Long Range Interactions and the Geometry of the Four-Dimensional Space

COLBER G. OLIVEIRA

Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro GB

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In this work we propose to study the problem of the measurement of long range interactions which may exist due to the geometrical properties of spaces with curvature. All such fields can possibly be detected by letting a test particle travel along the geodesic of the space. Usually this gives the geometrical structure of the gravitational field, translated by the presence of the Christoffel symbols, in such a way that the principle of equivalence is satisfied. Actually, this is interpreted as part of more general situations, by supposing that the test particle has a charge and is also able to interact with other fields, as for instance with a scalar field. These extra interactions are then studied from the geometrical point of view, similar to the gravitational interaction with the mass of the test particle. However, the gravitational interaction is peculiar among these several fields since it satisfies the principle of equivalence as a direct consequence of the covariance of the equations of motion under the manifold mapping group.

Propomo-nos a estudar o problema da medida de interações de longo alcance que possam existir devido as propriedades geométricas de espaços curvos. Todos os campos dêsse tipo poderiam ser detetados pelo movimento de uma partícula de prova ao longo de uma geodésica do espaço. Usualmente isso propicia o conhecimento da estrutura geométrica do campo gravitacional (caracterizada pelos símbolos de Christoffel), o princípio de equivalência sendo satisfeito. Tal situação pode ser considerada como caso especial de situações mais gerais, supondo-se que a partícula de prova tenha carga e possa interagir com outros campos, como por exemplo um campo escalar. Essas interações extras são então estudadas de um ponto de vista geométrico, análogamente ao que se faz com a interação gravitacional. A interação gravitacional, todavia, tem um caráter especial entre todas as outras interações, visto que satisfaz o princípio de equivalência, consequência direta da covariância das equações de movimento sob o grupo de transformações gerais de coordenadas.

I. The Affine Geodesic in Four-Space

The equation of the affine geodesic is obtained by the condition that the tangent vector to the curve is always displaced parallel to itself. This condition, which is the natural generalization of the concept of a straight line in Euclidian space, is mathematically expressed by the covariant set of equations,

$$u^{\nu} u^{\alpha}_{;\nu} = 0, \tag{1}$$

where $u^{\alpha} = dx^{\alpha}/ds$ is the tangent to the curve, and all indices run from 1 to 4. Without modifying the covariant structure of (1), we may multiply the equation by the inertial mass m_i of the test particle, that is, the particle which follows the geodesic,

$$m_i u^\nu u^\alpha_{;\nu} = 0. \tag{2}$$

The equation (1) in explicit form is

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx''}{ds} = 0, \tag{3}$$

where the $\Gamma^\alpha_{\rho\sigma}$ are the components of the affinity. Since both $d^2 x^\alpha/ds^2$ and $\Gamma^\alpha_{\rho\sigma} dx^\rho/ds \cdot dx^\sigma/ds$ are not vectors, the Eq. (3) is covariant only due to the fact that the non-tensor terms arising from both terms cancel each other. This means that at most we are free to multiply the left hand side by a constant. In Eq. (2) we have taken this constant to be the inertial mass of the particle.

This peculiar property of the four-acceleration in curved spaces, which is related in part to the possibility of relative accelerated motions, implies the statement of the principle of equivalence. Indeed, if we interpret $\Gamma^\alpha_{\rho\sigma} u^\rho u^\sigma$ as the gravitational interaction on the test particle, the gravitational mass which should act as the coupling parameter in front of $\Gamma^\alpha_{\rho\sigma} u^\rho u^\sigma$ is equal to the inertial mass, upon imposition of covariance of the equation of motion under the transformations of the manifold mapping group.

We use now a theorem of tensor calculus which says that any affinity is determined up to an additive tensor with the same index structure. Exploiting this result, we introduce a new affinity $\tilde{\Gamma}^\alpha_{\rho\sigma}$ by the following relation,

$$\tilde{\Gamma}^\alpha_{\rho\sigma} = \Gamma^\alpha_{\rho\sigma} + \Delta^\alpha_{\rho\sigma}. \tag{4}$$

The **A;** is a tensor field to be determined. Its explicit form is free to be chosen. Before working out the possible relevant choices for this field, it is interesting to analyse the covariance of (3) for this new choice of the affinity. We have now,

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} + \Delta^\alpha_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0. \tag{5}$$

