = -4(70/\pi^3)^{1/2}(x_+^4 - 6x_+^2 y_+^2 + y_+^4)(x_+ y_0 - x_0 y_+);\ u^2 = 16,\ w^2 = (120)^2$$

$$F'_{55} = \frac{1}{\sqrt{2}}(P_{(42)55} + P_{(24)55}) = 16(70/\pi^3)^{1/2}(x_+^2 + y_+^2)x_+ y_+(x_+ y_0 - x_0 y_+);\ u^2 = 4,\ w^2 = (60)^2$$

$$G'_{55} = \frac{1}{\sqrt{2}}(P_{(51)55} - P_{(15)55}) = 8(70/\pi^3)^{1/2}(x_+^4 - y_+^4)(x_+ y_0 - x_0 y_+);\ u^2 = 4,\ w^2 = (60)^2$$

$$A_{55} = \frac{i}{2\sqrt{2}}(\sqrt{3}\,P_{(51)55} + \sqrt{3}\,P_{(15)55} + \sqrt{2}\,P_{(33)55}) = 4i(210/\pi^3)^{1/2}(x_+^2 + y_+^2)^2(x_+ y_0 - x_0 y_+);\ u^2 = w^2 = 0$$

$$S_{44} = \frac{1}{4\sqrt{21}}[2\sqrt{5}(P_{(42)44} - P_{(24)44}) - 7(P_{(40)44} + P_{(04)44}) + 3\sqrt{22}\,P_{(22)44}] = 2(210/11\pi^3)^{1/2}[(\mathbf{x}^2 - \mathbf{y}^2)(x_+^4 - 6x_+^2 y_+^2 + y_+^4) - 8(\mathbf{x}\cdot\mathbf{y})(x_+^2 - y_+^2)x_+ y_+];\quad u^2 = 36,\ w^2 = (488)^2$$

$$A_{44} = \frac{i}{12\sqrt{5}}[5(P_{(51)44} + P_{(15)44}) + 2\sqrt{77}(P_{(31)44} - P_{(13)44}) - 3\sqrt{6}\,P_{(33)44}] = 4i(210/11\pi^3)^{1/2}[-(\mathbf{x}\cdot\mathbf{y})(x_+^4 - 6x_+^2 y_+^2 + y_+^4) - 2(\mathbf{x}^2 - \mathbf{y}^2)(x_+^2 - y_+^2)x_+ y_+];\quad u^2 = 36,\ w^2 = (488)^2$$

$$F_{44} = \frac{1}{2\sqrt{3}}(P_{(42)44} + P_{(24)44} - \sqrt{5}\,P_{(40)44} - \sqrt{5}\,P_{(04)44}) = 4(6/11\pi^3)^{1/2}[\mathbf{x}^2(5x_+^4 + 12x_+^2 y_+^2 - 9y_+^4) + \mathbf{y}^2(-9x_+^4 + 12x_+^2 y_+^2 + 5y_+^4) - 28(\mathbf{x}\cdot\mathbf{y})(x_+^2 + y_+^2)x_+ y_+];\quad u^2 = 16,\ w^2 = (340)^2$$

$$G_{44} = -\frac{1}{6}[\sqrt{7}(P_{(51)44} - P_{(15)44}) + \sqrt{11}(P_{(31)44} - P_{(13)44})] = 8(6/11\pi^3)^{1/2}[7(\mathbf{x}\cdot\mathbf{y})(x_+^4 - y_+^4) + (3\mathbf{x}^2 - 11\mathbf{y}^2)x_+^3 y_+ + (11\mathbf{x}^2 - 3\mathbf{y}^2)x_+ y_+^3];\quad u^2 = 16,\ w^2 = (340)^2$$

$$F'_{44} = -\frac{1}{2\sqrt{74}}(3\sqrt{11}\,P_{(40)44} + 3\sqrt{11}\,P_{(04)44} + 7\sqrt{2}\,P_{(22)44}) = 12(5/37\pi^3)^{1/2}[(3\mathbf{x}^2 - 11\mathbf{y}^2)x_+^4 + (11\mathbf{x}^2 - 3\mathbf{y}^2)y_+^4 - 14(\mathbf{x}^2 - \mathbf{y}^2)x_+^2 y_+^2];\ u^2 = 4$$