Calling the sum of the first two terms V^α, we know that this quantity is a vector. The third term on the left hand side of (5) is also a vector. Let us denote it by J". In this form, we separate the three factors on the left hand side of (5) into two terms which transform as vectors, and which are in principle not connected to each other. We have then

$$V^{\alpha} + J^{\alpha} = 0, \tag{6}$$

with

$$V^{\alpha} = d^2 x^{\alpha}/ds^2 + \Gamma^{\alpha}_{\rho\sigma} u^{\rho} u^{\sigma}, \tag{7}$$

$$J^{\alpha} = \Delta^{\alpha}_{\rho\sigma} u^{\rho} u^{\sigma}. \tag{8}$$

Since J^{α} transforms by itself covariantly independent of V^{α}, we are free to introduce the following sum as a new possible expression for the J^{α},

$$J^{\alpha} = \sum_j b_j J^{\alpha(j)} = u^{\rho} u^{\sigma} \sum_j b_j \Delta^{\alpha(j)}_{\rho\sigma} \tag{9}$$

where the b_j are scalars and each term $J^{\alpha(j)}$ is a vector. The affinity (4) is written as,

$$\tilde{\Gamma}^{\alpha}_{\rho\sigma} = \Gamma^{\alpha}_{\rho\sigma} + (1/m_i) \sum_j b_j \Delta^{\alpha(j)}_{\rho\sigma}$$

giving rise to the covariant equations of motion,

$$m_i (d^2 x^{\alpha}/ds^2 + \Gamma^{\alpha}_{\rho\sigma} u^{\rho} u^{\sigma}) + \sum_j b_j \Delta^{\alpha(j)}_{\rho\sigma} u^{\rho} u^{\sigma} = 0. \tag{10}$$

The fact that, necessarily, a part of the affinity, $\Gamma^{\alpha}_{\rho\sigma}$, is of the same geometrical nature as the four-acceleration, is, as we remarked previously, connected to the equivalence principle, which states that locally we cannot separate the effects of inertia from those of gravitation. We now set equal $\Gamma^{\alpha}_{\rho\sigma}$ to the Christoffel symbols as a definition. In this form we are taking into account the usual results of general relativity:

$$\Gamma^{\alpha}_{\rho\sigma} = \left\{ \begin{matrix} \alpha \\ \rho\sigma \end{matrix} \right\}. \tag{11}$$

As was mentioned in the introduction, the present work intends to interpret other fields besides the gravitational as geometrical properties of the space-time. Thus, we intend to go beyond the usual results of general relativity but nevertheless all results of this theory are taken as valid in the limit where those extra fields do not exist. That was the reason for taking the relation (11) as valid.

The covariant derivative of $g_{\mu\nu}$ is given by

$$g_{\mu\nu;\sigma} = g_{\mu\nu,\sigma} - \Gamma^{\lambda}_{\mu\sigma} g_{\lambda\nu} - \Gamma^{\lambda}_{\nu\sigma} g_{\mu\lambda} - \Delta^{\lambda}_{\mu\sigma} g_{\lambda\nu} - \Delta^{\lambda}_{\nu\sigma} g_{\mu\lambda},$$

which according to (11) takes the form

$$g_{\mu\nu;\sigma} = -\Delta^{\lambda}_{\mu\sigma} g_{\lambda\nu} - \Delta^{\lambda}_{\nu\sigma} g_{\mu\lambda}. \tag{12}$$

2. The Motion For Charged Test Particles

In this section we establish a first possible choice for the $\Delta^{\alpha(j)}_{\rho\sigma}$. This geometrical object will be determined by physical considerations. In each situation where we need the $\Delta^{\alpha(j)}_{\rho\sigma}$, we want to describe a deviation from the pure geodesic of the gravitational field. To clarity the situation, we have to say that the sources of the gravitatíonal field may be a certain mass distribution which eventually may have an overall electric charge. A test particle moving into this field is acted on by the gravitational force, represented by the Christoffel symbols. However, such a force will contain the contribution coming from the net charge of the source which contributes to the $g_{\mu\nu}$. If the test particle has no charge, this will be the net force acting on it. But if the test particle has a charge, it will interact not only with the gravitational field given by the $\left\{{\lambda \atop \mu\nu}\right\}$, but also with the electromagnetic field generated by the relative motion of the charges of the source. It is this last situation which is treated here as a deviation from the pure geodesic of the gravitational field.