$$G'_{44} = \frac{1}{2\sqrt{1110}}[5\sqrt{77}(P_{(51)44} + P_{(15)44}) + \sqrt{462}\,P_{(33)44} - 8(P_{(31)44} - P_{(13)44})] = 24(5/37\pi^3)^{1/2}[4(\mathbf{x}^2 + \mathbf{y}^2)x_+ y_+ - 7(\mathbf{x}\cdot\mathbf{y})(x_+^2 + y_+^2)](x_+^2 + y_+^2);\quad u^2 = 4$$

$$S_{44} = -\frac{1}{2\sqrt{3}}[\sqrt{5}(P_{(42)44} + P_{(24)44}) + (P_{(40)44} - P_{(04)44})] = 4(30/11\pi^3)^{1/2}[(\mathbf{x}^2 - 13\mathbf{y}^2)x_+^4 + (\mathbf{y}^2 - 13\mathbf{x}^2)y_+^4 - 12(\mathbf{x}^2 + \mathbf{y}^2)x_+^2 y_+^2 + 28(\mathbf{x}\cdot\mathbf{y})(x_+^2 + y_+^2)x_+ y_+];\ u^2 = 0,\ w^2 = 6160$$

$$A_{44} = \frac{i}{6}[\sqrt{11}(P_{(51)44} - P_{(15)44}) - \sqrt{7}(P_{(31)44} + P_{(13)44})] = 8i(42/\pi^3)^{1/2}[(\mathbf{x}^2 - \mathbf{y}^2)(x_+^2 + y_+^2)x_+ y_+ - (\mathbf{x}\cdot\mathbf{y})(x_+^4 - y_+^4)];\ u^2 = 0,\ w^2 = 6160$$

$$F''_{44} = \frac{1}{4\sqrt{39}}[-2\sqrt{55}(P_{(42)44} - P_{(24)44}) + \sqrt{11}(P_{(40)44} + P_{(04)44}) + 9\sqrt{2}\,P_{(22)44}] = 2(30/13\pi^3)^{1/2}[56(\mathbf{x}\cdot\mathbf{y})(x_+^2 - y_+^2)x_+ y_+ - (\mathbf{x}^2 - 15\mathbf{y}^2)x_+^4 - 42(\mathbf{x}^2 - \mathbf{y}^2)^2 x_+^2 y_+^2 + (15\mathbf{x}^2 + \mathbf{y}^2)y_+^4];\ u^2 = 4$$

$$G''_{44} = \frac{1}{12\sqrt{65}}\left[-5\sqrt{77}\left(P_{(51)44} + P_{(15)44}\right) + 3\sqrt{462}\,P_{(33)44} + 26\left(P_{(31)44} - P_{(13)44}\right)\right] = 120(6/2171\pi^3)^{1/2}\left[7(\mathbf{x}\cdot\mathbf{y})(x_+^4 - 6x_+^2 y_+^2 + y_+^4) - \right.$$
$$\left. - (6\mathbf{x}^2 - 22\mathbf{y}^2)x_+^3 y_+ + (22\mathbf{x}^2 - 6\mathbf{y}^2)x_+ y_+^3\right];\; u^2 = 4$$

$$F_{33} = -\frac{1}{2\sqrt{7}}\left[\sqrt{5}\left(P_{(42)33} - P_{(24)33}\right) + 3\sqrt{2}\,P_{(22)33}\right] = 8(70/\pi^3)^{1/2}\left[(\mathbf{x}^2 - \mathbf{y}^2)x_+ y_+ + (\mathbf{x}\cdot\mathbf{y})(x_+^2 - y_+^2)\right](x_+ y_0 - x_0 y_+);\; u^2 = 16,\; w^2 = (372)^2$$

$$G_{33} = \frac{1}{2\sqrt{5}}\left[-\sqrt{6}\,P_{(33)33} + \sqrt{7}\left(P_{(31)33} - P_{(13)33}\right)\right] = 4(70/\pi^3)^{1/2}\left[4(\mathbf{x}\cdot\mathbf{y})x_+ y_+ - (\mathbf{x}^2 - \mathbf{y}^2)(x_+^2 - y_+^2)\right](x_+ y_0 - x_0 y_+);\; u^2 = 16,\; w^2 = (372)^2$$

$$F'_{33} = -\frac{1}{\sqrt{2}}\left(P_{(42)33} + P_{(24)33}\right) = 16(1/\pi^3)^{1/2}\left[7(\mathbf{x}\cdot\mathbf{y})(x_+^2 + y_+^2) - 2(\mathbf{x}^2 + \mathbf{y}^2)x_+ y_+\right](x_+ y_0 - x_0 y_+);\; u^2 = 4,\; w^2 = (240)^2$$

$$G'_{33} = -\frac{1}{\sqrt{2}}\left(P_{(31)33} + P_{(13)33}\right) = 8(1/\pi^3)^{1/2}\left[(5\mathbf{x}^2 - 9\mathbf{y}^2)x_+^2 + (9\mathbf{x}^2 - 5\mathbf{y}^2)y_+^2\right](x_+ y_0 - x_0 y_+);\; u^2 = 4,\; w^2 = (240)^2$$

$$S_{33} = \frac{1}{2\sqrt{7}}\left[-3\left(P_{(42)33} - P_{(24)33}\right) + \sqrt{10}\,P_{(22)33}\right] = 24(14/\pi^3)^{1/2}\left[(\mathbf{x}\cdot\mathbf{y})(x_+^2 - y_+^2) - (\mathbf{x}^2 - \mathbf{y}^2)x_+ y_+\right](x_+ y_0 - x_0 y_+);\; u^2 = 0,\; w^2 = 15120$$

$$A_{33} = -\frac{i}{2\sqrt{5}}\left[\sqrt{3}\left(P_{(31)33} - P_{(13)33}\right) + \sqrt{14}\,P_{(33)33}\right] = 4i(10/3\pi^3)^{1/2}\left[(3\mathbf{x}^2 - 11\mathbf{y}^2)x_+^2 + (-11\mathbf{x}^2 + 3\mathbf{y}^2)y_+^2 + 28(\mathbf{x}\cdot\mathbf{y})x_+ y_+\right](x_+ y_0 - x_0 y_+);$$
$$u^2 = 0,\; w^2 = 15120$$

$$F_{22} = -\frac{1}{\sqrt{110}}\left[7\left(P_{(42)22} - P_{(24)22}\right) + \sqrt{6}\left(P_{(20)22} - P_{(02)22}\right)\right] = 4(10/231\pi^3)^{1/2}\left\{\left[-3\mathbf{x}^4 - 7\mathbf{y}^4 + 32\mathbf{x}^2\mathbf{y}^2 - 98(\mathbf{x}\cdot\mathbf{y})^2\right]x_+^2 + \left[7\mathbf{x}^4 + 3\mathbf{y}^4 - 32\mathbf{x}^2\mathbf{y}^2 + \right.\right.$$
$$\left.\left. + 98(\mathbf{x}\cdot\mathbf{y})^2\right]y_+^2 + 56(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2 - \mathbf{y}^2)x_+ y_+\right\};\; u^2 = 4$$

$$G_{22} = \frac{1}{5\sqrt{11}}\left[\sqrt{21}\,P_{(33)22} - 8\left(P_{(31)22} - P_{(13)22}\right) + 3\sqrt{14}\,P_{(11)22}\right] = 8(10/231\pi^3)^{1/2}\left\{4(\mathbf{x}\cdot\mathbf{y})\left[(-4\mathbf{x}^2 + 3\mathbf{y}^2)x_+^2 + (3\mathbf{x}^2 - 4\mathbf{y}^2)y_+^2\right] + \left[19(\mathbf{x}^4 + \mathbf{y}^4) - \right.\right.$$
$$\left.\left. - 60\mathbf{x}^2\mathbf{y}^2 + 42(\mathbf{x}\cdot\mathbf{y})^2\right]x_+ y_+\right\};\; u^2 = 4$$