This deviation from the geodesic generated by $\left\{{\lambda \atop \mu\nu}\right\}$ may in principle be due to various types of fields, as is suggested by the sum over fields which is present in Eq. (10). In this Section we consider just the contribution of one term in this sum, by writing b_1 as equal to the electric charge of the test particle and all remaining b_k equal to zero. Then, $b_1 \Delta^{\alpha(1)}_{\rho\sigma} u^\rho u^\sigma$ is equal to the Lorentz force in presence of gravitation,

$$\Delta^{\alpha(1)}_{\rho\sigma} u^\rho u^\sigma = -F_{\sigma\rho} g^{\sigma\alpha} u^\rho .$$

This relation may be solved for the **A;',",**

$$\Delta^{\alpha(1)}_{\rho\sigma} = -F_{\beta\rho} g^{\beta\alpha} g_{\sigma\lambda} u^\lambda + sF_{\rho\sigma} u^\alpha , \tag{13}$$

where s is a constant. In the solution (13) we wrote the terms which contain the presence of F,,, $g_{\mu\nu}$ and u^a, that is, the presence of the physical quantities under consideration. In other words, we neglect any other variable on which we have no information from the physics of the system[1].
The value for the constant s will be obtained by substitution of (13) into Eq. (12) giving

$$g_{\mu\nu;\sigma} = (F_{\mu\nu} + F_{\nu\mu}) u_\sigma - s(F_{\mu\sigma} u_\nu + F_{\nu\sigma} u_\mu).$$

Since we do not want the length of vectors to be varied under parallel displacement, as in the unitary field theory of Weyl[2] (since such a variation leads to unphysical consequences as was pointed out by Einstein), we set the constant s equal to zero. Then,

$$g_{\mu\nu;\sigma} = 0,$$

similar to the result of general relativity. Thus, for gravitational plus electromagnetic fields the affinity has the form, as measured by the test particle,

$$\tilde{\Gamma}^{\alpha}_{\rho\sigma} = \left\{ {\alpha \atop \rho\sigma} \right\} - (e/m_i) F_{\beta\rho} g^{\beta\alpha} g_{\sigma\lambda} u^{\lambda}. \tag{14}$$

The fact that the affinity varies due to the presence of the electromagnetic field, and that such variation is transmitted to the charged test particle is a violation of the principle of equivalence, But this is not a new result since it is known that charged test particles do not satisfy the law of free fall.

Therefore, the method treated here has up to now only the interest of being a general procedure for the geometrization of other fields besides the gravitational, but no new physical result was obtained.

3. The Motion for a Particle with Mesonic Charge

Following the method outlined previously, we give here a second possible choice for the affinity. We consider in this section that a long range scalar field is the agent giving rise to a correction in the expression for the affine connection.

The interest towards the consideration of long range scalar fields was recently raised by Dicke[3] in connection with a possible realization of Mach's program. As is also known, scalar interactions can also be treated relativistically in the framework of the five-dimensional unitary field theory of Thirry[4] and Kaluza[5]. The type of approach treated here does not require the introduction of extra dimensions into the manifold and, as we will see, leads naturally to the equations of motion and the field equations.

Similarly to the case for electromagnetic interactions, we consider that the next term in the sum over fields in Eq. (10) comes from the presence of a scalar field $\phi(x)$:

$$b_2 u^{\rho} u^{\sigma} \Delta^{\alpha(2)}_{\rho\sigma} = e_m \frac{\partial \phi}{\partial x^{\rho}} g^{\rho\alpha},$$

(see Note 6) which has as solution

$$\Delta^{\alpha(2)}_{\rho\sigma} = (1/2) \frac{\partial \phi}{\partial x^{\lambda}} g^{\lambda\alpha} u_{\rho} u_{\sigma} + (1/2) \frac{\partial \phi}{\partial x^{\lambda}} g^{\lambda\alpha} g_{\rho\sigma}. \tag{15}$$

Therefore, the affinity now takes the form

$$\tilde{\Gamma}^{\alpha}_{\rho\sigma} = \left\{ {\alpha \atop \rho\sigma} \right\} - (e/m_i) F_{\beta\rho} g^{\beta\alpha} g_{\sigma\lambda} u^{\lambda} + (e_m/2m_i) \left(\frac{\partial\phi}{\partial x^{\beta}} g^{\beta\alpha} g_{\sigma\lambda} g_{\rho\nu} u^{\nu} u^{\lambda} + \frac{\partial\phi}{\partial x^{\beta}} g^{\beta\alpha} g_{\rho\sigma} \right). \tag{16}$$

In this expression, the term coming from the electromagnetic field is not symmetrical, it has symmetric and skew symmetric parts. The term coming from the relativistic scalar field is symmetric.