$$S_{22} = -\frac{1}{\sqrt{10}}\left[\sqrt{2}\left(P_{(20)22} + P_{(02)22}\right) + \sqrt{3}\left(P_{(42)22} + P_{(24)22}\right)\right] = 4(10/7\pi^3)^{1/2}\left\{8(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2 + \mathbf{y}^2)x_+ y_+ - \left[\mathbf{x}^4 - 8\mathbf{x}^2\mathbf{y}^2 + 5\mathbf{y}^4 + 14(\mathbf{x}\cdot\mathbf{y})^2\right]x_+^2 - \right.$$
$$\left. - \left[5\mathbf{x}^4 - 8\mathbf{x}^2\mathbf{y}^2 + \mathbf{y}^4 + 14(\mathbf{x}\cdot\mathbf{y})^2\right]y_+^2\right\};\; u^2 = w^2 = 0$$

$$S'_{22} = \frac{1}{2\sqrt{35}}\left[-\sqrt{6}\left(P_{(42)22} - P_{(24)22}\right) - \sqrt{30}\,P_{(22)22} + 7\left(P_{(20)22} - P_{(02)22}\right)\right] = 2(10/\pi^3)^{1/2}\left\{\left[(\mathbf{x}^2 - \mathbf{y}^2)^2 - 4(\mathbf{x}\cdot\mathbf{y})^2\right](x_+^2 - y_+^2) - 8(\mathbf{x}^2 - \mathbf{y}^2)(\mathbf{x}\cdot\mathbf{y})x_+ y_+\right\};$$
$$u^2 = 36,\; w^2 = (684)^2$$

$$A_{22} = -\frac{i}{10}\left[-2P_{(33)22} + \sqrt{21}\left(P_{(31)22} - P_{(13)22}\right) + 3\sqrt{6}\,P_{(11)22}\right] = -4i(10/\pi^3)^{1/2}\left\{\left[(\mathbf{x}^2 - \mathbf{y}^2)^2 - 4(\mathbf{x}\cdot\mathbf{y})^2\right]x_+ y_+ + 2(\mathbf{x}^2 - \mathbf{y}^2)(\mathbf{x}\cdot\mathbf{y})(x_+^2 - y_+^2)\right\};$$
$$u^2 = 36,\; w^2 = (684)^2$$

$$F'_{22} = \frac{1}{\sqrt{10}}[-\sqrt{2}(P_{(42)22} + P_{(24)22}) + \sqrt{3}(P_{(20)22} + P_{(02)22})] = 4(5/21\pi^3)^{1/2}\{16(\mathbf{x}^2 + \mathbf{y}^2)(\mathbf{x}\cdot\mathbf{y})x_+ y_+ + [3\mathbf{x}^4 - 14\mathbf{x}^2\mathbf{y}^2 + 11\mathbf{y}^4 - 28(\mathbf{x}\cdot\mathbf{y})^2]x_+^2 +$$
$$+ [11\mathbf{x}^4 - 14\mathbf{x}^2\mathbf{y}^2 + 3\mathbf{y}^4 - 28(\mathbf{x}\cdot\mathbf{y})^2]y_+^2\};\ u^2 = 16,\ w^2 = (480)^2$$

$$G'_{22} = \frac{1}{\sqrt{2}}(P_{(31)22} + P_{(13)22}) = 16(5/21\pi^3)^{1/2}\{(\mathbf{x}\cdot\mathbf{y})[(5\mathbf{x}^2 - 9\mathbf{y}^2)x_+^2 + (9\mathbf{x}^2 - 5\mathbf{y}^2)y_+^2] - 2(\mathbf{x}^4 - \mathbf{y}^4)x_+ y_+\};\ u^2 = 16,\ w^2 = (480)^2$$

$$F''_{22} = \frac{1}{2\sqrt{145}}[-9\sqrt{2}(P_{(42)22} - P_{(24)22}) + 5\sqrt{10}P_{(22)22} + \sqrt{3}(P_{(20)22} - P_{(02)22})] = 2(30/203\pi^3)^{1/2}\{168(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2 - \mathbf{y}^2)x_+ y_+ + [\mathbf{x}^4 - 34\mathbf{x}^2\mathbf{y}^2 +$$
$$+ 49\mathbf{y}^4 - 84(\mathbf{x}\cdot\mathbf{y})^2]x_+^2 + [-49\mathbf{x}^4 + 34\mathbf{x}^2\mathbf{y}^2 - \mathbf{y}^4 + 84(\mathbf{x}\cdot\mathbf{y})^2]y_+^2\};\ u^2 = 4$$

$$G''_{22} = \frac{1}{10\sqrt{29}}[-6\sqrt{21}P_{(33)22} - 27(P_{(31)22} - P_{(13)22}) + 7\sqrt{14}P_{(11)22}] = 4(30/203\pi^3)^{1/2}\{(\mathbf{x}\cdot\mathbf{y})[(-18\mathbf{x}^2 + 66\mathbf{y}^2)x_+^2 + (66\mathbf{x}^2 - 18\mathbf{y}^2)y_+^2] +$$
$$+ [17(\mathbf{x}^4 + \mathbf{y}^4) - 50\mathbf{x}^2\mathbf{y}^2 - 84(\mathbf{x}\cdot\mathbf{y})^2]x_+ y_+\};\ u^2 = 4$$

$$A_{11} = -\frac{i}{\sqrt{5}}(\sqrt{3}P_{(33)11} + \sqrt{2}P_{(11)11}) = -8i(3/\pi^3)^{1/2}[\mathbf{x}^4 - 4\mathbf{x}^2\mathbf{y}^2 + \mathbf{y}^4 + 6(\mathbf{x}\cdot\mathbf{y})^2](x_+ y_0 - x_0 y_+);\ u^2 = w^2 = 0$$

$$F_{11} = P_{(22)11} = 48(2/\pi^3)^{1/2}(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}^2 - \mathbf{y}^2)(x_+ y_0 - x_0 y_+);\ u^2 = 16,\ w^2 = (512)^2$$

$$G_{11} = \frac{1}{\sqrt{5}}(\sqrt{2}P_{(33)11} - \sqrt{3}P_{(11)11}) = -12(2/\pi^3)^{1/2}[(\mathbf{x}^2 - \mathbf{y}^2)^2 - 4(\mathbf{x}\cdot\mathbf{y})^2](x_+ y_0 - x_0 y_+);\ u^2 = 16,\ w^2 = (512)^2$$

$$F_{00} = \frac{1}{\sqrt{2}}(P_{(22)00} + P_{(00)00}) = -2(2/\pi^3)^{1/2}[\mathbf{x}^4 - 10\mathbf{x}^2\mathbf{y}^2 + \mathbf{y}^4 + 12(\mathbf{x}\cdot\mathbf{y})^2](\mathbf{x}^2 - \mathbf{y}^2);\ u^2 = 4,\ w^2 = (256)^2$$

$$G_{00} = -\frac{1}{\sqrt{10}}(3P_{(33)00} + P_{(11)00}) = 4(2/\pi^3)^{1/2}[\mathbf{x}^4 - 10\mathbf{x}^2\mathbf{y}^2 + \mathbf{y}^4 + 12(\mathbf{x}\cdot\mathbf{y})^2](\mathbf{x}\cdot\mathbf{y});\ u^2 = 4,\ w^2 = (256)^2$$

$$S_{00} = \frac{1}{\sqrt{2}}(P_{(22)00} - P_{(00)00}) = 2(2/\pi^3)^{1/2}[(\mathbf{x}^2 - \mathbf{y}^2)^2 - 12(\mathbf{x}\cdot\mathbf{y})^2](\mathbf{x}^2 - \mathbf{y}^2);\ u^2 = 36,\ w^2 = (768)^2$$

$$A_{00} = \frac{i}{\sqrt{10}}(-P_{(33)00} + 3P_{(11)00}) = 4i(2/\pi^3)^{1/2}[4(\mathbf{x}\cdot\mathbf{y})^2 - 3(\mathbf{x}^2 - \mathbf{y}^2)^2](\mathbf{x}\cdot\mathbf{y});\ u^2 = 36,\ w^2 = (768)^2$$