In principle this process of summing over fields may be further extended in order to consider any other interaction which may be of interest. For our present purposes we stop in the second term of the sum.

4. The Curvature Tensor

In this Section we compute the Riemann tensor associated with the corrected affinity. By convenience we consider only the correction coming from the electromagnetic field. A look at Eq. (14) shows that, besides the field F,,, the four-velocity of the test particle also appears in the expression for the affínity. The reason for this comes from the fact that the test particle is the agent which measures the total field, and due to the structure of the Lorentz force the u^a appears in the affinity as long as we intend to interpret this interaction as geometry as we did here. As a consequence, the test particle will modify the curvature of the field, and this modification appears due to the interaction of the charged test particle with the electromagnetic field generated by the motion of the source particles. Since this type of interacttion is stronger than the interaction of the mass of the test particle with the gravitational field of the source, such a reaction of the test particle on the total field, translated geometrically, is natural. In other terms, we may say that the field $F_{\mu\nu}$ cannot be interpreted as geometry unless we have a charged test particle to measure this effect.

Nevertheless, we will see that such an interpretation leads to mathematical diffículties, and in order to remove these difficulties we have to introduce a new interpretation on the dynamics of the interaction.

First we calculate directly the components of the curvature tensor by taking the affinity given by (14). We have, for any given vector k^{μ},

$$k^{\mu}_{;\nu} = k^{\mu}_{,\nu} + \tilde{\Gamma}^{\mu}_{\nu\lambda} k^{\lambda}$$

The affinity given by (14) is decomposed into its symmetrical and skew-symmetrical parts, the latter representing the tensor of torsion:

$$\tilde{\Gamma}^{\mu}{}_{\nu\lambda} = S^{\mu}{}_{\nu\lambda} + T^{\mu}{}_{\nu\lambda}. \tag{17}$$

Accordingly, the tensor $k^{\mu}_{;\nu}$ decomposes into the sum of the two tensors,

$$k^{\mu}_{;\nu} = \overset{(1)}{k^{\mu}_{\nu}} + \overset{(2)}{k^{\mu}_{\nu}}$$

where

$$\overset{(1)}{k^{\mu}_{\nu}} = k^{\mu}_{,\nu} + S^{\mu}_{\nu\lambda}\, k^{\lambda} \tag{18a}$$

$$\overset{(2)}{k^{\mu}_{\nu}} = T^{\mu}_{\nu\lambda}\, k^{\lambda}. \tag{18b}$$

The values for the affinity $S^{\mu}{}_{\nu\lambda}$ and for the tensor $T^{\mu}{}_{\nu\lambda}$ are given by

$$S^{\mu}{}_{\nu\lambda} = \left\{{\mu \atop \nu\lambda}\right\} - (e/2m_i)\,(F^{p}{}_{\nu}\, u_{\lambda} + F^{\mu}{}_{\lambda}\, u_{\nu}), \tag{19a}$$

$$T^{\mu}{}_{\nu\lambda} = (e/2m_i)\,(F^{\mu}{}_{\lambda}\, u_{\nu} - F^{\mu}{}_{\nu}\, u_{\lambda}). \tag{19b}$$

In the case where k^{p} is equal to the fourvelocity of the test particle, we see that the tensor of torsion $T^{\mu}{}_{\nu\lambda}$ will not contribute to the equation of motion.[7] The Riemannian curvature is generated by the affinity $S^{\mu}{}_{\nu\lambda}$ according to the usual formula,'

$$\overset{(1)}{k^{\mu}_{\alpha;\rho}} - \overset{(1)}{k^{\mu}_{\rho;\alpha}} = \tilde{R}^{\mu}{}_{\lambda\alpha\rho}\, k^{\lambda}. \tag{20}$$

A similar expression may be formed for the tensor $\overset{(2)}{k^{\mu}_{\nu}}$:

$$\overset{(2)}{k^{\mu}_{\alpha;\rho}} - \overset{(2)}{k^{\mu}_{\rho;\alpha}} = B^{\mu}{}_{\lambda\rho\alpha}\, k^{\lambda} + L^{\nu\mu}{}_{\lambda\rho\alpha}\, k^{\lambda}_{;\nu}. \tag{21}$$

The expressions for $\tilde{R}^{\mu}{}_{\lambda\alpha\rho}$, $B^{\mu}{}_{\lambda\alpha\rho}$ and $L^{\nu\mu}{}_{\lambda\rho\alpha}$ are

$$\tilde{R}^{\mu}{}_{\lambda\alpha\rho} = S^{\mu}{}_{\lambda\alpha,\rho} - S^{\mu}{}_{\lambda\rho,\alpha} - S^{\mu}{}_{\tau\alpha}\, S^{\tau}{}_{\lambda\rho} + S^{\mu}{}_{\tau\rho}\, S^{\tau}{}_{\lambda\alpha}, \tag{22}$$

$$B^{\mu}{}_{\lambda\alpha\rho} = T^{\mu}{}_{\lambda\alpha;\rho} - T^{\mu}{}_{\lambda\rho;\alpha}, \tag{23}$$

$$L^{\nu\mu}{}_{\lambda\rho\alpha} = T^{\mu}{}_{\lambda\rho}\, \delta^{\nu}_{\alpha} - T^{\mu}{}_{\lambda\alpha}\, \delta^{\nu}_{\rho}. \tag{24}$$

After a straigtforward calculation, we find the following value for these tensors:

$$\tilde{R}^{\alpha}{}_{\rho\sigma\mu} = R^{\alpha}{}_{\rho\sigma\mu} + (e^2/4m_i^2)\{F^{\alpha}{}_{\lambda} F^{\lambda}{}_{\nu} (\delta^{\nu}_{\sigma} u_{\rho} u_{\mu} - \delta^{\nu}_{\mu} u_{\rho} u_{\sigma}) +$$
$$+ (F'', F^{\lambda}{}_{\rho} \delta^{\nu}_{\sigma} + F^{z}, F^{\lambda}{}_{\sigma} \delta^{\nu}_{\rho} - F^{\alpha}{}_{\sigma} F^{\lambda}{}_{\rho} \delta^{\nu}_{\mu} - F^{\alpha}{}_{\sigma} \qquad \delta^{\nu}_{\rho}) u_{\lambda} u_{\nu}\} + (e/2m_i)\{(F^{\alpha}{}_{\rho} u_{\mu} +$$
$$+ F^{\alpha}{}_{\mu\, l^*,)_{,,,}} - (F^{\alpha}{}_{\rho} u_{\sigma} + F^{\alpha}{}_{\sigma} u_{\rho})_{.\mu}\} + (e/2m_i)\{F^{\lambda}{}_{\rho}[\{{}^{\alpha}_{\lambda\sigma}\} u_{\mu} - \{{}^{\alpha}_{\lambda\mu}\} u_{\sigma}] +$$
$$+ F^{\alpha}{}_{\lambda}[\{{}^{\lambda}_{\rho\mu}\} u_{\sigma} - \{{}^{\lambda}_{\rho\sigma}\} u_{\mu}] + F^{\lambda}{}_{\tau}[\delta^{\tau}_{\mu}\{{}^{\alpha}_{\lambda\sigma}\} - \delta^{\tau}_{\sigma}\{{}^{\alpha}_{\lambda\mu}\}] u_{p} + F^{\alpha}{}_{\tau}[\delta^{\tau}_{\rho}\{{}^{\lambda}_{\rho\mu}\} - \delta^{\tau}_{\mu}\{{}^{\lambda}_{\rho\sigma}\}] \quad \} \tag{25}$$

$$B^{\mu}{}_{\lambda\alpha\rho} = T^{\mu}{}_{\lambda\alpha,\rho} - T^{\mu}{}_{\lambda\rho,\alpha} + S^{\mu}{}_{\rho\tau} T^{\tau}{}_{\lambda\alpha} - S^{\tau}{}_{\lambda\rho} T^{\mu}{}_{\tau\alpha} - S^{\tau}{}_{\rho\alpha} T^{\mu}{}_{\lambda\tau}$$
$$- S^{\mu}{}_{\alpha\tau} T^{\tau}{}_{\lambda\rho} + S^{\tau}{}_{\lambda\alpha} T^{\mu}{}_{\tau\rho} + S^{\tau}{}_{\alpha\rho} T^{\mu}{}_{\lambda\tau} + T^{\mu}{}_{\rho\tau} T^{\tau}{}_{\lambda\alpha} - T^{\tau}{}_{\lambda\rho} T^{\mu}{}_{\tau\alpha}$$
$$- T^{\tau}{}_{\rho\alpha} T^{\mu}{}_{\lambda\tau} - T^{\mu}{}_{\alpha\tau} T^{\tau}{}_{\lambda\rho} + T^{\tau}{}_{\lambda\alpha} T^{\mu}{}_{\tau\rho} + T^{\tau}{}_{\alpha\rho} T^{\mu}{}_{\lambda\tau}, \tag{26}$$

$$L^{\nu\mu}{}_{\lambda\rho\alpha} = (e/2m_i)\{\delta^{\nu}_{\alpha}[F^{\mu}{}_{\rho} u_{\lambda} - F^{\mu}{}_{\lambda} u_{\rho}] - \delta^{\nu}_{\rho}[F^{\mu}{}_{\alpha} u_{\lambda} - F^{\mu}{}_{\lambda} u_{\alpha}]\}. \tag{27}$$
$$F^{\lambda}{}_{\mu}$$