References and Notes

1. E. Chacón and M. Moshinsky, Rev. Mex. Fis. 14, 119 (1965).
2. A. J. Dragt, J. Math. Phys. **6**, 533 (1965).
3. V. V. Pustovalov and Yu. A. Simonov, Zurn. Eksp. Teor. Fiz 51, 345 (1966). (Sov. Phys. JETP, 24, 230 (1967)).
4. The Hamiltonian (2-1) is also invariant under reflection of any number of coordinate axes. It is then invariant under ***O*,,** the orthogonal group in six dimensions. We come back to this point in Section 6.
5. Homogeneous polynomials of degree λ are components of symmetric tensors of rank λ with respect to U_6 or, more generally, GL,. On the other hand, homogeneous and harmonic polynomials of degree λ are components of (irreducible) symmetric tensors wrt ***R*,,** i.e., tensors which are also traceless. Such tensors are associated to one-row Young diagrams, λ giving the length of the row.
6. The hyperspherical coordinates~0, , φ_1, θ_2, φ_2 in E, suitable for our problem aredefined by

$$\begin{array}{l|l} x_1 = r\cos\chi\sin\theta_1\cos\varphi_1 & y_1 = r\sin\chi\sin\theta_2\cos\varphi_2 \\ x_2 = r\cos\chi\sin\theta_1\sin\varphi_1 & y_2 = r\sin\chi\sin\theta_2\sin\varphi_2 \\ x_3 = r\cos\chi\cos 0, & y_3 = r\sin\chi\cos\theta_2, \end{array}$$

$$r = (\mathbf{x}^2 + \mathbf{y}^2)^{1/2}, 0 \le \theta_i \le \pi,\ 0 \le \varphi_i \le 2\pi,\ 0 \le \chi \le \pi/2.$$

In terms of these coordinates, the solid angle element reads $d\Omega_6 = \sin^2\chi\cos^2\chi\sin\theta_1 \sin\theta_2\, d\chi\, d\theta_1\, d\theta_2\, d\varphi_1\, d\varphi_2 = \sin^2\chi\cos^2\chi\, d\Omega_3(\hat{\mathbf{x}})\, d\Omega_3(\hat{\mathbf{y}})$.

7. By "accidental degeneracy" one means the existence of an extra degeneracy, i.e., a multiplet of E which cannot be accounted by the symmetry group one is using. In other words, it means the existence of a larger symmetry group having the symmetry group one is dealing with as a subgroup. Furthermore, the irreducible representation of the larger symmetry group, carried by the eigenfunctions of a multiplet corresponding to a given energy E, splits into multiplets of the symmetry subgroup, i.e., such a representation is reducible with respect to the subgroup. A well known example is provided by the potential $V(r) \sim \frac{1}{r}$: when r is the six-dimensional distance, the largest symmetry group is ***R*, ⊃ *R*,** and several representations ***[i,]***of ***R*,** are present in the same R, multiplet of given energy E.
8. A. Erdelyi et al.: Bateman Manuscript Project (Addison – Wesley, USA, 1962). E. D. Rainville: Special Functions (Macmillan, USA, 1960).
9. The linearly independent homogeneous polynomials of degree λ in six variables constitute a basis for the most degenerate ***IR*** $[\lambda]$ of U_6, the unitary group in six dimensions. Corresponding to the realization (2-3) for the ***R*,** generators, we have for the thirty six generators of U_6 the realization

$$\mathscr{C}_{ij}^{\alpha\beta} = x_i^\alpha \frac{\partial}{\partial x_j^\beta},$$

$i, j = 1, 2, 3;\ \alpha, \beta = 1, 2$. The first-order Casimir invariant is simply

$$\sum_{i\,\alpha} \mathscr{C}_{ii}^{\alpha\alpha} = \mathbf{r} \cdot \nabla,$$

and the higher order ones are functions of (r · V), not providing therefore new labels to distinguish different *IR*'s. For homogeneous polynomials of a given degree λ, the above Casimir

operator has the value λ and this shows that the representation they carry is irreducible, being characterized by a single label ▮ _Weyl's dimension formula for U_n gives then

$$\dim [A] = \binom{\lambda + 5}{\lambda}$$

for $n = 6$.