Of special importance is formula *(25)* giving the correction to the curvature tensor $R''_{,,,,}$ of the pure gravitational field. We next compute the Ricci tensor associated to $\tilde{R}^{\alpha}{}_{\rho\sigma\mu}$, and the corresponding scalar of curvature, $\tilde{R}$. We find,

$$\tilde{R}_{\rho\mu} = R_{,,} + (e^2/4m_i^2)\{F^{\alpha\lambda} F_{\lambda\alpha} u_{\rho\, ia,,} + F^{z}, F^{\lambda}{}_{\rho} u_{\lambda} u,) +$$
$$+ (e/2m_i)[(F'', u_{\mu} + F^{\alpha}{}_{\mu} u_{\rho})_{,\alpha} - (F'', u_{\alpha})_{,\mu}] + (e/2m_i)[(F^{\lambda}{}_{\rho} u_{\mu} + F^{\lambda}{}_{\mu} u_{\rho})\{{}^{\alpha}_{\lambda\alpha}\}$$
$$(F^{\alpha}{}_{\rho}\{{}^{\lambda}_{\alpha\mu}\} + F^{\alpha}{}_{\mu}\{{}^{\lambda}_{\rho\alpha}\}) u_{\lambda} + F^{\alpha}{}_{\lambda}\{{}^{\lambda}_{\rho\mu}\} u_{\alpha} - F^{\lambda}{}_{\alpha}(\{{}^{\alpha}_{\lambda\rho}\}/u_{\mu} + \{{}^{\alpha}_{\lambda u}\} u_{\rho}), \tag{28}$$

$$\tilde{R} = R + (e^2/4m_i^2) F^{\alpha\lambda} F_{\lambda\alpha} + (e/2m_i)[2F^{\lambda}{}_{\rho}\{{}^{\alpha}_{\lambda\alpha}\}\, \mathrm{ir}^{P} + F^{\alpha}{}_{\lambda} g^{\rho\mu}\{{}^{\lambda}_{\rho\mu}\} u_{\alpha} -$$
$$- 2F^{A}, \{{}^{\alpha}_{\lambda\rho}\} u^{\rho} + (e/2m_i) F^{\alpha}{}_{\mu} F^{\lambda\mu} u_{\lambda} u_{\alpha}] + (e/2m_i) g^{\mu\rho}[(F^{\alpha}{}_{\rho} u_{\mu} + F^{\alpha}{}_{\mu} u_{\rho})_{,\alpha} -$$
$$- (F^{\alpha}{}_{\rho\, i^*,),,}] \quad . \tag{29}$$

The present method is essentially a classical theory of measurement. By letting a test particle travel along the region under consideration we obtain the various interactions with the fields present. The structure of the geodesic gives the information on the existence of those fields. The gravitational field plays the fundamental role, since the remaining interactions are translated as part of the correction to the Christoffel symbols, and to the corresponding tensor of curvature, that means, they are taken as new fields in the framework of the geometry of the space.

At the same time, the motion of the test particle also gives us the structure of the interacting fields in this region. Indeed, from *(29)* a interaction term is present, and is proportional to the charge of the test particle. The four-velocity of the particle is also present.