10. By **[x]** we mean the greatest integer smaller than x. We believe that this notation will not be confused with the one used in the text to denote *IR*'s.

11. The argument of course is relevant only for $\lambda \leq 2$, since homogeneous polynomials of degrees zero and one are necessarily harmonic.

12. D. M. Brink and G. R. Satchler: Angular Momentum (2^{nd} ed., Clarendon Press, Oxford). We followed their definition of reduced matrix elements.

13. The mixed representation is a two-dimensional IR carried by a pair of functions which are neither completely symmetric nor antisymmetric under S_3 : they have what is called a niixed symmetry. See, e.g., M. Hamermesh, Group Theory (Addison-Wesley, 1962, USA). Using definitions (2-2) for x and y, it is simple to show that

$$(1,2)\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \quad (1,3)\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix},$$

where (i, j) is the transposition ($\mathbf{r}_i \leftrightarrow \mathbf{r}_j$). Since (1,2) and (1,3) are generators of S_3 (i.e., their products give rise to all elements of S_3), the above relations completely define the mixed representation we chose to adopt, and they show that the Jacobi vectors x and y transform like the "up" and "down" components of the mixed representation. More generally, in the text we denote by F and G a pair of functions which carry the mixed representation of S_3, with the 2 x 2 niatrices given above.

14. To show that, one determines the multiplicity of each Lvalue in the set (6-13) and the rcsult is that it coincides with the value given by (4-10).

15. By scalar we mean, in this paper, a scalar with respect to R_3 (L).

16. This operator is related to the Bargmann-Moshinsky operator Ω (cf. Ref. 1) by the relation

$$W = 4\Omega - \tfrac{2}{3}(6\mathscr{I}_2 - \mathbf{L}^2 + 12)U.$$

17. The linearly independent homogeneous polynomials, of given λ and L (A, in number), can be distributed into $\mathscr{M}_L^\lambda(S)$ symmetric, $\mathscr{M}_L^\lambda(A)$ antisymmetric and $\mathscr{M}_L^\lambda(M)$ mixed representations of S_3, where of course $\mathscr{M}_L^\lambda = \mathscr{M}_L^\lambda(S) + \mathscr{M}_L^\lambda(A) + 2\,\mathscr{M}_L^\lambda(M)$. The numbers $\mathscr{M}_L^\lambda$ are given in the paper of G. Karl and E. Obryk, Nucl. Phys. **B8**, 609 (1968). From the reduction (4 – 13), one sees at once that the corresponding numbers for ***harmonic*** polynomials are given by

$$\mathscr{M}_L^{[\lambda]}(S) = \mathscr{M}_L^\lambda(S) - \mathscr{M}_L^{\lambda-2}(S),$$

the same holding for the other representations of S_3.

18. The basic states of a general representation of O_6 can be labelled by a set of integers or half-integers

$$n_1,\ n_2,\ n_3\ ;\ m_{41},\ m_{42}\ ;\ m_{31},\ m_{32}\ ;\ m_{21},\ m_{11},$$

where n_1, n, and n, characterize the irreducible representation (IR) and the m's distinguish states within a given IR. These numbers are related by the following branching laws:

$$n_1 \geq m_{41} \geq n_2 \geq m_{42} \geq |n_3|,$$
$$m_{41} \geq m_{31} \geq m_{42} \geq m_{32} \geq -m_{32},$$
$$m_{31} \geq m_{21} \geq m_{32},$$
$$m_{21} \geq m_{11} \geq -m_{11}.$$

For the IR of O_6 carried by the harmonic polynowials (6-13), one has $n_1 = \lambda$ and $n_2 = n_3 = O$. The above branching laws then require that $m_{42} = m_{32} = O$ and we see that only four tabels are needed to specify completely a basic state of the most degenerate IR $[\lambda]$ of O_6, making five labels altogether. The above labels correspond to the mathematical chain $O_6 \supset O_5 \supset O_4 \supset O, \supset O,$ which has not been used in this paper. (cf. Supplement 1 of I. M. Gelfand, R. **A.** Minlos and Z. Ya. Shapiro: Representations of the Rotation and Lorentz groups and their Applications, Macmillan, USA, 1963).