The Lagrangian density for the whole system may be taken as the product of $\sqrt{-g}$ by $\tilde{R}$. It is given by the expression

$$L = \sqrt{-g}\, R + (e^2/4m_i^2)\sqrt{-g}\, F^{\alpha\lambda} F_{\lambda\alpha} + (e/2m_i)\sqrt{-g}\, I,$$

with the interaction term

$$I = 2F^{\lambda}{}_{\rho} \left\{ {\alpha \atop \lambda\alpha} \right\} ir^{P} + F^{\alpha}{}_{\lambda}\, g^{\rho\mu} \left\{ {\lambda \atop \rho\mu} \right\} u_{\alpha} - 2F^{i}, \left\{ {\alpha \atop \lambda\rho} \right\} u^{\rho} + (e/2m_i)\, F'', F^{\lambda\mu} u_{\lambda}\ \mathrm{ir}, -$$

$$- \frac{(\sqrt{-g}\, g^{\mu\rho})_{,\alpha}}{\sqrt{-g}} (F^{\alpha}{}_{\rho} u_{\mu} + F^{\alpha}{}_{\mu} u_{p}) + \frac{(\sqrt{-g}\, g^{\mu\rho})_{,\mu}}{\sqrt{-g}} F^{\alpha}{}_{\rho} u_{\alpha}, \tag{30}$$

where we have dropped a surface term, since $F_{\mu\nu}$ vanishes at infinity. The interaction Lagrangian density (30) contains the fourvelocity u^{λ} of the test particle. Thus, it may appear that there is no reason for taking $\tilde{R}\sqrt{-g}$ as the full Lagrangian, since we have to know the $\left\{ {\alpha \atop \beta\gamma} \right\}$ and the $F_{\alpha\beta}$ in the equation of motion for obtaining the u^{λ}, but u^{λ} is present in the field equations together with $\left\{ {\alpha \atop \beta\gamma} \right\}$ and $F_{\alpha\beta}$. What this means is that these equations have to be solved simultaneously. First we try to write the relation (30) in terms only of the fields. We integrate (10), for the case of the Lorentz force, along the geodesic of the particle,

$$u^{\alpha}(x(s)) = \int_{s_0}^{s} (F_{\beta\rho}\, g^{\beta\alpha} u^{\rho} - \left\{ {\alpha \atop \rho\sigma} \right\} u^{\rho} u^{\sigma})'\, ds' + \hat{u}^{\alpha}. \tag{31}$$

By substitution of this integral into (30), in principle we may get the interaction term as function only of the fields, now not only locally but also depending on the previous history of the particle. Equation (31) is an integral equation for the u^{α},. and may be solved by an iteration process (see Appendix). However, there are two difficulties. First the resulting field equations will be non linear integro-differential equations. In particular, the field equations for the $F_{\mu\nu}$ become non linear, since quadratic (and higher powers) terms in $F_{\mu\nu}$ appear as sources of the field. Consequently, this method is much too complicated for any practical application. Second, the integral written in Eq. (31) has no geometric meaning in curved spaces. The u^a written at the left hand side of (31) does not represent a geometrical object. As a consequence, the substitution of the solutions of (31) into (30) will generate field equations which are not covariant under the group of general coordinate transformations of general relativity. This last difficulty may be overcome by introducing a coordinate condition. In some sense we have substitututed the overall curvature of the space by a sum over histories, represented by the several terms of the Neumann-Liouville series giving the solution of the integral equation (31).

5. Conclusion

In this work we have treated the problem of introducing other long range interactions besides the gravitational as a part of the whole affinity of the space. As was shown, this may be done in a simple form. The further interpretation of this new affinity as the generator of the curvature tensor leads to certain difficulties. The resulting field equations become much too complicated, and the original geometrical idea has to be substituted by a different interpretation of the interaction. The four-dimensional space is not interpreted as possessing an overall curvature, but instead is flat and the interaction between the fields at a given point will depend on the history of the fields along a time-like "geodesic" arriving at this point at the instant of measurement. Nevertheless, the Riemann tensor coming from the gravitational field is still there, but it is not interpreted as a curvature. It can be, at most, interpreted as a local curvature since at the region where the field becomes infínity no coordinate condition can be consistently used. From the mathematical point of view we can say that the u^α of (31) is a pseudo-vector and therefore it possess an interpretation only by means of linear transformation groups.

The reason for obtaining a non-covariant formulation in the present treatment may also be seen from the following argument: We intend to measure the interactions by means of a test particle, but this particle presently is identical[g] to all the remaining particles forming the sources. Indeed, it generates an action back on the other elements of the source similarly to all the other particles of the source. If the process of measurement were instantaneous we could still hope to obtain a covariant formulation. But if the process calls for a finite extension in time (and space) as is the case in Eq. (31), we lose the covariance, since then we are taking a particular frame of reference, where the test particle is at rest (or is moving with constant velocity, as is the case for the first approximation of the series expansion of (31)).

Appendix

We give here the solution of the integral equation **(31).** This solution will be presented only to the first order of approximation of the Neumann-Liouville series. Writing,

$$\phi^{\alpha}{}_{\lambda} = F_{\beta\lambda} g^{\beta\alpha}, \ \psi^{\alpha}{}_{\nu\lambda} = -\left\{{}^{\alpha}_{\nu\lambda}\right\},$$

we obtain:

$$\begin{aligned} u^{\alpha}(s) = {} & \hat{u}^{\alpha} + \hat{u}^{\lambda}\int_{so}^{s} dz\, \phi^{\alpha}{}_{\lambda}(z) + \hat{u}^{\nu}\hat{u}^{\lambda}\int_{so}^{s} dz\, \psi^{\alpha}{}_{\nu\lambda}(z) + \int_{so}^{s} dz \int_{so}^{s} dz'\, \phi^{\alpha}{}_{\lambda}(z)\phi^{\lambda}{}_{\tau}(z')u^{\tau}(z') + \\ & + 2\hat{u}^{\nu}\int_{so}^{s} dz \int_{so}^{s} dz'\, \psi^{\alpha}{}_{\nu\lambda}(z)\psi^{\lambda}{}_{\kappa\nu}(z')u^{\kappa}(z')u^{\nu}(z') + \\ & + \int_{so}^{s} dz \int_{so}^{s} dz'\, \phi^{\alpha}{}_{\lambda}(z)\psi^{\lambda}{}_{\rho\sigma}(z')u^{\rho}(z')u^{\sigma}(z') + 2\hat{u}^{\nu}\int_{so}^{s} dz \int_{so}^{s} dz'\, \psi^{\alpha}{}_{\nu\lambda}(z)\phi^{\lambda}{}_{\tau}(z')\, u^{\tau}(z') + \\ & + \int_{so}^{s} dz \int_{so}^{s} dz' \int_{so}^{s} dz''\, \psi^{\alpha}{}_{\nu\lambda}(z)\phi^{\nu}{}_{\rho}(z')u^{\rho}(z')\phi^{\lambda}{}_{\tau}(z'')u^{\tau}(z'') + \\ & + 2\int_{so}^{s} dz \int_{so}^{s} dz' \int_{so}^{s} dz''\, \psi^{\alpha}{}_{\nu\lambda}(z)\phi^{\nu}{}_{\rho}(z')u^{\rho}(z')\psi^{\lambda}{}_{\kappa\beta}(z'')u^{\kappa}(z'')u^{\beta}(z'') + \\ & + \int_{so}^{s} dz \int_{so}^{s} dz' \int_{so}^{s} dz''\, \psi^{\alpha}{}_{\nu\lambda}(z)\psi^{\nu}{}_{\rho\sigma}(z')u^{\rho}(z')u^{\sigma}(z')\psi^{\lambda}{}_{\kappa\beta}(z'')u^{\kappa}(z'')u^{\beta}(z''). \end{aligned}$$

References and Notes

1. **A** vector Q_{α} such that $u^{\alpha}Q_{\alpha} = 0$ is allowed to appear on the right hand side of (13), for instance in the combination $\delta^{\alpha}_{\rho}Q_{\sigma}$. However, we do not take into account this possibility.
2. H. Weyl, Sitz. Preuss. Akad. Wiss. **(1918).** Raum-Zeit-Materie, fourth ed., **1921.**
3. R. **H.** Dicke, "The Many Faces of Mach", in Gravitation and Relativity, H. Y. Chiu and W. F. Hoffmann, eds. (Benjamin, **1964).**
4. Y. Thirry, Journal de Mathématiques Pures et Appliquées, série IX, **30,** 275 **(1951).**
5. **T.** Kaluza, Sitz. Preuss. Akad. Wiss., **966 (1921).**
6. If we want to get a correspondence with the metrical formulation, where the scalar interaction may be obtained from the geodesic of the metric $g_{\mu\nu}(1 + e_m\psi)$, we have to replace the equation preceding Eq. **(15)** by

$$b_2\, u^{\rho}\, u^{\sigma}\, \Delta^{\alpha(2)}_{\rho\sigma} = e_m(g^{\rho\alpha} - u^{\rho}\, u^{\alpha})\frac{\partial\phi}{\partial x^{\rho}},$$

where $\phi = \log(1 + e_m\psi)$.
7. Recall that this equation has the form $k^{\nu} k^{\sigma}_{;\nu} = 0$.
8. In this formula only the symmetrical part of the affinity is considered.
9. In order to have a true test particle we should have an affinity independent of the u^{α}, but the correction coming from the Lorentz force depends on the velocities.